CS 2802: Homework 4

September 21, 2020

Handed out Sept. 21; due Sept. 28

- In Chapter 10, read 10.1, 10.2, 10.6, 10.7, 10.10
- Do the following problems:
  - 10.32
  - 10.35 (Just give the answer and a short, 1-2 line explanation. No need for a long formal proof.)
  - 10.38 [Part (a) will be done in recitation, if there's time.]
  - 10.39 (again, just give short explanations here). The answer to part (c) depends in part on how you interpret =. (That is, what does it mean that a propositional formula G is equal to a propositional formula H. If you view a formula as a function from truth assignments to truth values (so that a formula like $(P \wedge Q) \vee \neg R$ maps TTT to T, FTT to F, and so on), then two formulas are equal as functions iff they're logically equivalent (since they give the same answer for all truth assignments). Alternatively, you can view formulas as syntactic objects, so that $G = H$ iff they are identical as syntactic objects. You'll get different answers depending on your viewpoint. Give the answer for both viewpoints.) [10.39(b) and (c) will be done in recitation.]
  - Additional Problem 1: Let R be a relation on a set S. Recall that if $s \in S$, then $[s]_R = \{s' : (s, s') \in R$. That is, $[s]_R$ consists of all the elements in S related to s by R. [Part (a) will be done in recitation.]
    (a) Show that if R is an equivalence relation, then the sets $[s]_R$ form a partition of S (i.e., for all $s, s' \in S$, we have either $[s] = [s']$ or $[s] \cap [s'] = \emptyset$). That is, equivalence classes are either equal or disjoint.
    (b) Show that R is an equivalence relation iff (a) the sets $[s]_R$ form a partition of S and (b) $s \in [s]_R$ for all $s \in S$.
  - Additional Problem 2: This problem will be done in recitation.] Say that vertices u, v in a digraph G are *mutually connected* and write $u \leftrightarrow v$ when there is a path from u to v and also a path from v to u.

(a) Prove that $\leftrightarrow$ is an equivalence relation on $V(G)$.

(b) The equivalence classes of the equivalence relation $\leftrightarrow$ are called the *strongly connected components* of G. Define a relation $\rightsquigarrow$ on the strongly connected components of G by the rule $C \rightsquigarrow D$ iff there is a path from some vertex in C to some vertex in D. Prove that $\rightsquigarrow$ is a weak partial order on the strongly connected components.

– Additional Problem 3: You're responsible for this problem iff we get to transitive closure by Friday. If not, you don't have to do this problem this week, although it will be assigned next week. Recall the inductive definition of transitive closure of R given in class:

  * Suppose that R is a relation on S. Let $R_0 = R$.
  * Let $R_{n+1} = R_n \cup \{(s,t) : \exists u \in S((s,u),\ (u,t) \in R_n)\}$.
  * Let $R' = \cup_{n=0}^{\infty} R_n$.

  Prove that R' is the transitive closure of R.

- Challenge Problem: (As usual, you don't have to hand this in.) Recall that F_n is the nth Fibonacci number. What is

$$\frac{1}{3} + \frac{1}{9} + \frac{2}{27} + \cdots + \frac{F_n}{3^n} + \cdots.$$

(Hint: if the sum is S, consider $S/3$ and $S/9$.)