Lecture 15: Calculus in OCAML

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Introduction The basic concepts of calculus are real numbers, functions from reals to reals, continuity of functions, and the derivatives and integrals of functions. Expressing these things in a functional programming language like OCAML lets us compute with these concepts and gives a concrete meaning for mathematical concepts that might seem very abstract. Thinking about the OCAML types for these things can give you a clearer understanding of them.

Integers The integers are represented in OCAML by the type `big_int`. For brevity, I'll write `Int` for the type of integers.

Real numbers We have seen that a real number x is a function from positive integers to rationals such that for all n and m, $|x(n) - x(m)| \leq (\frac{1}{n} + \frac{1}{m})$. We can think of this as saying that the n^{th} approximation $x(n)$ is a rational number that is within distance $\frac{1}{n}$ of the real number x.

We represent a rational number using pair of integers, so it can have OCAML type `Int * Int`. So a real number can have OCAML type `Int -> Int * Int`. But we could code a pair of integers into a single integer, or, even better, we can always normalize the n^{th} approximation $x(n) = \frac{a_n}{b_n}$ to $\frac{c_n}{2n}$ where $c_n = 2n * a_n \div b_n$. Then we can just represent the real number x by the function $\lambda n.c_n$ because the denominator of the n^{th} approximation will always be $2n$.

So, a real number can have the OCAML type `Int -> Int`. Lets write `real` for whichever type we have chosen for the real numbers.

Functions of reals What is the type of a function like $cosine(x)$? Is its OCAML type `real -> real`? The answer is "yes and no". Yes, because that is the best we can do in the OCAML type system. No, because to be a function from reals to reals it has to satisfy an additional property.

Two real numbers x and y are equal if their n^{th} approximations are always within $\frac{2}{n}$ of each other:

$$x = y \Leftrightarrow |x(n) - y(n)| \leq \frac{2}{n}, \text{ for all n}$$

We call an OCAML function `f: real -> real` an *operation* on real numbers. But to be a (mathematical) *function* on real numbers it must satisfy the property that if $x = y$ then $f(x) = f(y)$.

If that holds for all reals x and y in some interval $[a, b]$, then we say that the operation `f` is a *function* defined on interval $[a, b]$.

$$\text{FUN}(f, a, b) \Leftrightarrow (x = y \Rightarrow f(x) = f(y), \text{ for all x,y in } [a, b]$$

Continuous functions A function is *continuous* if its graph does not have any gaps or jumps. So, if x_1 and x_2 are very close together then $f(x_1)$ and $f(x_2)$ must also be close together (otherwise there would be a jump in the graph of f between x_1 and x_2). To say this precisely we need an "epsilon-delta" definition, but that is not complicated. We can represent an arbitrarily small ϵ or δ as $\frac{1}{n}$ where n is a positive integer. The operation `f: real -> real` is (uniformly) continuous on the interval $[a, b]$ if for any ϵ there is a δ such that for all reals x_1 and x_2 in the interval $[a, b]$, if x_1 and x_2 are within δ of each other then $f(x_1)$ and $f(x_2)$ are within ϵ of each other.

$$\text{CONT}(f, a, b) \Leftrightarrow \forall n. \exists m. |x - y| \leq \frac{1}{m} \Rightarrow |f(x) - f(y)| \leq \frac{1}{n}, \text{ for all } x, y \text{ in } [a, b]$$

We can represent the fact that operation `f` is continuous on $[a, b]$ by giving another function `mc: Int->Int`, called the *modulus of continuity* of `f`, that for each n gives the needed m. So an operation `f` is continuous on $[a, b]$ if there is a modulus of continuity `mc` such that, for any positive integer n, if x_1 and x_2 are within $\frac{1}{mc(n)}$ of each other then $f(x_1)$ and $f(x_2)$ will be within $\frac{1}{n}$ of each other.

Integral of a function If f is a continuous function on $[a, b]$ then $\int_a^b f(x)dx$ is the (signed) area under the graph of f between a and b. To get the n^{th} approximation of this real number, we partition the interval $[a, b]$ into parts of length $s = \frac{b-a}{k}$, by letting $p_0 \leq p_1 \leq p_2 \cdots \leq p_k$ be $p_i = a + (i * s)$ so $p_0 = a$ and $p_k = b$, Then we add up the areas of the rectangles $f(p_i) * s$ for $i = 0, 1, \ldots k-1$. We need to choose k big enough and approximate the $f(p_i)$ close enough so that what we get is withing $\frac{1}{n}$ of the true area.

If `mc` is a modulus of continuity for `f` and c is an integer c such that $2n*(b-a) \leq c$, then we let $m = mc(c)$ and choose k so that $s = \frac{b-a}{k} \leq \frac{1}{m}$. Then for any x in the small interval $[p_i, p_{i+1}]$ we will have $|f(x) - f(p_i)| \leq \frac{1}{c}$. So the difference between the area of the rectangle $f(p_i)*s$ and the true area under the curve between p_i and p_{i+1} will be at most $\frac{s}{c}$. If we add up all of these errors we get at most $k * \frac{s}{c}$ which equals $\frac{b-a}{c}$ which is $\leq \frac{1}{2n}$. The sum of the areas of the rectangles is the real number `riemann_sum f a b k` $= s*(f(a)+f(a+s)+\ldots f(a+(k-1)*s))$. The $(2n)^{th}$ approximation of that real number is within $\frac{1}{2n}$ of the area of the rectangles which is within $\frac{1}{2n}$ of the true area. So it is within $\frac{1}{n}$ of the true area.

OCAML code for integral

```
let riemann_sum f a b k =
   let x = rdiv_int (rsubtract b a) k in
   let g i =
      let aa = rmul (bigint2real (sub_big_int k i)) a in
      let bb = rmul (bigint2real i) b in
      rdiv_int (radd aa bb) k in
   let s = rsum (fun i -> f (p i)) zero_big_int (pred_big_int k) in
   rmul s x

let integral mc f a b:real =
 fun n ->
   let nn = mult_int_big_int 2 n in
   let len = canonical_bound (rsubtract b a) in
   let c = mult_big_int nn len in
   let m = mc c in
   let k = mult_big_int m len in
   riemann_sum f a b k nn:q
```

When we load this code into OCAML we get this:

```
val integral:
 (Big_int.big_int -> Big_int.big_int) ->
 (real -> real) -> real -> real -> real = <fun>
```

The inputs to the integral are the modulus of continuity, the operation, and the endpoints. The output is a real.

How to get a modulus of continuity If a function f has a derivative f' then the *Mean Value Theorem* says that $\frac{|f(x)-f(y)|}{|x-y|} = f'(c)$ for some c

between x and y. So, if the maximum of the absolute value of f' on the interval $[a, b]$ is bounded by an integer k, we can use `fun n -> k*n` for a modulus of continuity for f on $[a, b]$.

Running the integral code We can try this out. To calculate the area under the curve $y = x^2$ between 1 and 2 we use

```
integral (fun n -> 4*n) (fun x -> rmul x x) (int2real 1) (int2real 2)
```

Since we know that the derivative of x^2 is $2x$, the maximum of the derivative on the interval [1,2] is 4. That is why we can use `fun n -> 4*n` for the modulus of continuity.

To get two digits of accuracy, we need to approximate the integral within $\frac{1}{100}$ so we apply it to 100. We get 2.33, reasonably fast. But to get three digits accuracy we apply to 1000, and it takes more than a minute to get 2.333.

Let's compute the area under the curve $y = sine(x)$ between 0 and 3. Since the derivative of $sine(x)$ is $cosine(x)$ and that is bounded by 1, we can use the modulus of continuity `fun n -> n`. So we use

```
integral (fun n -> n) (fun x -> sine x) (int2real 0) (int2real 3)
```

We get two digits of accuracy by applying it to 100 and get 1.98 but it takes about a minute.

Fundamental theorem of Calculus Function F is an *anti-derviative* of function of f if $F'(x) = f(x)$.

The fundamental theorem of calculus says that in that case, $\int_a^b f(x)dx = F(b) - F(a)$.

We can use this to compute $\int_1^2 x^2 dx$ quickly because an anti-derivative is $\frac{x^3}{3}$ so the answer is $\frac{2^3}{3} - \frac{1^3}{3} = \frac{7}{3} = 2.3333333333333333...$

For $\int_0^3 sine(x)dx$, the anti-deriviative is $-cosine(x)$, so the answer is $-cosine(3) + cosine(0)$ which is $1 - cosine(3)$. We can compute 20 digits of this very quickly and get 1.98999249660044545727. This agrees with the measly two digits of accuracy we got after a minute using the integral code.

Moral: *The fundamental theorem of calculus is an efficiency result.* It says that a labor-intensive summation can be computed by evaluating an anti-derivative at two points.