CS 3110

Lecture 24: Efficiency

Prof. Clarkson

Fall 2014

Today's music: Opening theme from *The Big O*
(THE ビッグオ)
by Toshihiko Sahashi
And a bonus: Pokémon Theme

Review

Course so far:

- Introduction to functional programming
- Modular programming
- Advanced topics in functional programming
- Reasoning about correctness

Next:

- Reasoning about performance
- Today:
  - What it means to be efficient

Question #1

Which is your favorite Steammon?

A. Blastoise

B. Mewtwo

C. Pikachu

D. Charizard

E. What's a Steammon?

Performance

- You've built beautiful, elegant, functional code
- You've organized it into modules with clear, sufficiently general, sufficiently restrictive specifications
- You've established assurance through a combination of testing and verification
- **Finally**, you begin to worry about performance
  - Some part of code is too slow
  - You want to find a more *efficient* algorithm

What is "efficiency"?

Attempt #1: An algorithm is efficient if, when implemented, it runs quickly on particular input instances

...problems with that?

What is "efficiency"?

- **Attempt #1:** An algorithm is efficient if, when implemented, it runs quickly on particular input instances
- Problems:
  - Inefficient algorithms can run quickly on small test cases
  - Fast processors and optimizing compilers can make inefficient algorithms run quickly
  - Good algorithms can run slowly when coded sloppily
  - Some input instances are harder than others
  - Efficiency on small inputs doesn't imply efficiency on large inputs
  - Some clients can afford to be more patient than others; quick for me might be slow for you

Lessons learned from attempt #1

Lesson 1: Time as measured by a clock is not the right metric

- Want a metric that is reasonably independent of hardware, compiler, other software running, etc.
- **idea:** number of steps taken by dynamic semantics during evaluation of program
  - steps are independent of implementation details
  - each step might really take a different amount of time?
    - creating a closure, looking up a variable, computing an addition
  - in practice, the difference isn't really big enough to matter

Lessons learned from attempt #1

Lesson 2: Running time on particular input instances is not the right metric

- Want a metric that can predict running time on **any** input instance
- **idea:** size of the input instance
  - make metric be a function of input size
  - (combined with lesson 1) specifically, the maximum number of steps for an input of that size
  - particular inputs of the same size might really take a different amount of time?
    - multiplying arbitrary matrices vs. multiplying by all zeros
  - in practice, size matters more

Lessons learned from attempt #1

Lesson 3: Quickness is not the right metric

- Want a metric that is reasonably objective; independent of subjective notions of what is fast
- **idea:** beats brute-force search
  - enumerate all the answers one by one, check and see whether the answer is right
    - the simply, dumb solution to nearly any algorithmic problem
    - related idea: guess an answer, check whether correct e.g., bogosort
  - but *by how much* is enough to beat brute-force search?

Lessons learned from attempt #1

Lesson 3: Quickness is not the right metric

- Want a metric that is reasonably objective; independent of subjective notions of what is fast
- **better idea:** polynomial time
  - (combined with ideas from previous two lessons)
    can express maximum number of steps as a polynomial function of the size N of input, e.g.,
    - aN^2 + bN + c
  - some polynomials might be too big to be quick (N^100)?
  - some non-polynomials might be quick enough (N^(1+.02*(log N)))?
  - in practice, polynomial time really does work

What is "efficiency"?

- **Attempt #2**: An algorithm is efficient if its maximum number of steps of execution is polynomial in the size of its input.
- **In brief**: *efficient* means worst-case polynomial running time.

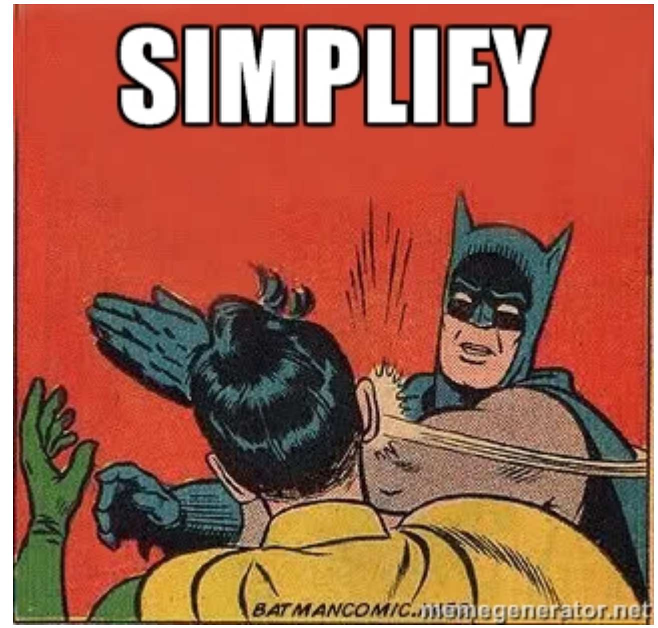
Precision of running time

INSERTION-SORT(A)	*cost*	*times*
for j = 2 **to** A.length	c_1	n
key = *A*[*j*]	c_2	$n - 1$
// Insert *A*[*j*] into the sorted sequence *A*[1 .. *j* - 1]	0	$n - 1$
i = *j* - 1	c_4	$n - 1$
while i > 0 and *A*[*i*] < *key*	c_5	$\sum_{j=2}^{n} t_j$
A[*i* + 1] = *A*[*i*]	c_6	$\sum_{j=2}^{n} (t_j - 1)$
i = *i* - 1	c_7	$\sum_{j=2}^{n} (t_j - 1)$
A[*i* + 1] = *key*	c_8	$n - 1$

[Cormen et al. *Introduction to Algorithms*, 3rd ed, 2009]

Precision of running time

INSERTION-SORT(A)	*cost*	*times*
1 **for j = 2 to A.length**	c_1	n
2 *key* = *A*[*j*]	c_2	$n - 1$
3 // Insert *A*[*j*] into the sorted sequence *A*[1 .. *j* - 1]	0	$n - 1$
4 *i* = *j* - 1	c_4	$n - 1$
5 **while** i > 0 and *A*[*i*] < *key*	c_5	$\sum_{j=2}^{n} t_j$
6 *A*[*i* + 1] = *A*[*i*]	c_6	$\sum_{j=2}^{n} (t_j - 1)$
7 *i* = *i* - 1	c_7	$\sum_{j=2}^{n} (t_j - 1)$
8 *A*[*i* + 1] = *key*	c_8	$n - 1$



The running time of the algorithm is the sum of running times for each statement executed; a statement that takes c_i steps to execute and executes n times will contribute $c_i n$ to the total running time.[6] To compute $T(n)$, the running time of INSERTION-SORT on an input of n values, we sum the products of the *cost* and *times* columns, obtaining

$$T(n) = c_1 n + c_2(n-1) + c_4(n-1) + c_5 \sum_{j=2}^{n} t_j + c_6 \sum_{j=2}^{n} (t_j - 1) + c_7 \sum_{j=2}^{n} (t_j - 1) + c_8(n-1) .$$

[Cormen et al. *Introduction to Algorithms*, 3rd ed, 2009]

Precision of running time

- Precise bounds are exhausting to find
- Precise bounds are to some extent meaningless
  - Are those constants c1..c8 really useful?
  - If it takes 25 steps in high level language, but compiled down to assembly would take 10x more steps, is the precision useful?
  - Caveat: if you're building code that flies an airplane or controls a nuclear reactor, you do care about precise, real-time guarantees

Some simplified running times

max # steps as function of N

size of input

	N	N^2	N^3	2^N
N=10	< 1 sec	< 1 sec	< 1 sec	< 1 sec
N=100	< 1 sec	< 1 sec	1 sec	10^17 years
N=1,000	< 1 sec	1 sec	18 min	very long
N=10,000	< 1 sec	2 min	12 days	very long
N=100,000	< 1 sec	3 hours	32 years	very long
N=1,000,000	1 sec	12 days	10^4 years	very long

assuming 1microsecond/step

very long = more years than estimated number of atoms in universe

Simplifying running times

- Rather than 1.62N^2 + 3.5N + 8 steps, we would rather say that running time "grows like N^2"
  - identify broad classes of algorithm with similar performance
- Ignore the *low-order terms*
  - e.g., ignore 3.5N+8
  - Why? For big N, N^2 is much, much bigger than N
- Ignore the *constant factor* of high-order term
  - e.g., ignore 1.62
  - Why? For classifying algorithms, constants aren't meaningful
  - Caveat: Performance tuning real-world code actually can be about getting the constants to be small!
- Abstraction to an imprecise quantity

Imprecise abstractions

- OCaml's **`int`** type is an abstraction of a subset of Z
  - don't know which int when reasoning about the type of an expression
- ±1 is an abstraction of {1,-1}
  - don't know which when manipulating it in a formula
- Here's a new one: Big Ell
  - L(e) represents an integer whose absolute value is less than or equal to the absolute value of e
  - precisely, L(e) = {m | abs(m) <= abs(e)}
  - e.g., L(5) = {-5,-4,-3,...,3,4,5}

Manipulating Big Ell

- What is 1 + L(5)?
- Trick question!
  - Replace L(5) with set: 1 + {-5,-4,-3,...,3,4,5}
  - But + is defined on ints, not sets of ints
- We could distribute the + over the set:
  {1-5,1-4,...,1+4,1+5} = {-4,-2,...,4,6}
  - That is, a set of values, one for each possible instantiation of L(5)
- Note that {-4,-3,...,5,6} ⊆ {-6,-5,-4-3,-2,...,4,5,6} = L(6)
- So we could say that 1 + L(5) ⊆ L(6)
- Or, in a serious abuse of notation, we could say that
  1 + L(5) = L(6)

Question #2

What is L(2) + L(3)?

Hint: set of values, one for each possible instantiation of L(2) and of L(3)

A. L(2) + L(3) ⊆ L(2)
B. L(2) + L(3) ⊆ L(3)
C. L(2) + L(3) ⊆ L(4)
D. L(2) + L(3) ⊆ L(5)
E. L(2) + L(3) ⊆ L(6)

Question #2

What is L(2) + L(3)?

Hint: set of values, one for each possible instantiation of L(2) and of L(3)

A. $L(2) + L(3) \subseteq L(2)$

B. $L(2) + L(3) \subseteq L(3)$

C. $L(2) + L(3) \subseteq L(4)$

D. $L(2) + L(3) \subseteq L(5)$

E. $L(2) + L(3) \subseteq L(6)$

Question #3

What is L(5) - L(3)?

A. $L(5) - L(3) \subseteq L(2)$
B. $L(5) - L(3) \subseteq L(3)$
C. $L(5) - L(3) \subseteq L(5)$
D. $L(5) - L(3) \subseteq L(7)$
E. $L(5) - L(3) \subseteq L(8)$

Question #3

What is L(5) - L(3)?

A. L(5) - L(3) ⊆ L(2)

B. L(5) - L(3) ⊆ L(3)

C. L(5) - L(3) ⊆ L(5)

D. L(5) - L(3) ⊆ L(7)

E. L(5) - L(3) ⊆ L(8)

Even harder…

What is 2^L(3)?

- L(3) = {-3,-2,…2,3}
- So 2^L(3) could be any of {2^(-3), 2^(-2),…,2^2,2^3} = {1/8, 1/4, …, 4, 8}
- And {1/8, 1/4, …, 4, 8} ⊆ L(8) = L(2^3)
- Therefore 2^L(3) ⊆ L(2^3)

…we can use this idea of Big Ell to invent an imprecise abstraction for running times

Big Oh, take 1

- Recall: we're interested in running time as a function of input size
- "New" imprecise abstraction: Big Oh
  - O(g) represents a function f whose absolute value is less than or equal to the absolute value of function g, for every input n.
  - precisely, O(g) = {f | forall n, abs(f(n)) <= abs(g(n))}
  - e.g., O(fun n -> 2n) = {f | forall n, abs(f(n)) <= abs(2n)}
    - (fun n -> n) ∈ O(fun n -> 2n)
    - (fun n -> n) ∈ O(fun n -> n^100)

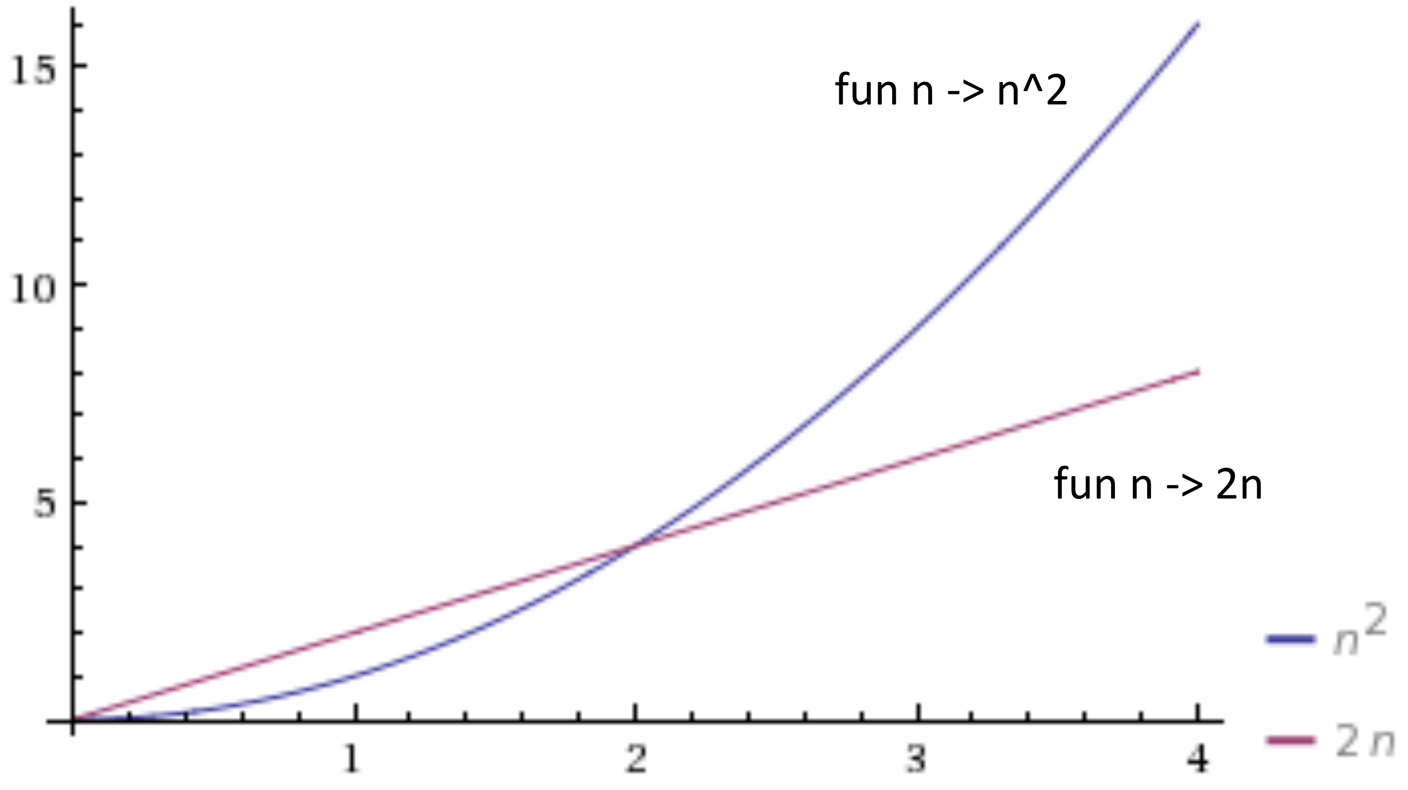
Big Oh, take 2

Recall: we want to ignore constant factors

- O(g) represents a function f whose absolute value is less than or equal to the absolute value of function g times some positive constant c, for every input n.
- precisely, O(g) = {f | exists c>0, forall n,
  abs(f(n)) <= c * abs(g(n))}
- e.g., O(fun n -> n^3) = {f | exists c>0, forall n,
  abs(f(n)) <= c * abs(n^3)}
  - (fun n -> 3*n^3) ∈ O(fun n -> n^3)
    because 3*n^3 <= c * n^3, where c = 3

Big Oh, take 3

Recall: we care about what happens at scale



Big Oh, take 3

Recall: we care about what happens at scale

- O(g) represents a function f whose absolute value is less than or equal to the absolute value of function g times some positive constant c, for every input n greater than or equal to some positive constant n0.
- precisely, O(g) = {f | exists c>0, n0>0, forall n >= n0,
  abs(f(n)) <= c * abs(g(n))}
- e.g., O(fun n -> n^2) = {f | exists c>0, n0>0, forall n >= n0,
  abs(f(n)) <= c * abs(n^2)}
  - (fun n -> 2n) ∈ O(fun n -> n^2)
    because 2n <= c * n^2, where c = 1, for all n >= 2
  - (fun n -> 3110) ∈ O(fun n -> 1)
    because 3110 <= c * n, where c = 3110, for all n >= 1

Big Oh

```
O(g) = {f | exists c>0, n0>0,
            forall n >= n0,
            abs(f(n)) <= c * abs(g(n))}
```

- Most authors write "O(g(n)) = {f(n) | ..." in definitions
- They don't really mean g applied to n; they mean a function g parameterized on input n but not yet applied
- Maybe they never studied functional programming ☺

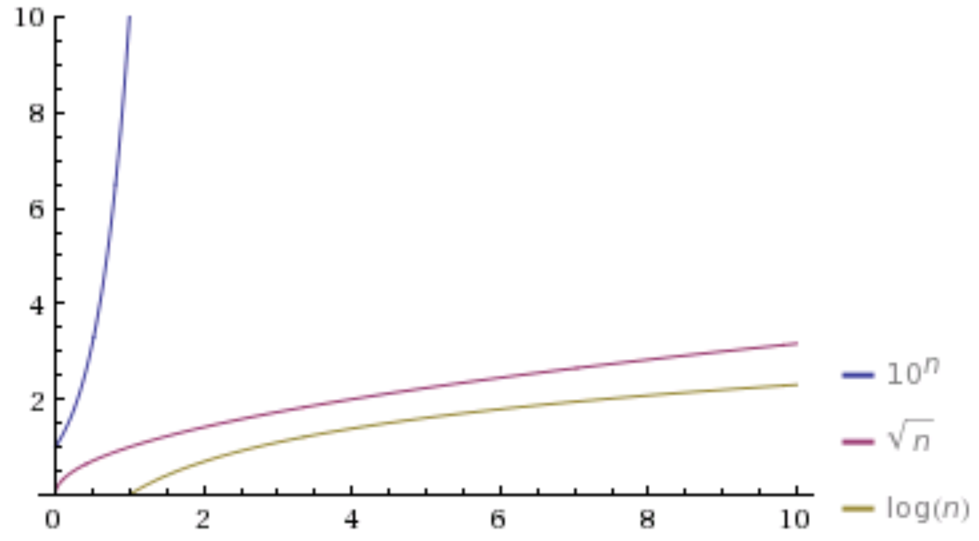
Big Oh

O(g) = {f | exists c>0, n0>0,
forall n >= n0,
abs(f(n)) <= c * abs(g(n))}

- All authors write, e.g.,
  - 2n = O(n^2) instead of
  - (fun n -> 2n) ∈ O(fun n -> n^2)
- Your instructor has always found this abusage distressing
- Yet henceforth he will follow the convention ☺
  - The standard defense is that = should be read here as "is" not as "equals"
- You must be careful with quantity is on the RHS: one-directional equality!

Question #4

Arrange these functions in *ascending order of growth*: if f is immediately before g, then f=O(g).



(fun n -> 10^n)

(fun n -> sqrt(n))

(fun n -> ln(n))

A. 10^n, sqrt(n), ln(n)
B. 10^n, ln(n), sqrt(n)
C. sqrt(n), ln(n), 10^n
D. sqrt(n), 10^n, ln(n)
E. ln(n), sqrt(n), 10^n
F. ln(n), 10^n, sqrt(n)

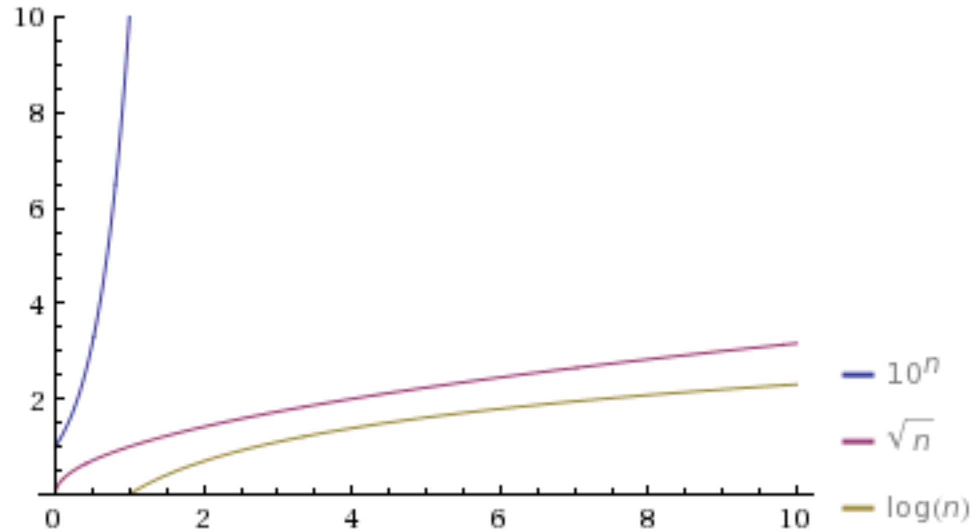
Question #4

Arrange these functions in *ascending order of growth*: if f is immediately before g, then f=O(g).

(fun n -> 10^n)

(fun n -> sqrt(n))

(fun n -> ln(n))



A. 10^n, sqrt(n), ln(n)
B. 10^n, ln(n), sqrt(n)
C. sqrt(n), ln(n), 10^n
D. sqrt(n), 10^n, ln(n)
E. **ln(n), sqrt(n), 10^n**
F. ln(n), 10^n, sqrt(n)

A Theory of Big Oh

- reflexivity: f = O(f)
- *(no symmetry condition for Big Oh; there is one for Big Theta)*
- transitivity: f = O(g) /\ g = O(h) => f = O(h)
- c * O(f) = O(f)
- O(c * f) = O(f)
- O(f) + O(g) = O(|f| + |g|)
  - where |f| + |g| means (fun n -> abs(f(n)) + abs(g(n)))
- O(f) * O(g) = O(f * g)
  - where f * g means (fun n -> f(n)*g(n))
- …

Competency with Big Oh requires knowing at least this much of its theory

What is "efficiency"?

Final attempt: An algorithm is efficient if its worst-case running time is O(N^d) for some constant d.

Running times of some algorithms

- **O(1):** access an element of an array (of length n)
- **O(log n):** binary search through sorted array of length n
- **O(n):** maximum element of list of length n
- **O(n log n):** mergesort a list of length n
- **O(n^2):** bubblesort an array of length n
- **O(n^3):** matrix multiplication of n-by-n matrices
- **O(2^n):** enumerate all integers of bit length n

…some of these are not obvious, require proof

Want to learn more?

- Take CS 4820 Algorithms
- Much of today's material from:
  - *Algorithm Design* by Jon Kleinberg and Éva Tardos
  - *Concrete Mathematics* by Graham, Knuth, Patashnik
  - *Introduction to Algorithms* by Cormen, Leiserson, Rivest, and Stein

Please hold still for 1 more minute

WRAP-UP FOR TODAY

Upcoming events

- Clarkson office hours cancelled today
- **Thanksgiving Break**: no class, consulting hours, or office hours Wed. or Thur.
- **PS6 due in 9 days**, no late passes

This is efficient.

THIS IS 3110