Suppose that you toss a coin that has success pobability of p n times. What's the expected number of heads? I gave an intuitive induction argument in class today whose formalization may not have been so clear. Here are the details.

I'm actually using a general fact that was proved formally in Friday's class, which should be pretty intuitive:

$$E(X) = \Pr(A)E(X \mid A) + \Pr(\overline{A})E(X \mid \overline{A}).$$

Here, the A is "get heads on the first coin toss", so $\Pr(A) = p$, and $\Pr(\overline{A}) = 1 - p$. I want to show that $E(B_{n,p} \mid \overline{A}) = B_{n-1,p}$ and $E + (B_{n,p} \mid A) = 1 + B_{n-1,p}$. This will give me

$$E(B_{n,p}) = p(1 + E(B_{n-1,p})) + (1 - p)E(B_{n-1,p}),$$

which is what I claimed in the class notes.

The argument that $E(B_{n,p} \mid \overline{A}) = B_{n-1,p}$ is straightforward. Conditional on getting tails in the first coin toss, the probability of getting k heads is exactly the probability of getting k heads in the last $n - 1$ coin tosses, so the expected number of heads is $\sum_{k=0}^{n-1} k \Pr(k \text{ heads in last } n-1 \text{ tosses})$; this is exactly $B_{n-1,p}$.

Showing that $E(B_{n,p} \mid \overline{A}) = B_{n-1,p}$ is slightly harder. Conditional on getting heads in the first coin toss, the probability of getting k heads is exactly the probability of getting $k - 1$ heads in the last $n - 1$ coin tosses, so the expected number of heads conditional on getting heads in the first coin toss is

$$\begin{aligned}
& \sum_{k=0}^{n} k \Pr(k-1 \text{ heads in last } n-1 \text{ tosses}) \\
= & \sum_{k=0}^{n} (1 + (k-1)) \Pr(k-1 \text{ heads in last } n-1 \text{ tosses}) \\
= & \sum_{k=1}^{n} \Pr(k-1 \text{ heads in last } n-1 \text{ tosses}) + \sum_{k=1}^{n} (k-1) \Pr(k-1 \text{ heads in last } n-1 \text{ tosses}) \\
= & \sum_{k=0}^{n-1} \Pr(k \text{ heads in last } n-1 \text{ tosses}) + \sum_{k=0}^{n-1} k \Pr(k \text{ heads in last } n-1 \text{ tosses}) \\
= & 1 + E(B_{n-1,p}).
\end{aligned}$$

As I said in class, once you show that expectation is linear (which is not at all hard, and was done in class) that provides an even simpler argument ...