CS 2802: Homework 3

September 13, 2020

Handed out Sept. 14, due Sept. 21

- Read Chapter 5
- Do the following problems:
  - 5.5 (in recitation)
  - 5.6(a)
  - 5.7
  - 5.16(a), (c), (d), (e), (h) ((a), (c), and (d) will be done in rectation)
  - 5.21 ((a) and (b) will be done in recitation)
  - 5.24
  - 5.30

  and the following:

- Additional problem 1: (Based on Problem 8.16 in MCS.) In this problem you'll prove what might seem quite surprising: there is a a bijection from $(0,1]$ to $[0,\infty)\times[0,\infty)$. (Some notation: (a,b) denotes the interval $\{x \in \mathbb{R} : a < x < b\}$; similarly, $(a,b] = \{x \in \mathbb{R} : a < x \leq b\}$; $[a,b) = \{x \in \mathbb{R} : a \leq x < b\}$; $(a,\infty) = \{x \in \mathbb{R} : a < x\}$; and $[a,\infty) = \{x \in \mathbb{R} : a \leq x\}$.)

  (a) Describe a bijection from $(0,1]$ to $[0,\infty)$. (Hint: $1/x$ almost works.)

  b) An infinite sequence $(a_1, a_2, a_3, \ldots)$ where each element a_i is a decimal digits (i.e., $a_i \in \{0,1,\ldots,9\}$) will be called *long* if it does not end with all 0s. An equivalent way to say this is that a long sequence is one that has infinitely many occurrences of nonzero digits. Let L be the set of all long sequences. Describe a bijection from L to the half-open real interval $(0,1]$. (Hint: Put a decimal point at the beginning of the sequence.) (This will be discussed in recitation. The key issue is why we're restricting to long sequences.)

(c) Describe an injection from L to L^2 and an injection from L^2 to L. (For the second injection, consider alternating digits.)

(d) Show that if there is a bijection from A to B, then there is a bijection from $A \times A$ to $B \times B$.

(e) Complete the argument (i.e., show that there is a bijection from $(0, 1]$ to $[0, \infty) \times [0, \infty)$) using the Schröder-Bernstein Theorem and problem 5 from HW2.

Note that the parts of this problem are independent. For example, you can do (d) even if you don't do (a), (b), and (c). When you're doing a later part, you can use the earlier parts, even if you didn't prove them.

NOTE: When you do these problems, make clear what $P(n)$ is, what the base case is, and what the inductive step is. We will take off points for bad presentation!

Here's a challenge problem (not to be handed in), from the 2019 prelim: Imagine a Grand Hotel with a countably infinitely of rooms, numbered 1, 2, 3, ... The hotel is full. When you arrive at the hotel late in the evening, the manager says that there is no problem accommodating you, although the hotel is full. You can go in room 1, and the previous occupant of room n moves to room $n + 1$.

(a) Now a bus with a countable infinity of passengers arrives. Show how they can be accommodated (you have to give an explicit room assignment).

(b) Now countably infinitely many buses arrive, each with a countable infinity of passengers. Show explicitly how they can be accommodated, making use of the fact that there are infinitely many primes. Explain carefully why different passengers are in different rooms.

(c) Could you accommodate everyone if uncountably many cars arrived, each containing only the driver? Explain carefully why or why not. Would it have helped if the drivers carpooled? (Again, explain why or why not.)