Example 1: $((A \Rightarrow B) \wedge \neg B) \Rightarrow \neg A$

Let's suppose we're interested in proving that the following is true:

$$((A \Rightarrow B) \wedge \neg B) \Rightarrow \neg A$$

(This is known as the law of *modus tollens*.) Since the statement we want to prove is an implication, it would make sense to begin by assuming the left side, and seeing if we can derive the right side. So, let's begin with the following:

1.	$(A \Rightarrow B) \wedge \neg B$	(assum)

Now, we have a conjunction, which isn't really doing us much good right now, so let's break it apart.

2.	$A \Rightarrow B$	(1, $\wedge$ elim)
3.	$\neg B$	(1, $\wedge$ elim)

Now, recall that our goal is to derive $\neg A$, given what we just assumed. Since nothing else seems to be jumping out at us right now, let's try to derive it by contradiction. Assume the opposite:

4.	A	(assum)

It just so happens that we have $A \Rightarrow B$, so let's use that.

5.	B	(2,4,$\Rightarrow$ elim)

Now we have B and $\neg B$, which can surely get us somewhere, right? Let's annihilate these together:

6.	$B \Rightarrow \bot$	(3,$\neg$ elim)
7.	$\bot$	(5,6,$\Rightarrow$ elim)

Now we've got $\bot$, which signifies that we've reached a contradiction. If we've derived a contradiction, we can conclude anything we want — this is the purpose of the $\bot$ elimination rule. In this case, we're trying to conclude

8.	$\neg A$	(4,$\bot$ elim)

This also discharges one of our assumptions. Now we've gotten $\neg A$, which is what we wanted, so we can go ahead and discharge our other assumption to get the implication.

9.	$((A \Rightarrow B) \wedge \neg B) \Rightarrow \neg A$	(1,8,$\Rightarrow$ intr)

And now we're done! The full proof, written in sequential steps, is as follows.

1.	$(A \Rightarrow B) \wedge \neg B$	(assum)
2.	$A \Rightarrow B$	(1, $\wedge$ elim)
3.	$\neg B$	(1, $\wedge$ elim)
4.	A	(assum)
5.	B	(2,4,$\Rightarrow$ elim)
6.	$B \Rightarrow \bot$	(3, $\neg$ intr)
7.	$\bot$	(5,6,$\Rightarrow$ elim)
8.	$\neg A$	(4, $\bot$ elim)
9.	$((A \Rightarrow B) \wedge \neg B) \Rightarrow \neg A$	(1,8,$\Rightarrow$ intr)

And presented as a tree, the proof looks like this.

$$
\begin{prooftree}
\AxiomC{}
\LeftLabel{(assum)}
\UnaryInfC{$(A \Rightarrow B) \wedge \neg B, A \vdash (A \Rightarrow B) \wedge \neg B$}
\LeftLabel{($\wedge$ elim)}
\UnaryInfC{$(A \Rightarrow B) \wedge \neg B, A \vdash A \Rightarrow B$}
\AxiomC{}
\LeftLabel{(assum)}
\UnaryInfC{$(A \Rightarrow B) \wedge \neg B, A \vdash A$}
\LeftLabel{($\Rightarrow$ elim)}
\BinaryInfC{$(A \Rightarrow B) \wedge \neg B, A \vdash B$}
\AxiomC{}
\RightLabel{(assum)}
\UnaryInfC{$(A \Rightarrow B) \wedge \neg B, A \vdash (A \Rightarrow B) \wedge \neg B$}
\RightLabel{($\wedge$ elim)}
\UnaryInfC{$(A \Rightarrow B) \wedge \neg B, A \vdash \neg B$}
\RightLabel{($\neg$ elim)}
\UnaryInfC{$(A \Rightarrow B) \wedge \neg B, A \vdash B \Rightarrow \bot$}
\LeftLabel{($\Rightarrow$ elim)}
\BinaryInfC{$(A \Rightarrow B) \wedge \neg B, A \vdash \bot$}
\LeftLabel{($\Rightarrow$ intr)}
\UnaryInfC{$(A \Rightarrow B) \wedge \neg B \vdash A \Rightarrow \bot$}
\LeftLabel{($\neg$ intr)}
\UnaryInfC{$(A \Rightarrow B) \wedge \neg B \vdash \neg A$}
\LeftLabel{($\Rightarrow$ intr)}
\UnaryInfC{$\vdash ((A \Rightarrow B) \wedge \neg B) \Rightarrow \neg A$}
\end{prooftree}
$$

Notice that in the above proof tree, the assumptions are copied from line to line for most of the rules. To help avoid repetition and clutter, we sometimes write $\cdots$ in the assumptions to mean "the assumptions from the line below". Here is the same proof using this convention:

$$
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        \dfrac{
          \dfrac{\dfrac{}{(A \Rightarrow B) \land \neg B, A \vdash (A \Rightarrow B) \land \neg B}\,(\text{assum})}{\cdots \vdash A \Rightarrow B}\,(\land\ \text{elim})
          \qquad
          \dfrac{}{(A \Rightarrow B) \land \neg B, A \vdash A}\,(\text{assum})
        }{\cdots \vdash B}\,(\Rightarrow \text{elim})
        \qquad
        \dfrac{\dfrac{\dfrac{}{(A \Rightarrow B) \land \neg B, A \vdash (A \Rightarrow B) \land \neg B}\,(\text{assum})}{\cdots \vdash \neg B}\,(\land\ \text{elim})}{\cdots \vdash B \Rightarrow \bot}\,(\neg\ \text{elim})
      }{\cdots, A \vdash \bot}\,(\Rightarrow \text{elim})
    }{\cdots \vdash A \Rightarrow \bot}\,(\Rightarrow \text{intr})
  }{(A \Rightarrow B) \land \neg B \vdash \neg A}\,(\neg\ \text{intr})
}{\vdash ((A \Rightarrow B) \land \neg B) \Rightarrow \neg A}\,(\Rightarrow \text{intr})
$$

Example 2: $(A \Rightarrow (B \vee (A \Rightarrow B))) \Rightarrow (A \Rightarrow B)$

Here, the claim to prove is

$$(A \Rightarrow (B \vee (A \Rightarrow B))) \Rightarrow (A \Rightarrow B)$$

We give a briefer description of how to prove it.

As before, we're trying to prove an implication, so let's start by assuming the left side, (1) $(A \Rightarrow (B \vee (A \Rightarrow B)))$. We're looking to derive $A \Rightarrow B$ from this assumption. So let's now assume the left side of the smaller implication, (2) A.

Now, we see that (2) is the left side of (1), so we can use our implication rules and derive (3) $(B \vee (A \Rightarrow B))$. We have a disjunction now, and we want to eliminate it, which means we need to prove that both sides imply the same thing. This amounts to assuming both sides of the disjunction (both B and $A \Rightarrow B$), and showing that they both lead to the same thing. It just so happens that one half of the disjunction is just B, so let's see if $A \Rightarrow B$ implies B as well.

Firstly, it's easy to derive $B \Rightarrow B$ from thin air: assume (4) B, and then discharge the assumption to get (5) $B \Rightarrow B$.

So now let's assume the other part, (6) $A \Rightarrow B$. Recall that statement (2) is just A, and that assumption hasn't been discharged yet. So we can obtain (7) B, and discharging (6), we get $(A \Rightarrow B) \Rightarrow B$.

Now we're in good shape, because both sides of (3) imply B. So using disjunction elimination, we can just conclude (8) B. From discharging our assumption (2), we get (9) $A \Rightarrow B$. And finally, from discharging our assumption (1), we get (10) $(A \Rightarrow (B \vee (A \Rightarrow B))) \Rightarrow (A \Rightarrow B)$.

The proof tree is below.

$$
\text{(}\Rightarrow\text{ intr)}\;\dfrac{\text{(}\Rightarrow\text{ intr)}\;\dfrac{\text{(}\vee\text{ elim)}\;\dfrac{\text{(}\Rightarrow\text{ elim)}\;\dfrac{\text{(assum)}\;\dfrac{}{\cdots \vdash A \Rightarrow (B \vee (A \Rightarrow B))} \quad \dfrac{}{\cdots \vdash A}\;\text{(assum)}}{\cdots \vdash B \vee (A \Rightarrow B)} \quad \dfrac{\dfrac{}{\cdots , B \vdash B}\;\text{(assum)}}{\cdots \vdash B \Rightarrow B}\;\text{(}\Rightarrow\text{ intr)} \quad \dfrac{\dfrac{\text{(assum)}\;\dfrac{}{\cdots \vdash A \Rightarrow B} \quad \dfrac{}{\cdots \vdash A}\;\text{(assum)}}{\cdots , A \Rightarrow B \vdash B}\;\text{(}\Rightarrow\text{ elim)}}{\cdots \vdash (A \Rightarrow B) \Rightarrow B}\;\text{(}\Rightarrow\text{ intr)}}{\cdots , A \vdash B}}{(A \Rightarrow (B \vee (A \Rightarrow B))) \vdash A \Rightarrow B}}{\vdash (A \Rightarrow (B \vee (A \Rightarrow B))) \Rightarrow (A \Rightarrow B)}
$$

Example 3: $\neg((\neg A \lor \neg B) \land (A \land B))$

We only give the proof tree below. See if you can walk yourself through the proof as we did for the previous 2 examples. Interestingly, in this proof, we are not proving an implication, so the way we start is by assuming the opposite of what we want, and deriving $\bot$.

$$\frac{}{\cdots \vdash (\neg A \lor \neg B) \land (A \land B)}\ (\text{assum}) \qquad \frac{\cdots \vdash (\neg A \lor \neg B) \land (A \land B)}{\cdots \vdash \neg A \lor \neg B}\ (\land\ \text{elim})$$

$$\frac{}{\cdots \vdash \neg A}\ (\text{assum}) \qquad \frac{\cdots \vdash \neg A}{\cdots \vdash A \Rightarrow \bot}\ (\neg\ \text{elim})$$

$$\frac{}{\cdots \vdash (\neg A \lor \neg B) \land (A \land B)}\ (\text{assum}) \qquad \frac{\cdots \vdash (\neg A \lor \neg B) \land (A \land B)}{\cdots \vdash A \land B}\ (\land\ \text{elim}) \qquad \frac{\cdots \vdash A \land B}{\cdots \vdash A}\ (\land\ \text{elim})$$

$$\frac{\cdots \vdash A \Rightarrow \bot \qquad \cdots \vdash A}{\cdots , B \vdash \bot}\ (\Rightarrow \text{elim}) \qquad \frac{\cdots , B \vdash \bot}{\cdots \vdash B \Rightarrow \bot}\ (\Rightarrow \text{intro}) \qquad \frac{\cdots \vdash B \Rightarrow \bot}{\cdots , \neg A \vdash \neg B}\ (\neg\ \text{intro}) \qquad \frac{\cdots , \neg A \vdash \neg B}{\cdots \vdash \neg A \Rightarrow \neg B}\ (\Rightarrow \text{intr})$$

$$\frac{}{\cdots , \neg B \vdash \neg B}\ (\text{assum}) \qquad \frac{\cdots , \neg B \vdash \neg B}{\cdots \vdash \neg B \Rightarrow \neg B}\ (\Rightarrow \text{intr})$$

$$\frac{\cdots \vdash \neg A \lor \neg B \qquad \cdots \vdash \neg A \Rightarrow \neg B \qquad \cdots \vdash \neg B \Rightarrow \neg B}{\cdots \vdash \neg B}\ (\lor\ \text{elim}) \qquad \frac{\cdots \vdash \neg B}{\cdots \vdash B \Rightarrow \bot}\ (\neg\ \text{elim})$$

$$\frac{}{\cdots \vdash (\neg A \lor \neg B) \land (A \land B)}\ (\text{assum}) \qquad \frac{\cdots \vdash (\neg A \lor \neg B) \land (A \land B)}{\cdots \vdash A \land B}\ (\land\ \text{elim}) \qquad \frac{\cdots \vdash A \land B}{\cdots \vdash B}\ (\land\ \text{elim})$$

$$\frac{\cdots \vdash B \Rightarrow \bot \qquad \cdots \vdash B}{((\neg A \lor \neg B) \land (A \land B)) \vdash \bot}\ (\Rightarrow \text{elim})$$

$$\frac{((\neg A \lor \neg B) \land (A \land B)) \vdash \bot}{\vdash ((\neg A \lor \neg B) \land (A \land B)) \Rightarrow \bot}\ (\Rightarrow \text{intr}) \qquad \frac{\vdash ((\neg A \lor \neg B) \land (A \land B)) \Rightarrow \bot}{\vdash \neg((\neg A \lor \neg B) \land (A \land B))}\ (\neg\ \text{intr})$$

Section 4: $(\forall x.\forall y.(P(x) \Rightarrow Q(y)) \wedge \exists x.P(x)) \Rightarrow \exists y.Q(y)$

The following two examples are proofs in predicate logic. Pay close attention to how the $\forall$ and $\exists$ quantifiers are treated.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\cdots \vdash \forall y.\forall x.(P(x) \Rightarrow Q(y)) \wedge \exists x.P(x)}\ (\text{assum})}{\cdots \vdash \exists x.P(x)}\ (\wedge\ \text{elim}) \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\cdots \vdash \forall y.\forall x.(P(x) \Rightarrow Q(y)) \wedge \exists x.P(x)}\ (\text{assum})}{\cdots \vdash \forall y.\forall x.(P(x) \Rightarrow Q(y))}\ (\wedge\ \text{elim})}{\cdots \vdash \forall x.(P(x) \Rightarrow Q(b))}\ (\forall\ \text{elim})}{\cdots \vdash P(a) \Rightarrow Q(b)}\ (\forall\ \text{elim})}{\cdots \vdash Q(b)}\ (\exists\ \text{elim})}{\forall x.\forall y.(P(x) \Rightarrow Q(y)) \wedge \exists x.P(x) \vdash \exists y.Q(y)}\ (\exists\ \text{intro})}{\vdash (\forall x.\forall y.(P(x) \Rightarrow Q(y)) \wedge \exists x.P(x)) \Rightarrow \exists y.Q(y)}\ (\Rightarrow\ \text{intr})
$$

Example 5: $\forall x.P(x) \Rightarrow \forall x.Q(x) \Rightarrow \forall x.(P(x) \Leftrightarrow Q(x))$

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\dfrac{}{\cdots \vdash P(n)}\,(\text{assum})
\quad
\dfrac{\dfrac{}{\cdots \vdash \forall x.Q(x)}\,(\text{assum})}{\cdots \vdash Q(n)}\,(\forall\ \text{elim})
}{\cdots \vdash P(n) \land Q(n)}\,(\land\ \text{intr})
}{\cdots , P(n) \vdash Q(n)}\,(\land\ \text{elim})
}{\cdots \vdash P(n) \Rightarrow Q(n)}\,(\Rightarrow\ \text{intr})
\quad
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\dfrac{
\dfrac{
\dfrac{}{\cdots \vdash Q(n)}\,(\text{assum})
\quad
\dfrac{\dfrac{}{\cdots \vdash \forall x.P(x)}\,(\text{assum})}{\cdots \vdash P(n)}\,(\forall\ \text{elim})
}{\cdots \vdash Q(n) \land P(n)}\,(\land\ \text{intr})
}{\cdots , Q(n) \vdash P(n)}\,(\land\ \text{elim})
}{\cdots \vdash Q(n) \Rightarrow P(n)}\,(\Rightarrow\ \text{intr})
}{\cdots \vdash P(n) \Leftrightarrow Q(n)}\,(\Leftrightarrow\ \text{intr})
}{\cdots , \forall x.Q(x) \vdash \forall x.P(x) \Leftrightarrow Q(x)}\,(\forall\ \text{intr})
}{\cdots , \forall X.P(x) \vdash \forall x.Q(x) \Rightarrow (\forall x.P(x) \Leftrightarrow Q(x))}\,(\Rightarrow\ \text{intr})
}{\cdots \vdash \forall x.P(x) \Rightarrow \forall x.Q(x) \Rightarrow \forall x.(P(x) \Leftrightarrow Q(x))}\,(\Rightarrow\ \text{intr})
$$