Discrete Structures	*Prelim 1*— ***Solutions***
CS2800	

1. *[5 points] Recall that $[X \to Y]$ denotes the set of all (total) functions from X to Y, and Y^n denotes $Y \times Y \times \cdots \times Y$ (n times).*

   Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite set with cardinality n. Define a bijection from $[X \to Y]$ to Y^n.

   Solution Let $F : [X \to Y] \to Y^n$ be given by $F(f) = (f(x_1), f(x_2), \ldots, f(x_n))$. This function is injective: if $F(f) = F(g)$ then $f(x_1) = g(x_1)$, $f(x_2) = g(x_2)$, and in fact $f(x_i) = g(x_i)$ for all $i \leq n$. Thus $f = g$.

   This function is surjective: given a tuple $t = (y_1, y_2, \ldots, y_n)$, let $f : X \to Y$ be given by $f(x_i) = y_i$. Then clearly $F(f) = t$. Thus F is bijective.

   A few comments on the proof: some poeople observed that $|X \to Y| = |Y^n| = |Y|^n$, and thus there must be a bijection. This is true (and typically got 2/5 if there were no other problems) but the question asked you to *define* a bijection. If you just say that there has to be a bijection, that isn't good enough; you have to say what it is.

2. *[5 points] Let $\{1, 2, 3\}^\omega$ be the set of infinite sequences containing only the numbers 1, 2, and 3. For example, some sequences of this kind are:*

$$(1, 1, 1, 1, 1, 1, \ldots)$$
$$(2, 2, 2, 2, 2, 2, \ldots)$$
$$(3, 2, 1, 3, 2, 1, \ldots)$$

   Prove that $\{1, 2, 3\}^\omega$ is uncountable.

   Solution Suppose (for the sake of contradiction) that $X = \{1, 2, 3\}^\omega$ is countable. Then there exists a surjection $f : \mathbb{N} \to X$. We will show that f is *not* a surjection by constructing a sequence s_D that is not in the image of f.

   Form the ith element of the sequence s_D by adding one to the ith element of $f(i)$ (wrapping around to 1 if $f(i)_i = 3$). The $s_D \neq f(i)$ for any i, because it differs in the ith place. Therefore s_D is not in the image of f, contradicting the assumption that f is surjective. Thus X is not countable.

3. *[4 points] Prove by induction that*

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}$$

   for all $n \geq 1$.

   Solution Let $P(n)$ be the statement

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}$$

   Base case: $P(1)$ is true, because $1 \cdot 2 = 1 \cdot 2 \cdot 3/3$.

   Inductive step: Assume $P(n)$. We wish to show $P(n+1)$, i.e.

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + (n+1)(n+2) = \frac{(n+1)(n+2)(n+3)}{3}$$

Well,

$$\begin{aligned}
1\cdot 2+2\cdot 3+3\cdot 4+\cdots+(n+1)(n+2) &= (1\cdot 2+2\cdot 3+3\cdot 4+\cdots+n(n+1))+(n+1)(n+2) \\
&= \frac{n(n+1)(n+2)}{3}+(n+1)(n+2) && \text{by } P(n) \\
&= \frac{n(n+1)(n+2)+3(n+1)(n+2)}{3} && \text{common denominator} \\
&= \frac{(n+3)(n+1)(n+2)}{3} && \text{grouping } (n+1)(n+2
\end{aligned}$$

as required.

To get full credit, you needed to have a clearly-stated induction hypothesis $P(n)$. You lost 1 point for the standard errors that we also deducted one point for on the homework: e.g., including the "for all n" inside the $P(n)$ or treating $P(n)$ as a number. ($P(n)$ is *not* a number! It's an English statement that has an n in it. You can't write, for example, "$P(n) = n(n+1)$". That doesn't make sense.)

4. *[6 points: 2+4] (a) Explain carefully what the flaw is with the following "proof" that every postage of three cents or more can be formed using just three-cent and four-cent stamps.*

   We use regular induction to prove that $P(n)$ holds for all $n \geq 3$, where $P(n)$ says that postage of n cents cents can be formed using just three-cent and four-cent stamps.

   Base Case: *We can form postage of three cents with a single three-cent stamp and we can form postage of four cents using a single four-cent stamp.*

   Inductive Step: *Assume that we can form postage of j cents for all nonnegative integers j with $j \leq k$ using just three-cent and four-cent stamps. We can then form postage of $k+1$ cents by replacing one three-cent stamp with a four-cent stamp or by replacing two four- cent stamps by three three-cent stamps.*

   (b) Show that every postage of six cents or more can be formed using just three-cent and four-cent stamps.

   Solution (a) In doing the inductive step, you don't know that a three-cent stamp or two four-cent stamps were used in paying for k cents. In particular, if $k = 4$, only one four-cent stamp is used. So the argument going from 4 to 5 fails. (We also reluctantly accepted as an error in the proof that the inductive step should have said "Assume that we can form postage of j cents for all nonnegative integers j with $3 \leq j \leq k$"; that is, we mut start at 3 (because that's the base case), not 0 (which is what we're doing if we just say "for all nonnegative integers j with $j \leq k$".)

   (b) Use strong induction to prove $P(n)$ (where $P(n)$ is as above) for $n \geq 6$.

   Base Case: We can form postage of six cents using two three-cent stamps.

   Inductive Step: Suppose that $P(j)$ holds for all $j \leq n$. We want to prove $P(n+1)$.

   If $n+1 = 7$, we can use a three- and a four-cent stamp; if $n+1 = 8$, we can use two four-cent stamps. (It's OK to make 7 and 8 part of the base case.) If $n+1 \geq 9$, then $n \geq 8$, so $n+1-3 = n-2 \geq 6$. That means that, using the induction hypothesis, we can pay for $n-2$ cents of postage using three- and four-cent stamps. Add one more three-cent stamp. That will pay for $n+1$ cents of postage.

   As in question 3, we deducted points for including "for all n" in the statement $P(n)$ or somehow treating $P(n)$ as a number.

5. *[7 points: 4+3] (a) Six women and nine men are on the faculty of a schools CS department. The individuals are distinguishable. How many ways are there to select a committee of 5 members if at least 1 woman must be on the committee?*

   (b) How many ways can Mr. and Mrs. Sweettooth distribute 13 identical pieces of candy to their three children so that each child gets at least one piece of candy?

Solution (a) The easiest way to do this is to first consider how many a committee of 5 can be formed without any women. This is clearly $C(9,5)$ (we choose 5 people among the 9 men). All the remaining committees of 5 must have at least one woman. Since there are $C(15,5)$ committees altogether, there are $C(15,5) - C(9,5)$ committees with at least one woman.

An alternative approach considers all the committes with exactly 1 woman (and thus, 4 men), which is $C((5,1)C(9,4)$ (by the Product Rule), all the committees with exactly 2 women: $C(5,2)C(9,3)$; all the committees with exactly 3 women: $C(5,3)C(9,2)$; all the committees with exactly 4 women: $C(5,4)C(9,1)$; and all the committees with exactly 5 women: $C(5,5)C(9,0) = 1$. Thus, there are $C(5,1)C(9,4)+C(5,2)C(9,3)+C(5,3)C(9,2)+C(5,4)C(9,1)+1$ such committees. (Of course, it follows that $C(15,5) - C(9,5) = C(5,1)C(9,4) + C(5,2)C(9,3) + C(5,3)C(9,2) + C(5,4)C(9,1) + 1$.)

(b) First, give each child one piece of candy. That leaves 10 pieces of candy that can be distributed arbitrarily among three children. This is a balls and urns problem with the candies are the balls (and are indistinguishable) and the urns the children (so are distinguishable). Thus, there are $C(12,2)$ (or, equivalently, $C(12,10)$) ways of doing this.

6. *[3 points] Give a combinatorial proof of the following identity:*

$$\binom{n}{r+a}\binom{r+a}{r} = \binom{n}{r}\binom{n-r}{a}.$$

(Note: no credit will be given for an algebraic proof.)

Solution Consider two ways of choosing two disjoint committees out of n people, one of size r and one of size a. One way is to first choose the $r+a$ who will be on one of the committees, and then choosing the r people who will be on the first committee. (The remaining a people will be on the second committee.) There are $C(n, r+a)C(r+a, r)$ ways of doing this. A second approach is to first choose the committee of size r, then choose the committee of size a out of the remaning $n-r$ people. There are $C(n,r)C(n-r,a)$ ways of doing this. Thus, $C(n, r+a)C(r+a, r) = C(n,r)C(n-r,a)$.