Discrete Structures *Prelim 2— **Solutions***
CS2800

1. ***(3 points)*** *How many solutions* (x_1, x_2, x_3, x_4) *are there to the equation* $x_1 + x_2 + x_3 + x_4 = 35$*, where* x_1, x_2, x_3, x_4 *are natural numbers? (Just write down the combinatorial expression that describes the answer; there's no need to calculate it numerically. Also, recall that 0 is a natural number.)*

   Solution This is a balls and urns problem. There are 35 indstinguishable balls and four urns (corresponding to $x_1, \ldots, x_4$). There are $C(35 + 4 - 1, 35) = C(38, 35)$

2. ***(3 points)*** *How many integers between 1 and 100 (inclusive) are divisible by either 3 or 4?*

   Solution Let A be the number of integers between 1 and 100 divisible by 3. Let B be the number of integers between 1 and 100 divisible by 4. We are interested in $|A \cup B|$. Clearly $|A| = 33 (= \lfloor 100/3 \rfloor)$ and $|B| = 25$. A number is divisible by both 3 and 4 if it is divisible by 12, so $|A \cap B| = 8 \lfloor 100/12 \rfloor)$. By the inclusion-exclusion rule, $|A \cup B| = |A| + |B| - |A \cap B| = 33 + 25 - 8 - 50$.

3. ***(5 points)*** *Let* S *be a sample space. Give a formal mathematical definition of the following:*
   (a) *Probability measure on* S

   Solution A function $\Pr : 2^S \to \mathbb{R}$ satisfying (1) for all $E \subseteq S$, $0 \leq \Pr(E) \leq 1$, (2) $\Pr(\emptyset) = 0$ and $\Pr(S) = 1$, and (3) if $E_1 \cap E_2 = \emptyset$ then $\Pr(E_1 \cup E_2) = \Pr(E_1) + \Pr(E_2)$.

   (b) *Event of* S

   Solution A subset $E \subseteq S$.

   (c) *Random variable on* S

   Solution A function $X : S\mathbb{R}$.

   (d) *Independent random variables of* S

   Solution Random variables X and Y such that for all x and y, the events $X = x$ and $Y = y$ are independent.

   (e) $\Pr(A \mid B)$*, where* A *and* B *are events.*

   Solution $\Pr(A \cap B)/\Pr(B)$

4. ***(4 points)*** *A group of 10 women and 9 men are in a room. If 3 of the 19 are selected at random, what is the probability that all 3 are of the same sex? (Again, just write down the combinatorial expression that describes the answer; there's no need to calculate it numerically.)*

   Solution There are two ways of thinking about this. The first is to treat all the men and women as distinct, in which case there are $C(19, 3)$ ways of choosing 3 of 19. The second is to think of them as indistinguishable (which they might not be too happy about!). In this case,

   - If all 3 are women, then there are are $C(10, 3)$ ways
   - If 2 are women and 1 is a man, then there are are $C(10, 2)C(9, 1)$ ways
   - If 1 is a woman and 2 are men, then there are are $C(10, 1)C(9, 2)$ ways.
   - If all 3 are men, then there are $C(9, 3)$ ways.

Thus, there are $C(10,3)+C(10,2)C(9,1)+C(10,1)C(9,2)+C(9,3)$ of choosing 3 out of 19.

Thus, the probability that all three are the same sex is either $\frac{C(10,3)+C(9,3)}{C(19,3)}$ or $\frac{C(10,3)+C(9,3)}{C(10,3)+C(10,2)C(9,1)+C(10,1)C(9,2)+C(9,3)}$, depending on which approach you took.

5. ***(3 points)*** *Roughly 1% of Cornell's undergraduate students take CS 2800. 95% of the CS 2800 students know the correct definition of "injective", while only 20% of students who didn't take 2800 know the definition. While walking down the hall, you overhear one undergraduate student saying to another "... since f is injective, we know that if $x \neq y$ then $f(x) \neq f(y)$, so ... ". What is the probability that the student has taken CS 2800? (Assume that you heard a randomly chosen Cornell undergrad.)*

   Solution Let C be the event that a randomly selected student took CS 2800, and let I be the event that a randomly selected student knows what injective means.

   We are given $P(C) = 0.01$, $P(I|C) = 0.95$ and $P(I|\bar{C}) = 0.20$. By Bayes's rule:

$$P(C|I) = \frac{P(I|C)P(C)}{P(I|C)P(C) + P(I|\bar{C})P(\bar{C})} = \frac{.95 \cdot 0.01}{0.95 \cdot 0.01 + 0.20 \cdot 0.99}$$

6. ***(8 points)*** *You toss a blue coin that lands heads 1/2 of the time, and a red coin that lands heads 1/3 of the time. You toss the blue coin 5 times and the red coin 3 times. Suppose you get \$2 every time the blue coin lands heads, and you lose \$1 every time the red coin lands heads. (You get nothing if either coin lands tails.)*

   (a) *Write down a sample space that describes this situation. How many elements does it have?*

   Solution There are many possible answers, but the most obvious sample space consists of all possible combinations of results for the 8 flips. For example, $HHHHHTTT$ and $HTHTHHTH$ are all outcomes. There are 2^8 outcomes in this space.

   (b) *Let E be the event that the third toss of the red coin lands tails. What is* $\Pr(E)$*?*

   Solution 2/3

   (c) *Let F be the event that the red coin lands heads at least twice. What is* $\Pr(F)$*?*

   Solution The probability that it lands heads 3 times is $(1/3)^3$. The probability that it lands heads twice is the sum of the probabilities of the outcomes where it lands twice. There are $C(3,2)$ such outcomes, and the probability of each of them is $(1/3)^2(2/3)$. Since landing 3 heads and landing 2 heads are mutually exclusive, the total probability is thus $(1/3)^3 + C(3,2)(1/3)^2(2/3) = 7/27$.

   (d) *Are E and F independent?*

   Solution Note that $E \cap F$ is the event that the third toss of the red coin is tails and the red coin lands at least heads twice. Since there are only three coin tosses (ignoring the blue coin), the events in $E \cap F$ are HHT. THis happens with probability 2/27.

   From above, we know $\Pr(E) = 2/3$ and $\Pr(F) = 7/27$. Since $\Pr(E)\Pr(F) = 7/81 \neq 2/27 = \Pr(E \cap F)$, E and F are not independent.

   (e) *Define a random variable that describes how much money you win. Find its expected value.*

Solution Let $X(s)$ be two times the number of blue heads in s minus one times the number of red heads in s. If X_i is one if coin i comes up heads and 0 otherwise, then $X = 2(X_1 + \cdots + X_5) - (X_6 + X_7 + X_8)$. Note that $E(X_1) = E(X_2) = \cdots = E(X_5) = 1/2$ and $E(X_6) = E(X_7) = E(X_8) = 1/3$. Thus by linearity of expectation we have $E(X) = 5 - 1 = 4$.

7. ***(4 points)*** *Suppose X is the constant random variable c (that is $X(s) = c$ for all s in the sample space). Show that (a) $E(X) = c$ and (b) $\mathrm{Var}(X) = 0$.*

Solution (a) $E(X) = \sum_{s\in S} X(s)\Pr(\{s\}) = c\sum_{s\in S}\Pr(\{s\}) = c$.

(b) First note that $X^2(s) = c^2$ for all s, so that X^2 is also a constant. Thus $E(X^2) = c^2$ Then we have $\mathrm{Var}(X) = E(X^2) - E(X)^2 = E(c^2) - c^2 = c^2 - c^2 = 0$ as required.

Alternatively, we know $\mathrm{Var}(X) = E((X - E(X))^2) = E((X - c)^2) = E(0) = 0$.

8. ***(3 points)*** *The average height of an adult American is about 5.5 feet, and the standard deviation is about 0.2 feet. You wish to build a door that guarantees that 90% of American adults can enter without ducking. Using Chebychev's inequality, how tall must the door be? (*Hint: *If $h \le E(H) + d$ then $|h - E(H)| \le d$. Thus $\Pr(H \ge E(H) + d) \le \Pr(|H - E(H)| \ge d)$).*

Solution If $H \ge E(H) + d$ then $|H - E(H)| \ge d$. Thus $\Pr(H \ge E(H) + d) \le \Pr(|H - E(H)| \ge d)$.

By Chebychev's inequality, $\Pr(|H - E(H)| \ge d) \le \mathrm{Var}(H)/d^2$. We wish to find d such that $\mathrm{Var}(H)/d^2 \le 0.1$. Note that $\mathrm{Var}(H) = 0.2^2 = 0.04$. Thus our requirement is satisfied if $d^2 \ge 0.04/0.1 = 0.4$, so $d \ge \sqrt{.4}$.

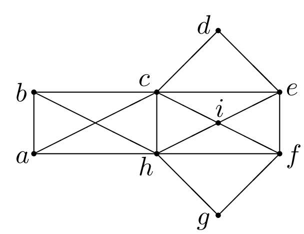
Thus the door must be $E(h) + d$ feet (which is about 6.1 feet).

I will not fit in your door ☹ (but most of you will ☺!)

9. ***(3 points)*** *Is there an undirected graph with 5 vertices all of which have degree 3? (If you think there is such a graph, draw it. If not, explain why not. Just a "yes" or "no" gets no credit.)*

Solution This is not possible. If this were the case, the sum of the degrees of all the vertices would be 15. But the sum of the degrees is twice the number of edges, so must be even.

10. ***(3 points)*** *Does the graph to the right have a Eulerian path? Does it have an Eulerian cycle? (If you think the answer is yes, say what it is. If not, explain why not.)*



Solution Since exactly two of the nodes have odd degree, a and b, there is an Eulerian path, which is $b - c - d - e - f - g - h - a - b - h - f - i - e - c - i - h - c - a$ (other answers are possible). Because not all vertices have even degree, there is no Eulerian cycle.