Combinatorics

Problem: How to count without counting.

- How do you figure out how many things there are with a certain property without actually enumerating all of them.

Sometimes this requires a lot of cleverness and deep mathematical insights.

But there are some standard techniques.

- That's what we'll be studying.

Bijection Rule

The Bijection Rule: If $f : A \to B$ is a bijection, then $|A| = |B|$.

- We used this rule in defining cardinality for infinite sets.
- Now we'll focus on finite sets.

Bijection Rule

The Bijection Rule: If $f : A \rightarrow B$ is a bijection, then $|A| = |B|$.

- We used this rule in defining cardinality for infinite sets.
- Now we'll focus on finite sets.

We sometimes use the bijection rule without even realizing it:

I count how many people voted are in favor of something by counting the number of hands raised:

- I'm hoping that there's a bijection between the people in favor and the hands raised!

Sum and Product Rules

Example 1: In New Hampshire, license plates consisted of two letters followed by 3 digits. How many possible license plates are there?

(a) $26^2 \times 10^3$?

(b) $26 \times 25 \times 10 \times 9 \times 8$?

(c) No idea.

Sum and Product Rules

Example 1: In New Hampshire, license plates consisted of two letters followed by 3 digits. How many possible license plates are there?

Answer: 26 choices for the first letter, 26 for the second, 10 choices for the first number, the second number, and the third number:

$$26^2 \times 10^3 = 676,000$$

Sum and Product Rules

Example 1: In New Hampshire, license plates consisted of two letters followed by 3 digits. How many possible license plates are there?

Answer: 26 choices for the first letter, 26 for the second, 10 choices for the first number, the second number, and the third number:

$$26^2 \times 10^3 = 676,000$$

Example 2: A traveling salesman wants to do a tour of all 50 state capitals. How many ways can he do this?

(a) 50^{50}
(b) $50 \times 49 \times 48 \times \cdots \times 1 = 50!$
(c) No clue.

Sum and Product Rules

Example 1: In New Hampshire, license plates consisted of two letters followed by 3 digits. How many possible license plates are there?

Answer: 26 choices for the first letter, 26 for the second, 10 choices for the first number, the second number, and the third number:

$$26^2 \times 10^3 = 676,000$$

Example 2: A traveling salesman wants to do a tour of all 50 state capitals. How many ways can he do this?

Answer: 50 choices for the first place to visit, 49 for the second, ...: 50! altogether.

There are two general techniques for solving problems. Two of the most important are:

The Sum Rule: If there are $n(A)$ ways to do A and, distinct from them, $n(B)$ ways to do B, then the number of ways to do A *or* B is $n(A) + n(B)$.

- This rule generalizes: there are $n(A) + n(B) + n(C)$ ways to do A *or* B *or* C

The Product Rule: If there are $n(A)$ ways to do A and $n(B)$ ways to do B, then the number of ways to do A *and* B is $n(A) \times n(B)$. This is true if the number of ways of doing A and B are independent; the number of choices for doing B is the same regardless of which choice you made for A.

- Again, this generalizes. There are $n(A) \times n(B) \times n(C)$ ways to do A *and* B *and* C

Some Subtler Examples

Example 3: If there are n Senators on a committee, in how many ways can a subcommittee be formed?

Two approaches:

1. Let N_1 be the number of subcommittees with 1 senator (n), N_2 the number of subcommittees with 2 senator ($n(n-1)/2$), . . .

   According to the sum rule:

$$N = N_1 + N_2 + \cdots + N_n$$

- It turns out that $N_k = \frac{n!}{k!(n-k)!}$ (n choose k) – proved later.
- A subtlety: What about N_0? Do we allow subcommittees of size 0? How about size n?
  - The problem is somewhat ambiguous.

  If we allow subcommittees of size 0 and n, then there are 2^n subcommittees altogether.
  - This is just the number of subsets of the set of n Senators: there is a bijection between subsets and subcommittees.

Number of subsets of a set

Claim: $\mathcal{P}(S)$ (the set of subsets of S) has $2^{|S|}$ elements (i.e, a set S has $2^{|S|}$ subsets).

Number of subsets of a set

Claim: $\mathcal{P}(S)$ (the set of subsets of S) has $2^{|S|}$ elements (i.e, a set S has $2^{|S|}$ subsets).

Proof #1: By induction on $|S|$.

Base case: If $|S| = 0$, then $S = \emptyset$. The empty set has one subset (itself).

Inductive Step; Suppose $S = \{a_1, \ldots, a_{n+1}\}$. Let $S' = \{a_1, \ldots, a_n\}$. By the induction hypothesis, $|\mathcal{P}(S')| = 2^n$.

Partition $\mathcal{P}(S)$ into two subsets:

$A =$ the subsets of S that don't contain a_{n+1}.

$B =$ the subsets of S that do contain a_{n+1}.

It's easy to see that $A = \mathcal{P}(S')$: T is a subset of S that doesn't contain a_{n+1} if and only if T is a subset of S'. Thus $|A| = 2^n$.

Claim: $|A|$ and $|B|$, since there is a bijection from A to B.

Proof: Let $f : A \to B$ be defined by $f(T) = T \cup \{a_{n+1}\}$. Clearly if $T \neq T'$, then $f(T) \neq f(T')$, so f is an injection. And if $T' \in B$, then $a_{n+1} \in T$, $T' = T - \{a_{n+1}\} \in A$, and $f(T') = T$, so f is a surjection. Thus, f is a bijection.

Thus, $|A| = |B|$, so $|B| = 2^n$. Since $\mathcal{P}(S) = A \cup B$, by the Sum Rule, $|S| = |A| + |B| = 2 \cdot 2^n = 2^{n+1}$.

Thus, $|A| = |B|$, so $|B| = 2^n$. Since $\mathcal{P}(S) = A \cup B$, by the Sum Rule, $|S| = |A| + |B| = 2 \cdot 2^n = 2^{n+1}$.

Proof #2: Suppose $S = \{a_1, \ldots, a_n\}$. We can identify $\mathcal{P}(S)$ with the set of bitstrings of length n. A bitstring $b_1 ... b_n$, where $b_i \in \{0, 1\}$, corresponds to the subset T where $a_i \in T$ if and only if $b_i = 1$.

Example: If $n = 5$, so $S = \{a_1, a_2, a_3, a_4, a_5\}$, the bitstring 11001 corresponds to the set $\{a_1, a_2, a_5\}$. It's easy to see this correspondence defines a bijection between the bitstrings of length n and the subsets of S.

Thus, $|A| = |B|$, so $|B| = 2^n$. Since $\mathcal{P}(S) = A \cup B$, by the Sum Rule, $|S| = |A| + |B| = 2 \cdot 2^n = 2^{n+1}$.

Proof #2: Suppose $S = \{a_1, \ldots, a_n\}$. We can identify $\mathcal{P}(S)$ with the set of bitstrings of length n. A bitstring $b_1 ... b_n$, where $b_i \in \{0, 1\}$, corresponds to the subset T where $a_i \in T$ if and only if $b_i = 1$.

Example: If $n = 5$, so $S = \{a_1, a_2, a_3, a_4, a_5\}$, the bitstring 11001 corresponds to the set $\{a_1, a_2, a_5\}$. It's easy to see this correspondence defines a bijection between the bitstrings of length n and the subsets of S.

Why are there 2^n bitstrings?

Thus, $|A| = |B|$, so $|B| = 2^n$. Since $\mathcal{P}(S) = A \cup B$, by the Sum Rule, $|S| = |A| + |B| = 2 \cdot 2^n = 2^{n+1}$.

Proof #2: Suppose $S = \{a_1, \ldots, a_n\}$. We can identify $\mathcal{P}(S)$ with the set of bitstrings of length n. A bitstring $b_1 ... b_n$, where $b_i \in \{0, 1\}$, corresponds to the subset T where $a_i \in T$ if and only if $b_i = 1$.

Example: If $n = 5$, so $S = \{a_1, a_2, a_3, a_4, a_5\}$, the bitstring 11001 corresponds to the set $\{a_1, a_2, a_5\}$. It's easy to see this correspondence defines a bijection between the bitstrings of length n and the subsets of S.

Why are there 2^n bitstrings?
That's the product rule: two choices for b_1 (0 or 1), two choices for b_2, ..., two choices for b_n. We're also using the bijection rule. How?

Back to the senators:

2. Simpler method: Use the product rule, just like above.
   - Each senator is either in the subcommittee or out of it: 2 possibilities for each senator:
     - $2 \times 2 \times \cdots \times 2 = 2^n$ choices altogether

General moral: In many combinatorial problems, there's more than one way to analyze the problem.

Coping with Ambiguity

If you think a problem is ambiguous:

1. Explain why
2. Choose one way of resolving the ambiguity
3. Solve the problem according to your interpretation
   - Make sure that your interpretation doesn't render the problem totally trivial

More Examples

Example 4: How many legal configurations are there in Towers of Hanoi with n rings?

More Examples

Example 4: How many legal configurations are there in Towers of Hanoi with n rings?

More Examples

Example 4: How many legal configurations are there in Towers of Hanoi with n rings?

Answer: The product rule again: Each ring gets to "vote" for which pole it's on.

- Once you've decided which rings are on each pole, their order is determined.
- The total number of configurations is 3^n

Example 5: How many distinguishable ways can the letters of "computer" be arranged? How about "discrete"?

More Examples

Example 4: How many legal configurations are there in Towers of Hanoi with n rings?

Answer: The product rule again: Each ring gets to "vote" for which pole it's on.

- Once you've decided which rings are on each pole, their order is determined.
- The total number of configurations is 3^n

Example 5: How many distinguishable ways can the letters of "computer" be arranged? How about "discrete"?

More Examples

Example 4: How many legal configurations are there in Towers of Hanoi with n rings?

Answer: The product rule again: Each ring gets to "vote" for which pole it's on.

- Once you've decided which rings are on each pole, their order is determined.
- The total number of configurations is 3^n

Example 5: How many distinguishable ways can the letters of "computer" be arranged? How about "discrete"?

For computer, it's 8!:

- 8 choices for the first letter, for the second, ...

Question: Is it also 8! for "discrete"?

(a) Yes
(b) No
(c) no idea

Hint: there are 2 e's. Does that make a difference?

Question: Is it also 8! for "discrete"?

Suppose we called the two e's e_1 and e_2:

- There are two "versions" of each arrangement, depending on which e comes first: discre$_1$te$_2$ is the same as discre$_2$te$_1$.
- Thus, the right answer is $8!/2!$

Division Rule: If there is a k-to-1 correspondence between of objects of type A with objects of type B, and there are $n(A)$ objects of type A, then there are $n(A)/k$ objects of type B.

A *k-to-1 correspondence* is an onto mapping in which every B object is the image of exactly k A objects.

Permutations

A *permutation* of n things taken r at a time, written $P(n, r)$, is an arrangement in a row of r things, taken from a set of n distinct things. Order matters.

Example 6: How many permutations are there of 5 things taken 3 at a time?

Permutations

A *permutation* of n things taken r at a time, written $P(n, r)$, is an arrangement in a row of r things, taken from a set of n distinct things. Order matters.

Example 6: How many permutations are there of 5 things taken 3 at a time?

Answer: 5 choices for the first thing, 4 for the second, 3 for the third: $5 \times 4 \times 3 = 60$.

- If the 5 things are a, b, c, d, e, some possible permutations are:

$$\begin{array}{llllll} abc & abd & abe & acb & acd & ace \\ adb & adc & ade & aeb & aec & aed \\ \ldots \end{array}$$

In general

$$P(n, r) = \frac{n!}{(n-r)!} = n(n-1)\cdots(n-r+1)$$

Combinations

A *combination* of n things taken r at a time, written $C(n, r)$ or $\binom{n}{r}$ ("n choose r") is any subset of r things from n things. Order makes no difference,

Example 7: How many ways can we choose 3 things from 5?

(a) $5 \times 4 \times 3$?
(b) $5 \times 4 \times 3/6$?
(c) Something else?

Combinations

A *combination* of n things taken r at a time, written $C(n, r)$ or $\binom{n}{r}$ ("n choose r") is any subset of r things from n things. Order makes no difference,

Example 7: How many ways can we choose 3 things from 5?

Answer: If order mattered, then it would be $5 \times 4 \times 3$. Since order doesn't matter,

$$abc, acb, bac, bca, cab, cba$$

are all the same.

- For way of choosing three elements, there are $3! = 6$ ways of ordering them.

Therefore, the right answer is $(5 \times 4 \times 3)/3! = 10$:

$$\begin{array}{ccccc} abc & abd & abe & acd & ace \\ ade & bcd & bce & bde & cde \end{array}$$

Combinations

A *combination* of n things taken r at a time, written $C(n, r)$ or $\binom{n}{r}$ ("n choose r") is any subset of r things from n things. Order makes no difference,

Example 7: How many ways can we choose 3 things from 5?

Answer: If order mattered, then it would be $5 \times 4 \times 3$. Since order doesn't matter,

$$abc, acb, bac, bca, cab, cba$$

are all the same.

- For way of choosing three elements, there are $3! = 6$ ways of ordering them.

Therefore, the right answer is $(5 \times 4 \times 3)/3! = 10$:

$$\begin{array}{ccccc} abc & abd & abe & acd & ace \\ ade & bcd & bce & bde & cde \end{array}$$

In general, it's $C(n, r) = \frac{n!}{(n-r)!r!} = n(n-1)\cdots(n-r+1)/r!$.

More Examples

Example 8: How many full houses are there in poker?

- A full house has 5 cards, 3 of one kind and 2 of another.
- E.g.: 3 5's and 2 K's.

More Examples

Example 8: How many full houses are there in poker?

- A full house has 5 cards, 3 of one kind and 2 of another.
- E.g.: 3 5's and 2 K's.

Answer: You need to find a systematic way of counting:

- Choose the denomination for which you have three of a kind: 13 choices.
- Choose the three: $C(4, 3) = 4$ choices
- Choose the denomination for which you have two of a kind: 12 choices
- Choose the two: $C(4, 2) = 6$ choices.

Altogether, there are:

$$13 \times 4 \times 12 \times 6 = 3744 \text{ choices}$$

0!

It's useful to define $0! = 1$.

Why?

1. Then we can inductively define

$$(n+1)! = (n+1)n!,$$

   and this definition works even taking 0 as the base case instead of 1.
2. A better reason: Things work out right for $P(n,0)$ and $C(n,0)$!

0!

It's useful to define $0! = 1$.

Why?

1. Then we can inductively define
$$(n+1)! = (n+1)n!,$$
and this definition works even taking 0 as the base case instead of 1.
2. A better reason: Things work out right for $P(n, 0)$ and $C(n, 0)$!

How many permutations of n things from n are there?

$$P(n, n) = \frac{n!}{(n-n)!} = \frac{n!}{0!} = n!$$

How many ways are there of choosing n out of n?
0 out of n?

$$\binom{n}{n} = \frac{n!}{n!0!} = 1; \quad \binom{n}{0} = \frac{n!}{0!n!} = 1$$

More Questions

Q: How many ways are there of choosing k things from $\{1, \ldots, n\}$ if 1 and 2 can't both be chosen? (Suppose $n, k \geq 2$.)

More Questions

Q: How many ways are there of choosing k things from $\{1, \ldots, n\}$ if 1 and 2 can't both be chosen? (Suppose $n, k \geq 2$.)

Method #1: There are $C(n, k)$ ways of choosing k things from n with no constraints. There are $C(n-2, k-2)$ ways of choosing k things from n where 1 and 2 are definitely chosen:

- This amounts to choosing $k-2$ things from $\{3, \ldots, n\}$: $C(n-2, k-2)$.

Thus, the answer is

$$C(n, k) - C(n-2, k-2)$$

Method #2: There are

- $C(n-2, k-1)$ ways of choosing n from k where 1 is chosen, but 2 isn't from n where 1 is chosen, but 2 isn't;
  - choose $k-1$ things from $\{3, \dots, n\}$ (which, together with 1, give the choice of k things)
- $C(n-2, k-1)$ ways of choosing k things from n where 2 is chosen, but 1 isn't;
- $C(n-2, k)$ ways of choosing k things from n where neither 1 nor 2 are

So the answer is $2C(n-2, k-1) + C(n-2, k)$.

Method #2: There are

- $C(n-2, k-1)$ ways of choosing n from k where 1 is chosen, but 2 isn't from n where 1 is chosen, but 2 isn't;
  - choose $k-1$ things from $\{3, \ldots, n\}$ (which, together with 1, give the choice of k things)
- $C(n-2, k-1)$ ways of choosing k things from n where 2 is chosen, but 1 isn't;
- $C(n-2, k)$ ways of choosing k things from n where neither 1 nor 2 are

So the answer is $2C(n-2, k-1) + C(n-2, k)$.

Why is
$C(n, k) - C(n-2, k-2) = 2C(n-2, k-1) + C(n-2, k)$?

- That's the next topic!

Combinatorial Identities

There all lots of identities that you can form using $C(n, k)$. They seem mysterious at first, but there's usually a good reason for them.

Theorem 1: If $0 \leq k \leq n$, then

$$C(n, k) = C(n, n-k).$$

Proof:

$$C(n, k) = \frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)!(n-(n-k))!} = C(n, n-k)$$

Q: Why should choosing k things out of n be the same as choosing $n-k$ things out of n?

Combinatorial Identities

There all lots of identities that you can form using $C(n,k)$. They seem mysterious at first, but there's usually a good reason for them.

Theorem 1: If $0 \leq k \leq n$, then

$$C(n,k) = C(n, n-k).$$

Proof:

$$C(n,k) = \frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)!(n-(n-k))!} = C(n, n-k)$$

Q: Why should choosing k things out of n be the same as choosing $n-k$ things out of n?

A: There's a 1-1 correspondence. For every way of choosing k things out of n, look at the things not chosen: that's a way of choosing $n-k$ things out of n.

This is a better way of thinking about Theorem 1 than the combinatorial proof.

Theorem 2: If $0 < k < n$ then

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

Theorem 2: If $0 < k < n$ then

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

Proof 1: (Combinatorial) Suppose we want to choose k objects out of $\{1, \ldots, n\}$. Either we choose the last one (n) or we don't.

1. How many ways are there of choosing k without choosing the last one? $C(n-1, k)$.
2. How many ways are there of choosing k including n? This means choosing $k-1$ out of $\{1, \ldots, n-1\}$: $C(n-1, k-1)$.

Proof 2: Algebraic ...

Note: If we define $C(n, k) = 0$ for $k > n$ and $k < 0$, Theorems 1 and 2 still hold.

Theorem 2: If $0 < k < n$ then

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

Proof 1: (Combinatorial) Suppose we want to choose k objects out of $\{1, \ldots, n\}$. Either we choose the last one (n) or we don't.

1. How many ways are there of choosing k without choosing the last one? $C(n-1, k)$.
2. How many ways are there of choosing k including n? This means choosing $k-1$ out of $\{1, \ldots, n-1\}$: $C(n-1, k-1)$.

Proof 2: Algebraic ...

Note: If we define $C(n, k) = 0$ for $k > n$ and $k < 0$, Theorems 1 and 2 still hold.

This explains why

$C(n, k) - C(n-2, k-2) = 2C(n-2, k-1) + C(n-2, k)$

$$\begin{aligned} & C(n,k) = C(n-1,k) + C(n-1,k-1) \\ & = C(n-2,k) + C(n-2,k-1) + C(n-2,k-1) + C(n-2,k-2) \\ & = C(n-2,k) + 2C(n-2,k-1) + C(n-2,k-2) \end{aligned}$$

Pascal's Triangle

Starting with $n = 0$, the nth row has $n + 1$ elements:

$$C(n, 0), \ldots, C(n, n)$$

Note how Pascal's Triangle illustrates Theorems 1 and 2.

Theorem 3: For all $n \geq 0$:

$$\Sigma_{k=0}^{n} \binom{n}{k} = 2^n$$

Proof 1: $\binom{n}{k}$ tells you all the way of choosing a subset of size k from a set of size n. This means that the LHS is *all* the ways of choosing a subset from a set of size n. The product rule says that this is 2^n.

Theorem 3: For all $n \geq 0$:

$$\Sigma_{k=0}^{n} \binom{n}{k} = 2^n$$

Proof 1: $\binom{n}{k}$ tells you all the way of choosing a subset of size k from a set of size n. This means that the LHS is *all* the ways of choosing a subset from a set of size n. The product rule says that this is 2^n.

Proof 2: By induction. Let $P(n)$ be the statement of the theorem.

Basis: $\Sigma_{k=0}^{0} \binom{0}{k} = \binom{0}{0} = 1 = 2^0$. Thus $P(0)$ is true.

Inductive step: How do we express $\Sigma_{k=0}^{n} C(n,k)$ in terms of $n-1$, so that we can apply the inductive hypothesis?

- Use Theorem 2!

The Binomial Theorem

We want to compute $(x+y)^n$.
Some examples:

$$(x+y)^0 = 1$$

$$(x+y)^1 = x+y$$

$$(x+y)^2 = x^2+2xy+y^2$$

$$(x+y)^3 = x^3+3x^2y+3xy^2+y^3$$

$$(x+y)^4 = x^4+4x^3y+6x^2y^2+4xy^3+y^4$$

The pattern of the coefficients is just like that in the corresponding row of Pascal's triangle!

Binomial Theorem:

$$(x+y)^n = \Sigma_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

Proof 1: By induction on n. $P(n)$ is the statement of the theorem.

Basis: $P(1)$ is obviously OK. (So is $P(0)$.)

Inductive step:

$$\begin{aligned}
& (x+y)^{n+1} \\
= \; & (x+y)(x+y)^n \\
= \; & (x+y)\Sigma_{k=0}^{n} \tbinom{n}{k} x^{n-k} y^k \\
= \; & \Sigma_{k=0}^{n} \tbinom{n}{k} x^{n-k+1} y^k + \Sigma_{k=0}^{n} \tbinom{n}{k} x^{n-k} y^{k+1} \\
= \; & \ldots \qquad \text{[Lots of missing steps]} \\
= \; & y^{n+1} + \Sigma_{k=0}^{n} (\tbinom{n}{k} + \tbinom{n}{k-1}) x^{n-k+1} y^k \\
= \; & y^{n+1} + \Sigma_{k=0}^{n} (\tbinom{n+1}{k} x^{n+1-k} y^k \\
= \; & \Sigma_{k=0}^{n+1} \tbinom{n+1}{k} x^{n+1-k} y^k
\end{aligned}$$

Binomial Theorem:

$$(x+y)^n = \Sigma_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

Proof 2: What is the coefficient of the $x^{n-k}y^k$ term in $(x+y)^n$?

Using the Binomial Theorem

Q: What is $(x+2)^4$?

A:

$$\begin{aligned} & (x+2)^4 \\ = \;& x^4 + C(4,1)x^3(2) + C(4,2)x^2 2^2 + C(4,3)x 2^3 + 2^4 \\ = \;& x^4 + 8x^3 + 24x^2 + 32x + 16 \end{aligned}$$

Q: What is $(1.02)^7$ to 4 decimal places?

A:

$$\begin{aligned} & (1+.02)^7 \\ = \;& 1^7 + C(7,1)1^6(.02) + C(7,2)1^5(.0004) + C(7,3)(.000008) + \cdots \\ = \;& 1 + .14 + .0084 + .00028 + \cdots \\ \approx \;& 1.14868 \\ \approx \;& 1.1487 \end{aligned}$$

Note that we have to go to 5 decimal places to compute the answer to 4 decimal places.

Inclusion-Exclusion Rule

Remember the Sum Rule:

The Sum Rule: If there are $n(A)$ ways to do A and, distinct from them, $n(B)$ ways to do B, then the number of ways to do A *or* B is $n(A) + n(B)$.
What if the ways of doing A and B aren't distinct?

Example: If 112 students take CS2800, 85 students take CS2110, and 45 students take both, how many take either CS2800 or CS2110.

$A =$ students taking CS2800
$B =$ students taking CS2110

$$|A \cup B| = |A| + |B| - |A \cap B| = 112 + 85 - 45 = 152$$

This is best seen using a Venn diagram:

What happens with three sets?

$$|A \cup B \cup C| =$$
$$|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

Example: There are 300 engineering majors. 112 take CS2800, 85 take CS 2110, 95 take AEP 3560, 45 take both CS2800 and CS 2110, 30 take both CS 2800 and AEP 3560, 25 take both CS 2110 and AEP 3560, and 5 take all 3. How many don't take any of these 3 courses?

A = students taking CS 2800
B = students taking CS 2110
C = students taking AEP 3560

$$\begin{aligned} & |A \cup B \cup C| \\ = \; & |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| \\ & \quad + |A \cap B \cap C| \\ = \; & 112 + 85 + 95 - 45 - 30 - 25 + 5 \\ = \; & 197 \end{aligned}$$

We are interested in $\overline{A \cup B \cup C} = 300 - 197 = 103$.

The General Rule

More generally,

$$| \cup_{i=1}^{n} A_i| = \sum_{k=1}^{n} \sum_{\{I | I \subseteq \{1,\dots,n\}, |I|=k\}} (-1)^{k-1} | \cap_{i \in I} A_i|$$

Why is this true? Suppose $a \in \cup_{k=1}^{n} A_k$, and is in exactly m sets. a gets counted once on the LHS. How many times does it get counted on the RHS?

- a appears in m sets (1-way intersection)
- a appears in $C(m, 2)$ 2-way intersections
- a appears in $C(m, 3)$ 3-way intersections
- ...

Thus, on the RHS, a gets counted

$$\sum_{k=1}^{m} (-1)^{k-1} C(m, k) = 1 \text{ times.}$$

Why is $\sum_{k=1}^{m}(-1)^{k-1}C(m,k) = 1$?

- That certainly doesn't seem obvious!

What theorems do we have that give expressions like $\sum_{k=1}^{m}(-1)^{k-1}C(m,k)$?

Why is $\sum_{k=1}^{m}(-1)^{k-1}C(m,k) = 1$?

- That certainly doesn't seem obvious!

What theorems do we have that give expressions like $\sum_{k=1}^{m}(-1)^{k-1}C(m,k)$?

By the binomial theorem:

$$\begin{aligned} 0 = (-1+1)^m &= \textstyle\sum_{k=0}^{m}(-1)^k 1^{m-k} C(m,k) \\ &= 1 + \textstyle\sum_{k=1}^{m}(-1)^k C(m,k) \end{aligned}$$

Thus, $\sum_{k=1}^{m}(-1)^k C(m,k) = -1$, so

$$\sum_{k=1}^{m}(-1)^{k-1}C(m,k) = 1.$$

Sometimes math is amazing :-)

[This result can also be proved by induction, without using the binomial theorem. There's also a nice proof using Theorem 2 ($\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$).]

Balls and Urns

[Not in text:] "Balls and urns" problems are paradigmatic. Many problems can be recast as balls and urns problems, once we figure out which are the balls and which are the urns.

How many ways are there of putting b balls into u urns?

- That depends whether the balls are distinguishable and whether the urns are distinguishable

How many ways are there of putting 5 balls into 2 urns?

Balls and Urns

[Not in text:] "Balls and urns" problems are paradigmatic. Many problems can be recast as balls and urns problems, once we figure out which are the balls and which are the urns.

How many ways are there of putting b balls into u urns?

- That depends whether the balls are distinguishable and whether the urns are distinguishable

How many ways are there of putting 5 balls into 2 urns?

- If both balls and urns are distinguishable:

Balls and Urns

[Not in text:] "Balls and urns" problems are paradigmatic. Many problems can be recast as balls and urns problems, once we figure out which are the balls and which are the urns.

How many ways are there of putting b balls into u urns?

- That depends whether the balls are distinguishable and whether the urns are distinguishable

How many ways are there of putting 5 balls into 2 urns?

- If both balls and urns are distinguishable: $2^5 = 32$
  - Choose the subset of balls that goes into the first urn
  - Alternatively, for each ball, decide which urn it goes in
  - This assumes that it's OK to have 0 balls in an urn.

Balls and Urns

[Not in text:] "Balls and urns" problems are paradigmatic. Many problems can be recast as balls and urns problems, once we figure out which are the balls and which are the urns.

How many ways are there of putting b balls into u urns?

- That depends whether the balls are distinguishable and whether the urns are distinguishable

How many ways are there of putting 5 balls into 2 urns?

- If both balls and urns are distinguishable: $2^5 = 32$
  - Choose the subset of balls that goes into the first urn
  - Alternatively, for each ball, decide which urn it goes in
  - This assumes that it's OK to have 0 balls in an urn.
- If urns are distinguishable but balls aren't:

Balls and Urns

[Not in text:] "Balls and urns" problems are paradigmatic. Many problems can be recast as balls and urns problems, once we figure out which are the balls and which are the urns.

How many ways are there of putting b balls into u urns?

- That depends whether the balls are distinguishable and whether the urns are distinguishable

How many ways are there of putting 5 balls into 2 urns?

- If both balls and urns are distinguishable: $2^5 = 32$
  - Choose the subset of balls that goes into the first urn
  - Alternatively, for each ball, decide which urn it goes in
  - This assumes that it's OK to have 0 balls in an urn.
- If urns are distinguishable but balls aren't: 6
  - Decide how many balls go into the first urn: 0, 1, . . . , 5

Balls and Urns

[Not in text:] "Balls and urns" problems are paradigmatic. Many problems can be recast as balls and urns problems, once we figure out which are the balls and which are the urns.

How many ways are there of putting b balls into u urns?

- That depends whether the balls are distinguishable and whether the urns are distinguishable

How many ways are there of putting 5 balls into 2 urns?

- If both balls and urns are distinguishable: $2^5 = 32$
  - Choose the subset of balls that goes into the first urn
  - Alternatively, for each ball, decide which urn it goes in
  - This assumes that it's OK to have 0 balls in an urn.
- If urns are distinguishable but balls aren't: 6
  - Decide how many balls go into the first urn: 0, 1, . . . , 5
- If balls are distinguishable but urns aren't:

Balls and Urns

[Not in text:] "Balls and urns" problems are paradigmatic. Many problems can be recast as balls and urns problems, once we figure out which are the balls and which are the urns.

How many ways are there of putting b balls into u urns?

- That depends whether the balls are distinguishable and whether the urns are distinguishable

How many ways are there of putting 5 balls into 2 urns?

- If both balls and urns are distinguishable: $2^5 = 32$
  - Choose the subset of balls that goes into the first urn
  - Alternatively, for each ball, decide which urn it goes in
  - This assumes that it's OK to have 0 balls in an urn.
- If urns are distinguishable but balls aren't: 6
  - Decide how many balls go into the first urn: 0, 1, ..., 5
- If balls are distinguishable but urns aren't: $2^5/2 = 16$
- If balls and urns are indistinguishable:

Balls and Urns

[Not in text:] "Balls and urns" problems are paradigmatic. Many problems can be recast as balls and urns problems, once we figure out which are the balls and which are the urns.

How many ways are there of putting b balls into u urns?

- That depends whether the balls are distinguishable and whether the urns are distinguishable

How many ways are there of putting 5 balls into 2 urns?

- If both balls and urns are distinguishable: $2^5 = 32$
  - Choose the subset of balls that goes into the first urn
  - Alternatively, for each ball, decide which urn it goes in
  - This assumes that it's OK to have 0 balls in an urn.
- If urns are distinguishable but balls aren't: 6
  - Decide how many balls go into the first urn: 0, 1, . . . , 5
- If balls are distinguishable but urns aren't: $2^5/2 = 16$
- If balls and urns are indistinguishable: $6/2 = 3$

Seems straightforward. But what if we had 6 balls and 2 urns?

- If balls and urns are distinguishable:

Seems straightforward. But what if we had 6 balls and 2 urns?

- If balls and urns are distinguishable: 2^6
- If urns are distinguishable and balls aren't:

Seems straightforward. But what if we had 6 balls and 2 urns?

- If balls and urns are distinguishable: 2^6
- If urns are distinguishable and balls aren't: 7
- If balls are distinguishable but urns aren't:

Seems straightforward. But what if we had 6 balls and 2 urns?

- If balls and urns are distinguishable: 2^6
- If urns are distinguishable and balls aren't: 7
- If balls are distinguishable but urns aren't:

$$2^6/2 = 2^5$$

- If balls and urns are indistinguishable:
  (a) $7/2$
  (b) 3
  (c) 4
  (d) something else?

Seems straightforward. But what if we had 6 balls and 2 urns?

- If balls and urns are distinguishable: 2^6
- If urns are distinguishable and balls aren't: 7
- If balls are distinguishable but urns aren't:

$$2^6/2 = 2^5$$

- If balls and urns are indistinguishable:
  (a) 7/2
  (b) 3
  (c) 4
  (d) something else?

  - It can't be 7/2, since that's not an integer
  - The problem is that if there are 3 balls in each urn, and you switch urns, then you get the same solution
  - The right answer is 4!

Distinguishable Urns

How many ways can b distinguishable balls be put into u distinguishable urns?

Distinguishable Urns

How many ways can b distinguishable balls be put into u distinguishable urns?

- By the product rule, this is u^b

How many ways can b indistinguishable balls be put into u distinguishable urns?

Distinguishable Urns

How many ways can b distinguishable balls be put into u distinguishable urns?

- By the product rule, this is u^b

How many ways can b indistinguishable balls be put into u distinguishable urns?

$C(u + b - 1, b)$

Indistinguishable Urns

How many ways can b distinguishable balls be put into u indistinguishable urns?

First view the urns as distinguishable: u^b

For every solution, look at all $u!$ permutations of the urns. That should count as one solution.

- By the Division Rule, we get: $u^b/u!$?

Indistinguishable Urns

How many ways can b distinguishable balls be put into u indistinguishable urns?

First view the urns as distinguishable: u^b

For every solution, look at all $u!$ permutations of the urns. That should count as one solution.

- By the Division Rule, we get: $u^b/u!$?

This can't be right! It's not an integer (e.g. $7^3/7!$).

What's wrong?

The situation is even worse when we have indistinguishable balls in indistinguishable urns.

Why Indistinguishable Balls and Urns is Complicated

Suppose that we have 6 balls and 3 urns. If the balls are indistinguishable and the urns are distinguishable, our earlier formula said that there are $C(8, 2) = 28$ ways of doing that. What happens if urns are indistinguishable?

- there are $3! = 6$ ways of ordering 3 urns, so the first thought might be $28/6$.
- But $28/6$ is not an integer, so that can't be right!

What goes wrong?

Why Indistinguishable Balls and Urns is Complicated

Suppose that we have 6 balls and 3 urns. If the balls are indistinguishable and the urns are distinguishable, our earlier formula said that there are $C(8, 2) = 28$ ways of doing that. What happens if urns are indistinguishable?

- there are $3! = 6$ ways of ordering 3 urns, so the first thought might be $28/6$.
- But $28/6$ is not an integer, so that can't be right!

What goes wrong?
The basic intuition for dividing by 6 is that we can group the 28 ways into groups of 6.

- For example, 1-3-2 (1 ball in the first urn, 3 in the second, 2 in the third) should be grouped with 1-2-3, 2-3-1, 2-1-3, 3-1-2, and 3-2-1

This intuition is correct if each urn has a different number of balls.

But it doesn't work if two are three urns have the same number of balls:

- 2-2-2 is a group of 1
- with two the same and one different, we get a group of 3:
  - 6-0-0, 0-6-0, 0-0-6
  - 4-1-1, 1-4-1, 1-1-4
  - 3-3-0, 3-0-3, 3-3-0

There is no closed-form expression for the number of ways of putting n indistinguishable balls into k indistinguishable urns

- The analysis for distinguishable balls and indistinguishable urns is similar
  - it's closely related to what are called *Stirling number of the second kind* (but these amount to requiring all urns to have at least 1 ball)

Reducing Problems to Balls and Urns

Q1: How many different configurations are there in Towers of Hanoi with n rings?

Reducing Problems to Balls and Urns

Q1: How many different configurations are there in Towers of Hanoi with n rings?

A: The urns are the poles, the balls are the rings. Both are distinguishable.

Q2: How many solutions are there to the equation $x + y + z = 65$, if x, y, z are nonnegative integers?

Reducing Problems to Balls and Urns

Q1: How many different configurations are there in Towers of Hanoi with n rings?

A: The urns are the poles, the balls are the rings. Both are distinguishable.

Q2: How many solutions are there to the equation $x + y + z = 65$, if x, y, z are nonnegative integers?

A: You have 65 indistinguishable balls, and want to put them into 3 distinguishable urns (x, y, z). Each way of doing so corresponds to one solution.

Q3: How many ways can 8 electrons be assigned to 4 energy states?

Reducing Problems to Balls and Urns

Q1: How many different configurations are there in Towers of Hanoi with n rings?

A: The urns are the poles, the balls are the rings. Both are distinguishable.

Q2: How many solutions are there to the equation $x + y + z = 65$, if x, y, z are nonnegative integers?

A: You have 65 indistinguishable balls, and want to put them into 3 distinguishable urns (x, y, z). Each way of doing so corresponds to one solution.

Q3: How many ways can 8 electrons be assigned to 4 energy states?

A: The electrons are the balls; they're indistinguishable. The energy states are the urns; they're distinguishable.

Reducing Problems to Balls and Urns

Q1: How many different configurations are there in Towers of Hanoi with n rings?

A: The urns are the poles, the balls are the rings. Both are distinguishable.

- 3^n

Q2: How many solutions are there to the equation $x + y + z = 65$, if x, y, z are nonnegative integers?

A: You have 65 indistinguishable balls, and want to put them into 3 distinguishable urns (x, y, z). Each way of doing so corresponds to one solution.

- $C(67, 65) = 67 \times 33 = 2211$

Q3: How many ways can 8 electrons be assigned to 4 energy states?

A: The electrons are the balls; they're indistinguishable. The energy states are the urns; they're distinguishable.

- $C(11, 8) = (11 \times 10 \times 9)/6 = 165$

The Pigeonhole Principle

The Pigeonhole Principle: If $n + 1$ pigeons are put into n holes, at least two pigeons must be in the same hole.

This seems obvious. How can it be used in combinatorial anlysis?

Q1: If you have only blue socks and brown socks in your drawer, how many do you have to pull out before you're sure to have a matching pair.

The Pigeonhole Principle

The Pigeonhole Principle: If $n + 1$ pigeons are put into n holes, at least two pigeons must be in the same hole.

This seems obvious. How can it be used in combinatorial anlysis?

Q1: If you have only blue socks and brown socks in your drawer, how many do you have to pull out before you're sure to have a matching pair.

A: The socks are the pigeons and the holes are the colors. There are two holes. With three pigeons, there have to be at least two in one hole.

- What happens if we also have black socks?

A more surprising use of the pigeonhole principle

Q2: Alice and Bob play the following game: Bob gets to pick any 10 integers from 1 to 40. Alice has to find two different sets of three numbers that have the same sum. Prove that Alice always wins.

A: So what are the pigeons and what are the holes?

A more surprising use of the pigeonhole principle

Q2: Alice and Bob play the following game: Bob gets to pick any 10 integers from 1 to 40. Alice has to find two different sets of three numbers that have the same sum. Prove that Alice always wins.

A: So what are the pigeons and what are the holes?

The pigeons are the possible sets of three numbers. There are $C(10, 3) = 120$ of them.

The holes are the possible sums. The sum is at least 6, and at most $38 + 39 + 40 = 117$. So there are 112 holes.

A more surprising use of the pigeonhole principle

Q2: Alice and Bob play the following game: Bob gets to pick any 10 integers from 1 to 40. Alice has to find two different sets of three numbers that have the same sum. Prove that Alice always wins.

A: So what are the pigeons and what are the holes?

The pigeons are the possible sets of three numbers. There are $C(10, 3) = 120$ of them.

The holes are the possible sums. The sum is at least 6, and at most $38 + 39 + 40 = 117$. So there are 112 holes.

- There are more pigeons that holes!

Therefore, no matter which set of 10 numbers Bob picks, Alice can find two subsets of size three that have the same sum!

Generalized Pigeonhole Principle

Theorem: If $|A| > k|B|$ and $f : A \to B$, then for at least one element $b \in B$, $|f^{-1}(b)| > k$

- $f^{-1}(b) = \{a \in A : f(a) = b\}$.

The Pigeonhole Principle is the special case with $k = 1$.

- A is the set of pigeons, B is the set of holes.

How can we prove the generalized Pigeonhold Principle formally?

Generalized Pigeonhole Principle

Theorem: If $|A| > k|B|$ and $f : A \to B$, then for at least one element $b \in B$, $|f^{-1}(b)| > k$

- $f^{-1}(b) = \{a \in A : f(a) = b\}$.

The Pigeonhole Principle is the special case with $k = 1$.

- A is the set of pigeons, B is the set of holes.

How can we prove the generalized Pigeonhold Principle formally?

Example: Suppose that the number of hairs on a person's head is at most 200,000 and the population of Manhattan is greater than 2,000,000. Then we are guaranteed there is a group of N people in Manhattan that have exactly the same number of hairs on their heads. What's the largest that N could be?

(a) 1
(b) 2
(c) 5
(d) 10
(e) 11

A Real-World Application of the Pigeonhole Principle

Thanks to Paul Ginsparg for the next two slides.

Is this voter fraud?

Were these vote counts fabricated (as some politicians suggested), and the fabricators too lazy to choose realistic numbers?

2016 Vote totals from https://www.nytimes.com/elections/2016/results/president

AK:	318608	MT:	501822
AL:	2123372	NC:	4741564
AR:	1130635	ND:	344360
AZ:	2604657	NE:	844227
CA:	14237884	NH:	744296
CO:	2780247	NJ:	3906723
CT:	1644920	NM:	798319
DC:	311268	NV:	1125385
DE:	443814	NY:	7721453
FL:	9501617	OH:	5536528
GA:	4141445	OK:	1452992
HI:	428937	OR:	2001336
IA:	1566031	PA:	6166708
ID:	690433	RI:	464144
IL:	5594420	SC:	2103027
IN:	2757828	SD:	370093
KS:	1194755	TN:	2508027
KY:	1924149	TX:	8969226
LA:	2029032	UT:	1143601
MA:	3325046	VA:	3982752
MD:	2781446	VT:	315067
ME:	747927	WA:	3316996
MI:	4824260	WI:	2976150
MN:	2945233	WV:	721231
MO:	2827673	WY:	255849
MS:	1211088		

Consider Michigan with 4,824,260 votes, note that:

318608 (AK) + 344360 (ND) + 370093 (SD) + 443814 (DE) + 501822 (MT) + 844227 (NE) + 2001336 (OR) = 4824260

Consider Pennsylvania with 6,166,708 votes, note that:

255849 (WY) + 315067 (VT) + 464144 (RI) + 747927 (ME) + 1143601 (UT) + 1211088 (MS) + 2029032 (LA) = 6166708

Consider Florida with 9,501,617 votes, note that:

311268 (DC) + 318608 (AK) + 344360 (ND) + 443814 (DE) + 844227 (NE) + 1644920 (CT) + 5594420 (IL) = 9501617

3315067 (VT) + 344360 (ND) + 428937 (HI) + 464144 (RI) + 1143601 (UT) + 1211088 (MS) + 5594420 (IL) = 9501617

OK, so we've seen:

```
 318608 AK
 344360 ND
 370093 SD
 443814 DE
 501822 MT
 844227 NE
2001336 OR
--------
4824260 MI
```

```
 255849 WY
 315067 VT
 464144 RI
 747927 ME
1143601 UT
1211088 MS
2029032 LA
--------
6166708 PA
```

```
 311268 DC
 318608 AK
 344360 ND
 443814 DE
 844227 NE
1644920 CT
5594420 IL
--------
9501617 FL
```

```
 315067 VT
 344360 ND
 428937 HI
 464144 RI
1143601 UT
1211088 MS
5594420 IL
--------
9501617 FL
```

But what about a serious state like NY, with 7,721,453?

```
 255849 WY
 315067 VT
 443814 DE
 464144 RI
 501822 MT
 690433 ID
1143601 UT
3906723 NJ
--------
7721453 NY
```

```
 311268 DC
 344360 ND
 464144 RI
 501822 MT
 798319 NM
1130635 AR
1194755 KS
2976150 WI
--------
7721453 NY
```

```
 255849 WY
 311268 DC
 315067 VT
 318608 AK
 501822 MT
 747927 ME
 798319 NM
1644920 CT
2827673 MO
--------
7721453 NY
```

```
 255849 WY
 311268 DC
 318608 AK
 443814 DE
 501822 MT
 744296 NH
 798319 NM
1566031 IA
2781446 MD
--------
7721453 NY
```

```
 318608 AK
 344360 ND
 370093 SD
 443814 DE
 464144 RI
 690433 ID
 721231 WV
 744296 NH
 844227 NE
2780247 CO
--------
7721453 NY
```

```
 443814 DE
 464144 RI
 501822 MT
 690433 ID
 844227 NE
1125385 NV
1143601 UT
2508027 TN
--------
7721453 NY
```

```
 255849 WY
 318608 AK
 370093 SD
 443814 DE
 464144 RI
 798319 NM
 844227 NE
2103027 SC
2123372 AL
--------
7721453 NY
```

```
 315067 VT
 318608 AK
 344360 ND
 443814 DE
 844227 NE
1125385 NV
1194755 KS
1211088 MS
1924149 KY
--------
7721453 NY
```

3982752 (VA): 1, 4824260 (MI): 1, 5536528 (OH): 2, 5594420 (IL): 1, 6166708 (PA): 1, 7721453 (NY): 8, 8969226 (TX): 40, 9501617 (FL): 64, 14237884 (CA): 1697

Is this voter fraud?

Were these vote counts fabricated (as some politicians suggested), and the fabricators too lazy to choose realistic numbers?

Apply the pigeonhole principle!

- The first 40 states (when sorted in order of number of people voting) each have under 4 million voters, and a total of 61,680,744 people who voted.
- How many different possible sums of populations can we get from these 40 states?

Is this voter fraud?

Were these vote counts fabricated (as some politicians suggested), and the fabricators too lazy to choose realistic numbers?

Apply the pigeonhole principle!

- The first 40 states (when sorted in order of number of people voting) each have under 4 million voters, and a total of 61,680,744 people who voted.
- How many different possible sums of populations can we get from these 40 states?
- 2^{40} (since that's how many subsets there are)
- But $2^{40} \approx 10^{12}$ (about a trillion)
- On average, each number less than 61,680,744 will be the sum of the populations of 17,826 combinations of states!

Is this voter fraud?

Were these vote counts fabricated (as some politicians suggested), and the fabricators too lazy to choose realistic numbers?

Apply the pigeonhole principle!

- The first 40 states (when sorted in order of number of people voting) each have under 4 million voters, and a total of 61,680,744 people who voted.
- How many different possible sums of populations can we get from these 40 states?
- 2^{40} (since that's how many subsets there are)
- But $2^{40} \approx 10^{12}$ (about a trillion)
- On average, each number less than 61,680,744 will be the sum of the populations of 17,826 combinations of states!

If we take all 51 states (including DC) into account:

- there are $2^{51} \approx 2 \cdot 10^{15}$ possible sums
- the total number of voteres is 137,098,601
- each number is hit on average 16,400,400 times!
  - e.g., 68,520,681 is hit 63,385,606 times