

CS 3220: HOMEWORK 1

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Due: September 14, 2020

POLICIES

You may discuss the homework problems freely with other students, but please refrain from looking at their code or writeups (or sharing your own). Ultimately, you must implement your own code and write up your own solution to be turned in. Remember, your solution should be typeset. Your solution, including plots and requested output from your code should be submitted via the CMS as a pdf file. Additionally, please submit any code written for the assignment via the CMS as well.

This homework is intended to be a short warm-up exercise for the linear algebra we will use extensively in this class. In addition, some of the problems may touch on material from prerequisite courses and will hopefully get you thinking about some of those concepts again.

QUESTION 1:

Given a matrix $A \in \mathbb{R}^{m \times n}$ show that the number of linearly independent rows is equal to the number of linearly independent columns (this should go beyond citing some of the results from class that are analogous to this fact).

QUESTION 2:

For $A \in \mathbb{R}^{m \times n}$ show that the null space of $A^T A$ is the same as the null space of A .

QUESTION 3:

Let $Q \in \mathbb{R}^{m \times n}$ with $m > n$ be a matrix with orthonormal columns, show that $Q Q^T \neq I$. (Recall that by definition $Q^T Q = I$. Because we will often deal with matrices that more rows than columns, it is important to keep this distinction in mind as it differs from what is true for square orthogonal matrices. In fact $Q Q^T$ is a very special matrix in this case, one we will discuss extensively later in this class.)

QUESTION 4 (A MORE INVOLVED AND INTERESTING QUESTION):

Consider the following three vector norms we talked about in class:

- $\|x\|_2 = \left(\sum_{i=1}^n x_i^2 \right)^{1/2}$
- $\|x\|_1 = \sum_{i=1}^n |x_i|$
- $\|x\|_\infty = \max_{i \in \{1, 2, \dots, n\}} |x_i|$

Now, let $n = 2$ and draw the “unit ball” for each of these norms. In other words, draw the set of vectors in \mathbb{R}^2 whose norm is equal to 1. Provide your plots and address the following questions:

1. From this plot can you derive any inequalities on $\|x\|_2$, $\|x\|_1$, and $\|x\|_\infty$? (In other words for any x can you provide an ordering on the actual values of the norms?)
2. What are the set of points for which $\|x\|_2 = \|x\|_1 = \|x\|_\infty$?
3. Say you are given a vector a in \mathbb{R}^2 that is non-zero in both entries and a scalar β , describe the set of solutions to the equation $a^T x = \beta$ both algebraically and geometrically (hint: recall from linear algebra that such a system has an infinite number of solutions so you will need to describe it in terms of a free parameter).
4. **(This problem is more challenging, but actually gets at a beautiful geometric idea related to quite recent developments in computer science, statistics, and applied math—specifically in so-called “sparse recovery” problems.)** Building on the previous problem, say someone now asked you the following question: of all the vectors that satisfy $a^T x = \beta$ find one of the “sparsest” ones. In other words, return the solution with the fewest possible non-zero entries. Furthermore, say you are restricted to solving the problem by phrasing it the following way:

$$\min \|x\|_* \quad \text{subject to} \quad a^T x = \beta$$

for $* = \{1, 2, \infty\}$. This means that you simply answer the question by returning the smallest point in the set of solutions to $a^T x = \beta$ and you get to determine how “smallest point” is defined (for the moment do not worry about how one would actually compute this).

Which norm should you use? (you may assume $|a_1| \neq |a_2|$.) Hint: think about this geometrically, consider what the set of solutions to $a^T x = \beta$ looks like and pair that with your understanding of what “level sets” of the norms look like. (They are just scalings of the unit balls you plotted earlier.)