Relations

- **Cartesian product**:
  $S \times T = \{(s,t) : s \in S, t \in T\}$
  - $\{1,2,3\} \times \{3,4\} =$
    $\{(1,3),(2,3),(3,3),(1,4),(2,4),(3,4)\}$
  - $|S \times T| = |S| \times |T|$.
- A *relation* on S and T (or, on $S \times T$) is a subset of $S \times T$
- A *relation* on S is a subset of $S \times S$
  - *Taller than* is a relation on people: (Joe,Sam) is in the Taller than relation if Joe is Taller than Sam
  - *Greater than* is a relation on $\boldsymbol{R}$ (the real numbers):

$$L = \{(x,y) : x,y \in R, x > y\}$$

  - *Divisibility* is a relation on $\boldsymbol{N}$ (the natural numbers):

$$D = \{(x,y) : x,y \in \boldsymbol{N}, x|y\}$$

Notation: the book writes $a\,R\,b$ to denote that the pair $(a,b) \in R$. The latter notation is more standard, and that's what I will use.

- You can use either one.

Functions; Composing and Inverting Relations

A *function* $f : A \to B$ is just a relation where for all $a \in A$, there is a unique $b \in B$ such that $(a, b) \in R$.

If R is a relation on $A \times B$, then R^{-1} is a relation on $B \times A$:

$$(a, b) \in R \text{ iff } (b, a) \in R^{-1}.$$

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If R is a relation on $B \times C$ and S is a relation on $A \times B$, then $R \circ S$ is a relation on $A \times C$:

$$(a, c) \in R \circ S \text{ iff } \exists b((a, b) \in S \text{ and } (b, c) \in R).$$

- Note the order of R and S on the right-hand side
- This is what we need to generalize the definition of function composition:
  - If $f : A \to B$, $g : B \to C$, then $g \circ f : A \to C$ (note that it's $g \circ f$, not $f \circ c$)
  - $g \circ f(a) = g(f(a)) = c$ if there exists a b such that $f(a) = b$ and $g(b) = c$ (i.e., $(a, b) \in f$ and $(b, c) \in g$).

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- Suppose that $x \in S$. Then $x = (n, n+2)$ for some n.
- Note that $(n, n+1) \in R$ and $(n+1, n+2) \in R$.
- Thus, by definition $(n, n+2) \in R \circ R$; that is, $x \in R \circ R$.

This shows that $S \subseteq R \circ R$.

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This shows that $S \subseteq R \circ R$.

The other direction works essentially the same way:

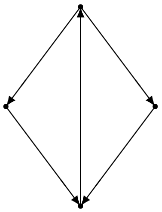
- Suppose that $(a, c) \in R \circ R$.
- Then (by definition), there is a b such that $(a, b) \in R$ and $(b, c) \in R$.
- Thus, $b = a + 1$ and $c = b + 1 = a + 2$.
- Thus $(a, c) = (a, a+2) \in S$.

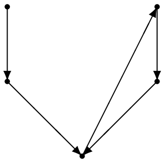
This shows that $R \circ R \subseteq S$.

Graphs

A *graph* consists of nodes and edges between nodes.

A *directed graph* (*digraph*) is one where the edges have a direction, usually denoted with an arrow.





Graphs come up everywhere.

- We can view the internet as a graph (in many ways)
  - who is connected to whom
- Web search views web pages as a graph
  - who points to whom
- Niche graphs (Ecology):
  - The vertices are species
  - Two vertices are connected by an edge if they compete (use the same food resources, etc.)

  Niche graphs give a visual representation of competitiveness.
- Influence Graphs
  - The vertices are people
  - There is an edge from a to b if a influences b

  Influence graphs give a visual representation of power structure.

There are lots of other examples in all fields ...

Terminology and Notation

An *undirected graph* G is a pair (V, E), where V is a set of *vertices* or *nodes* and E is a set of *edges* or *branches*; an edge is a set $\{v, v'\}$ of two not necessarily distinct vertices (i.e., $v, v' \in V$).

- We sometimes write $G(V, E)$ instead of G
- We sometimes write $V(G)$ and $E(G)$ if we want to emphasize the graph that the vertices and edges come from.

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A *digraph* is a pair (V, E) where E is a set of *directed edges*

- A directed edge is a pair (v, v'), where $v, v' \in G$
- The order matters!

Walks, Paths, and Cycles

- A *walk* in a graph G is an alternating sequence of vertices and edges, starting and ending with a vertex, where, for every edge (u, v) on the walk, u is the preceding vertex and v is the following vertex.

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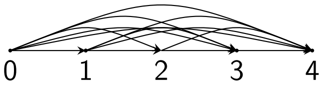
  - E.g., 1 (1,3), 3, (3,8), 8
  - Yuck! (The vertices are redundant)
    - It's more standard to leave them out; the text includes them
- The *length* of a walk is the number of vertices -1
- A *path* is a walk where all the vertices are different
- A cycle is a walk of positive length where all vertices are distinct except for the first and last one

Graphs and Relations

Given a relation R on $S \times T$, we can represent it by the directed graph $G(V, E)$, where

- $V = S \cup T$ and
- $E = \{(s, t) : (s, t) \in R\}$

Example: We can represent the $<$ relation on $\{0, 1, 2, 3, 4\}$ graphically.



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  - $\leq$ and $\geq$ are not
- A relation R on S is *antisymmetric* if $(x, y) \in R$ and $x \neq y$ implies $(y, x) \notin R$.
  - $\leq$ and $\geq$ are antisymmetric
- A relation R on S is *transitive* if $(x, y) \in R$ and $(y, z) \in R$ implies $(x, z) \in R$.
  - $\leq$, $<$, $\geq$, $>$ are all transitive;
  - "parent-of" is not transitive; "ancestor-of" is

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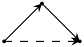
- transitive? 

Equivalence Relations

- A relation R is an *equivalence relation* if it is reflexive, symmetric, and transitive
  - $=$ is an equivalence relation
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An equivalence relation on S partitions S into *equivalence classes*:

- The equivalence class of s is denoted $[s]$.
  - $[s] = \{t : (s, t) \in R\}$

Theorem: Equivalences classes are either equal or disjoint: for all $s, s' \in S$, either $[s] = [s']$ or $[s] \cap [s'] = \emptyset$.

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- What are the equivalence classes of the parity relation?

Transitive Closure

The *transitive closure* of a relation R is the least relation R^* such that

1. $R \subseteq R^*$
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Take R^* to be the intersection of all the transitive relations that contain R.

- We must check that the intersection contains R and is transitive.

Clearly R^* is a subset of any transtive relation R' that contains R.

Example: Suppose $R = \{(1,2),(2,3),(1,4)\}$.

- $R^* = \{(1,2),(1,3),(2,3),(1,4)\}$
- we need to add $(1,3)$, because $(1,2),(2,3) \in R$

Note that we don't need to add (2,4).

- If (2,1), (1,4) were in R, then we'd need (2,4)
- (1,2), (1,4) doesn't force us to add anything (it doesn't fit the "pattern" of transitivity.

Note that if R is already transitive, then $R^* = R$.

An Inductive Definition of Transitive Closure

Given a relation R on S, here is a constructive inductive definition of transitive closure. Define $R_0, R_1, \ldots$ inductively:

- Let $R_0 = R$.
- Let $R_{n+1} = R_n \cup \{(s,t) : \exists u \in S((s,u) \in R_n,\ (u,t) \in R_n)\}$.
- Let $R' = \cup_{n=0}^{\infty} R_n$.

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- $R \subseteq R'$
- R' is transitive
- If R'' is transitive and $R \subseteq R''$, then $R' \subseteq R''$ (i.e., R' is the smallest transitive set that contains R).

This will be homework.

Partial Orders

A relation is *strict partial order* if it is irreflexive and transitive.

- $<$ and $>$ are strict partial orders

A relation is *weak partial order* if it is reflexive, transitive, and antisymmetric

- $\leq$ and $\geq$ are weak partial orders