CS3110 Spring 2016 Lecture 5 Modules for Rational Numbers

Mark Bickford and R. Constable

Abstract

This lecture begins our discussion of *modules* in OCaml and module *interfaces*. These language constructs allow us to collect definitions of types and related functions into one syntactic object called a module. The textbook discusses modules at length as do previous lecture notes. These notes for Lecture 5 will not stress the connection to files and compilation. Instead they present modules as a way to organize a related group of types and functions and apply them to defining the abstract type of rational numbers.

We will use modules to organize our implementation of rational numbers and later introduce another module to implement the computational real numbers. Modules provide the basis for *abstract types*. In Classic ML, the original ML programming language from which OCaml evolved, there were no modules, and the ideas discussed here were presented only in terms of *abstract data types*. We briefly mention the reason that Classic ML created these types. They are relevant to the theme of *problem specification* and correct programming that we have already discussed as motivation for many concepts taught in this course. OCaml was built in large part to support these ideas.

1 Lecture Topics

1. History of ML: Classic ML, Standard ML (SML), and OCaml
2. LCF - logic of computable functions
3. ML - meta-language for LCF
    - programming proofs for LCF
    - guarantee that proofs use only the rules of LCF

4. Type system for ML
5. Module for rational numbers
6. Normalizing rational numbers using the gcd algorithm

Top down proof style

In the 1970's computer scientists were exploring how to make programming much more reliable. Several of the most cited and honored research was devoted to this area. Two British computer scientists, C.A.R. Hoare (now Sir Hoare) and Robin Milner were leaders in these efforts as was the Stanford computer scientist John McCarthy (the creator of Lisp) and Dana Scott who had moved to Oxford University. There are many other very eminent researchers in this area that are not mentioned in this lecture. We will mention others as the course progresses. Robin Milner was visiting Stanford and designing a functional programming with features that supported reasoning about programs. He and his colleagues were implementing a logic for reasoning about computable functions based on ideas proposed by Dana Scott [10]. In the 1970's Robin moved to Edinburgh University in Scotland and continued developing his approach. It was called *Edinburgh LCF* [4] because Milner and his team built the system at the University of Edinburgh in Scotland. It became a very influential system.[1]

`http://amturing.acm.org/award_winners/milner_1569367.cfm`

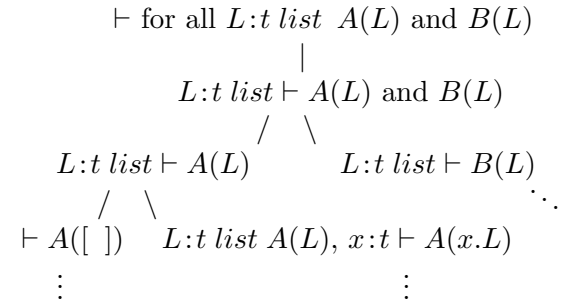
Edinburgh LCF provided a functional programming language for implementing the LCF logic and writing formal proofs in it. This functional language is often called *Classical ML*, the letters ML abbreviate Meta-Language, a central concept in logic. The ML type system includes types for logical formulas and a precise definition of *formal proofs*. ML implemented the concept of an *abstract type* to enforce that proofs can be built only using the formal rules of the logic as expressed in ML. There is no other way for normal users to create a proof than to build it in the abstract type of LCF proofs. Every such object will be a legal proof. The Classical ML type system makes precise the concept of an LCF proof. This

[1] Robin and the second author on these notes became close friends and colleagues, so I can tell stories about his personal role in computer science. He won the Turing Award for this and other fundamental contributions – see the url. He was advising the two authors of these notes on our logic of events when he died in 2010 and the CS community mourned.

implementation of proofs can be shown faithful to the notion of proof as defined by logicians studying the foundations of mathematics, such as Hilbert, Gödel , Church, and Turing among others.

The Edinburgh LCF approach is very general and is now used by other projects to implement more expressive logics and support the creation of formal proofs in them. Systems that support the creation of formal proofs are called *proof assistants*, and they have been built for various logics. Edinburgh LCF was one of the first; Cambridge LCF followed, creating the HOL proof assistant [8, 9, 7]. The Nuprl proof assistant for type theory uses Classic ML [2]. The Coq proof assistant [3, 6, 1] uses a related metalanguage and is implemented in OCaml as is the Meta-PRL [5] proof assistant.

Here is a diagram that shows the structure of a proof as a tree of objects called *sequents*.[2] The only way to build a proof tree is by applying the constructors for the abstract Classical ML type of LCF proofs.



The value of an abstract type is especially clear in the case of proofs. The LCF designers required a mechanism to ensure that *that only logically correct methods could be used to build formal proofs.* They reduced that idea to applying the constructors of the abstract type of LCF proofs. The Classic ML programming language was designed to write programs called *tactics* that are safe programming tools for building formal proofs. Nuprl tactics are Classic ML programs that build proof objects in the ML abstract type of Nuprl proofs.

[2]These trees are frequently drawn top down, with the root at the top and leaves at the bottom.

Algebraic properties of rational numbers

We now apply the notion of an abstract type to define the type of rational numbers and the basic functions to compute with them. We want to confirm that the implementation is correct by showing that the standard arithmetic laws apply. These can be expressed as equalities on the primitive operators, as we list below.

x, y, z: rational.

Commutative	$x + y = y + x$	$x * y = y * x$
Associative	$(x + y) + z = x + (y + z)$	$(x * y) * z = x * (y * z)$
Identity	$x + 0 = x$	$x * 1 = x$
Distributive	$(x + y) * z = x * z + y * z$	
Inverse	$x + (-x) = 0$	$x * \frac{1}{x} = 1$ if $x \neq 0$

<rational, +, $*$, -, $\frac{1}{x}$, 0, 1> is a "field".

Abstract 'Interface'

We now examine how to create a module that uses an interface, denoted *mli*, to provide the abstract signature that defines the module of rational numbers. These ideas are discussed and illustrated in Chapter 4 of *Real World OCaml*. The textbook explains the notion of signatures and abstract types. It also notes that in the context of OCaml, "the terms interface, signature, and module type are all used interchangeably." Here is the OCaml abstract type for rational numbers, *rat.mli*, defined using a module with an interface (indicated by mli).

rat.mli (You should really use *big_int* instead of int.)

```
type rat
val rat0 : rat
val rat1 : rat
val add_rat : rat -> rat -> rat
val mul_rat : rat -> rat-> rat
val inv_rat : rat -> rat
val aminus_rat: rat -> rat
val mk_rat : (int * int) -> rat
```

rat.ml

```
type rat = int * int
  let mk_rat(a,b) =
      if b = 0 then failwith "mk_rat: 0 denominator"
      else (a,b)
let rat0 = (0,1)
let rat1 = (1,1)
```

GCD algorithm

We want to present rational numbers in a canonical form that removes the common factors in the numerator and denominator. We do this by "dividing out" the greatest common divisor.

See 'An Algorithm for the Greatest Common Divisor' by Anne Trostle on the Nuprl website for a more detailed explanation, and the implementation in Nuprl.

```
http://www.nuprl.org/MathLibrary/gcd/
```

$$\frac{1}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{bc}{bd} = \frac{ad + bc}{bd}$$
$$(\frac{1}{2}) + (\frac{1}{2}) = \frac{1 * 2 + 2 * 1}{2 * 2} = \frac{4}{4}$$

We want to put $\frac{a}{b}$ in *lowest terms*.

```
let reduce_rat(a,b) =
    let g = gcd a b in
      (a/g, b/g)

let rec gcd a b =
    if b = 0 then a
    else gcd b (a mod) b
```

Example of the gcd algorithm:

`gcd` 70 18	$70 = 3 * 18 + 16$	$70 = 7 * 5 * 2$
`gcd` 18 16	$18 = 1 * 16 + 2$	$18 = 3 * 3 * 2$
`gcd` 16 2	$16 = 8 * 2 + 0$	
`gcd` 2 0		
2		

$g =$ `gcd` $a\ b$ is the greatest common divsor of a, b.

$g =$ `gcd` $a\ b \Rightarrow$ for some $u,\ v$: $g = ua + vb$.

Prove this by induction on $|b|$.

```
if |b| = 0
  then b = 0
    g = gcd a 0 = a
```

We need $a = \_\_\_ * a + \_\_\_ * 0$, (1,0) solves this.

```
if |b| > 0
   then b != 0
      g = gcd a b = gcd b (a mod b)
```

1. $a = qb + (a \bmod b)$, when $q = a \div b$.

   $|a \bmod b| < |b|$

   So for some u, v:

2. $g = ub + v * (a \bmod b)$

   $va = vqb + v * (a \bmod b)$

3. $g - va = ub - vqb$

   $(u - vq)b$

   $g = va + (u - vq)b$

References

[1] Yves Bertot and Pierre Castéran. *Interactive Theorem Proving and Program Development; Coq'Art: The Calculus of Inductive Constructions*. Texts in Theoretical Computer Science. Springer-Verlag, 2004.

[2] Robert L. Constable, Stuart F. Allen, H. M. Bromley, W. R. Cleaveland, J. F. Cremer, R. W. Harper, Douglas J. Howe, T. B. Knoblock, N. P. Mendler, P. Panangaden, James T. Sasaki, and Scott F. Smith. *Implementing Mathematics with the Nuprl Proof Development System*. Prentice-Hall, NJ, 1986.

[3] Thierry Coquand and G. Huet. The calculus of constructions. *Information and Computation*, 76:95–120, 1988.

[4] Michael Gordon, Robin Milner, and Christopher Wadsworth. *Edinburgh LCF: a mechanized logic of computation*, volume 78 of *Lecture Notes in Computer Science*. Springer-Verlag, NY, 1979.

[5] Jason Hickey, Aleksey Nogin, Robert L. Constable, Brian E. Aydemir, Eli Barzilay, Yegor Bryukhov, Richard Eaton, Adam Granicz, Alexei Kopylov, Christoph Kreitz, Vladimir N. Krupski, Lori Lorigo, Stephan Schmitt, Carl Witty, and Xin Yu. MetaPRL — A modular logical environment. In David Basin and Burkhart Wolff, editors, *Proceedings of the 16th International Conference on Theorem Proving in Higher Order Logics (TPHOLs 2003)*, volume 2758 of *Lecture Notes in Computer Science*, pages 287–303. Springer-Verlag, 2003.

[6] Gérard Huet, Gilles Kahn, and Christine Paulin-Mohring. The Coq proof assistant : A tutorial : Version 6.1. Technical report, INRIA-Rocquencourt, CNRS and ENS Lyon, August 1997.

[7] Tobias Nipkow, Lawrence C. Paulson, and Markus Wenzel. *Isabelle/HOL — A Proof Assistant for Higher-Order Logic*, volume 2283 of *Lecture Notes in Computer Science*. Springer, 2002.

[8] L. Paulson. *Logic and Computation: Interactive Proof with Cambridge LCF*. Cambridge University Press, NY, 1987.

[9] L. C. Paulson. *Standard ML for the Working Programmer*. Cambridge University Press, 1991.

[10] D. Scott. Outline of a mathematical theory of computation. In *Proceedings 4th Annual Princeton Conference on Information Sciences & Systems*, pages 169–176, Princeton, NJ, 1970.