CS 2802: Homework 2

September 5, 2020

Handed out Sept. 27; Sept. 14

- Read Chapter 4
- In Chapter 8, read 8.1 (including 8.1.1-8.1.4). (The rest of Chapter 8 is fun, but not required.)
- Do the following problems:
  1. MCS exercise 4.15
  2. Prove directly that
     (a) $A = (A - B) \cup (A \cap B)$. (Don't worry about going through propositional equivalences, as in the text.)
     (b) If $A \subseteq B$ then $C - B \subseteq C - A$.
  3. Prove that $\varphi \Rightarrow \psi$ is valid iff the set of truth assignments that make φ true is a subset of the set that makes ψ true. (It follows that φ and ψ are equivalent iff $\varphi \Leftrightarrow \psi$ is valid, although you don't have to explicitly prove this.)
  4. Prove that If $S \neq \emptyset$, then there is an injection from S to T iff there is a surjection from T to S.
  5. Show that if f is a bijection from A to B and g is a bijection from B to C, then $g \circ f$ is a bijection from A to C.
  6. Show that if $S \neq \emptyset$, then $f : S \rightarrow T$ is an injection iff f has a left inverse.
  7. Show that if A and B are countable, then so is $A \cup B$. [You may also want to think about the more general question: How does the cardinality of $A \cup B$ compared to the cardinality of A and B? You don't have to hand this in though.]
  8. If $A_0, A_1, A_2, \ldots$ are all countably infinite and disjoint (i.e., $A_i \cap A_j = \emptyset$), construct a bijection between $\cup_{i=0}^{\infty} A_i$ and $I\!N \times I\!N$ ($I\!N$ is the natural numbers). Carefully prove that your construction works.

9. Give a bijection between infinite binary sequences (i.e., infinite sequences of 0s and 1s) and subsets of the natural numbers.

- Challenge problem #1 (you don't have to hand in challenge problems): Prove that there is a bijection between the real numbers in $[0, 1]$ and subsets of the natural numbers. (Hint: problem #9 is useful here.)

- Challenge problem #2:

  (a) (An easy warmup:) Find an explicit bijection $f : \mathbb{N} \cup \{a, b\} \to \mathbb{N}$

  (b) Find an explicition bijection $f : [0, 1] \to (0, 1)$. (It easily follows from the Schröder-Bernstein Theorem that there is a bijection, but constructing it requires a bit of thought. Hint: use the ideas of part (a).)