Discrete Structures	Prelim 1
CS2800	Selected problems from past exams

1. True/false. For each of the following statements, indicate whether the statement is true or false. Give a one or two sentence explanation for your answer.
   (a) A proof that starts "Choose an arbitrary $y \in \mathbb{N}$, and let $x = y^2$" is likely to be a proof that $\forall y \in \mathbb{N}, \forall x \in \mathbb{N}, \dots$.
   (b) The set of real numbers ($\mathbb{R}$) is countable.
   (c) The set of rational numbers ($\mathbb{Q}$) is countable.
   (d) The sentence "everybody can fool Mike" is false if and only if the sentence "nobody can fool Mike" is true.
   (e) Recall that $[X \to Y]$ denotes the set of functions with domain X and codomain Y. Let $f : 2^S \to [S \to \{0,1\}]$ be given by $f(X) ::= h$ where $h : S \to \{0,1\}$ is given by $h(s) ::= 0$. f is injective.
   (f) f as just defined is surjective.

2. Prove the following claim using induction: for any $n \geq 0$, $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$

3. Complete the following diagonalization proof:

   Claim: $X = [\mathbb{N} \to \mathbb{N}]$ is uncountable.

   Proof: We prove this claim by contradiction. Assume that X is countable. Then there exists a function F : **FILL IN** that is **FILL IN**.

   Write $f_0 = F(0)$, $f_1 = F(1)$, and so on. We can write the elements of X in a table:

	0	1	2	$\cdots$
f_0	$f_0(0)$	$f_0(1)$	$f_0(2)$	$\cdots$
f_1	$f_1(0)$	$f_1(1)$	$f_1(2)$	$\cdots$
$\vdots$	$\vdots$	$\vdots$		$\ddots$

   Let f_D : **FILL IN** be given by $f_D : x \mapsto$ **FILL IN**

   Then **FILL IN**

   This is a contradiction because **FILL IN**.

4. Suppose you are given a function $f : \mathbb{N} \to \mathbb{N}$, and are told that $f(1) = 1$ and for all n, $f(n) \leq 2f(\lfloor n/2 \rfloor)+1$.

   Use strong induction on n to prove that for all $n \geq 2$, $f(n) \leq 2n \log_2 n$.

   You may write log to indicate $\log_2$. Here is a reminder of some facts about $\lfloor x \rfloor$ and $\log x$:

   - $\lfloor x \rfloor \leq x$
   - $\log 1 = 0$, $\log 2 = 1$
   - $\log(x/2) = \log x - 1$
   - $\log(2^x) = x$
   - $\log(x^2) = 2 \log x$
   - if $x \leq y$ then $\log x \leq \log y$

5. Which of the following sets are countably infinite and which are not countably infinite? Give a one to five sentence justification for your answer.
   (a) The set Σ^* containing all finite length strings of 0's and 1's.

(b) The set $2^{\mathbb{N}}$ containing all sets of natural numbers.

(c) The set $\mathbb{N} \times \mathbb{N}$ containing all pairs of natural numbers.

(d) The set $[\mathbb{N} \to \{0,1\}]$ containing all functions from $\mathbb{N}$ to $\{0,1\}$.

Be sure to include enough detail:

- If listing elements, be sure to clearly state how you are listing them;
- If diagonalizing, be sure it is clear what your diagonal construction is;
- If providing a function, make sure it is clear what the output is on a given input.

6. For any function $f : A \to B$ and a set $C \subseteq A$, define $f(C) = \{f(x) \mid x \in C\}$. That is, $f(C)$ is the set of images of elements of C. Prove that if f is injective, then $f(C_1 \cap C_2) = f(C_1) \cap f(C_2)$ for all $C_1, C_2 \subseteq A$.

   (*Hint:* one way to prove this is from the definition of set equality: $A = B$ iff $A \subseteq B$ and $B \subseteq A$.)

7. The Fibonacci numbers $F_0, F_1, F_2, \ldots$ are defined inductively as follows:

$$F_0 = 1$$
$$F_1 = 1$$
$$F_n = F_{n-1} + F_{n-2} \quad \text{for } n \geq 2$$

   That is, each Fibonacci number is the sum of the previous two numbers in the sequence. Prove by induction that for all natural numbers n (including 0):

$$\sum_{i=0}^{n} F_i \;=\; F_{n+2} - 1$$

8. Prove by induction that for any integer $n \geq 3$, $n^2 - 7n + 12$ is non-negative.

9. (a) Write the definition of "$f : A \to B$ is injective" using formal notation ($\forall$, $\exists$, $\wedge$, $\vee$, $\neg$, $\Rightarrow$, $=$, $\neq$, $\ldots$).

   (b) Similarly, write down the definition of "$f : A \to B$ is surjective".

   (c) Write down the definition of "A is countable". You may write "f is surjective" or "f is injective" in your expression.

10. Recall that the composition of two functions $f : B \to C$ and $g : A \to B$ is the function $f \circ g : A \to C$ defined as $(f \circ g)(x) = f(g(x))$. Prove that if f and g are both injective, then $f \circ g$ is injective.

11. For each of the following functions, indicate whether the function f is injective, whether it is surjective, and whether it is bijective. Give a one sentence explanation for each answer.

    (a) $f : \mathbb{N} \to \mathbb{N}$ given by $f : x \to x^2$

    (b) $f : \mathbb{R} \to \mathbb{R}$ given by $f : x \to x^2$

    (c) $f : X \to [Y \to X]$ given by $f : x \mapsto h_x$ where $h_x : Y \to X$ is given by $h_x : y \mapsto x$.

12. Recall that the Fibonacci numbers $F_1, F_2, F_3, \ldots$ are defined inductively as follows:

$$F_1 = 1$$
$$F_2 = 1$$
$$F_n = F_{n-1} + F_{n-2} \quad \text{for } n \geq 3$$

    That is, each Fibonacci number is the sum of the previous two numbers in the sequence. Prove that:

$$\sum_{i=1}^{n} F_i^2 \;=\; F_n F_{n+1}$$

13. A chocolate bar consists of n identical square pieces arranged in an unbroken rectangular grid. For instance, a 12-piece bar might be a 3×4, 2×6 or 1×12 grid. A single snap breaks the bar along a straight line separating the squares, into two smaller rectangular pieces. Prove that regardless of the initial dimensions of the bar, any n-piece bar requires exactly $n-1$ snaps to break it up into individual squares.

14. Briefly and clearly identify the errors in each of the following proofs:

    (a) **Proof that 1 is the largest natural number:** Let n be the largest natural number. Then n^2, being a natural number, is less than or equal to n. Therefore $n^2 - n = n(n-1) \leq 0$. Hence $0 \leq n \leq 1$. Therefore $n = 1$.

    (b) **Proof that 2 = 1:** Let $a = b$.

    $$\begin{aligned} \Rightarrow \quad & a^2 = ab \\ \Rightarrow \quad & a^2 - b^2 = ab - b^2 \\ \Rightarrow \quad & (a+b)(a-b) = b(a-b) \\ \Rightarrow \quad & a + b = b \end{aligned}$$

    Setting $a = b = 1$, we get $2 = 1$.

    (c) **Proof that** $(a+b)(a-b) = a^2 - b^2$**:**

    $$\begin{aligned} \text{To prove:} \quad & (a+b)(a-b) = a^2 - b^2 \\ \Rightarrow \quad & a^2 - ab + ab - b^2 = a^2 - b^2 \\ \Rightarrow \quad & a^2 - b^2 = a^2 - b^2 \end{aligned}$$

    ...which is true, hence the result is proved.

15. Prove that $7^m - 1$ is divisible by 6 for all positive integers m.

16. Prove that

    $$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$$

    for all positive integers n.

17. Prove by induction that the sum of the interior angles of a convex[1] polygon with n sides (and hence n vertices) is $180(n-2)$ degrees. You may use the fact that the sum of the interior angles of a triangle is 180 degrees. You do not need to prove straightforward geometrical facts rigorously (check with us if unsure).

18. In a permutation of the set $\{1, 2, \ldots, n\}$, a pair i, j is *out of order* if $i < j$ but i occurs after j in the permutation. In a random permutation of the set $\{1, 2, \ldots, n\}$ with all permutations equally likely, what is the expected number of pairs that are out of order?

19. Consider the following assertion.

    > Assuming everyone has three initials, there are at least 6 people in California (population 38 million) with the same initials and the same birthday (the same day of the year, but not necessarily the same year).

    Describe a simple calculation you could do to verify this assertion.

20. Give an expression describing the number of different ways the following things can happen. No credit will be given for just the value, even if correct.

[1] A polygon is convex if, for all vertices p and q of the polygon, the line joining p and q lies entirely within the polygon.

(a) During your pregnancy, you decided on a list of 23 girls' first names and 16 boys' first names, as well as a list of 11 gender-neutral middle names. To your surprise, you had quintuplets, two boys and three girls. Now you must select a first and a middle name for each child from the lists. The names must all be different.

(b) A professor teaching discrete structures is making up a final exam. He has a stash of 24 questions on probability, 16 questions on combinatorics, and 10 questions on logic. He wishes to put five questions on each topic on the exam.

(c) The very same professor wants to assign points to the 15 problems so that each problem is worth at least 5 points and the total number of points is 100.

(d) There are 30 graders to grade the final exam, and the professor would like to assign two graders to each of the 15 problems.

21. Give an expression describing the number of different ways the following things can happen. Your expression may involve binomial coefficients, multinomial coefficients, Stirling numbers of the second kind, factorial expressions, r-permutations, or whatever else you need, but do not evaluate the expression. No credit will be given for just the value, even if correct.

(a) You must choose a password consisting of 6, 7, or 8 letters from the 26-letter English alphabet $\{a, b, \ldots, z\}$.

(b) In a poker game, you are dealt a *full house*, a five-card hand containing three of a kind and a pair of another kind; for example, three kings and two sixes.

(c) Your team is in the championship game of a soccer tournament. The score is tied at full time and the winner will be decided by penalty kicks. As coach, you must choose a sequence of five different players out of 11 to take the kicks.

(d) You have ten rings, all with different gemstones. You wish to bequeath them to your five children so that each child inherits two of the rings.

(e) You have \$400 to donate to charity, which you would like to distribute among your five favorite charities so that each receives an integral number of dollars.

(f) The same as (e), but you also wish to ensure that each of the five charities receives at least \$20.