Discrete Structures	*Prelim 1 sample questions— **Solutions***
CS2800	*Selected problems from past exams and a few additional problems*

1. *Compute* $(10101)_2 + (101)_2$. *Express your answer in both binary and decimal.*

   Solution There are two approaches:

   Adding in binary, we have

$$\begin{array}{rr} & 1\ 1 \\ & 10101\text{b} \\ + & \quad 101\text{b} \\ \hline & 11010\text{b} \end{array}$$

   Converting this to decimal gives $0 + 2 + 0 + 8 + 16 = 26$.

   Alternatively, we could convert to decimal, then add, then convert back: $10101b = 1 + 4 + 16 = 21$ and $101 = 1 + 4 = 5$. Adding these gives $26 = 16 + 8 + 2 = 11010b$.

2. *In this problem, we are working mod 7, i.e.* $\equiv$ *denotes congruence mod 7 and* $[a]$ *is the equivalence of a mod 7.*

   (a) *What are the units of* $\mathbb{Z}_7$*? What are their inverses?*

   Solution

   - [1]'s inverse is [1]
   - [2]'s inverse is [4]
   - [3]'s inverse is [5]
   - [4]'s inverse is [2]
   - [5]'s inverse is [3]
   - [6]'s inverse is [6]

   (b) *Compute* $[2]^{393}$.

   Solution $[2]^{393} = ([2]^3)^{131} = [1]^{131} = [1]$

3. *Use Euler's theorem and repeated squaring to efficiently compute* $8^n \mod 15$ *for* $n = 5$, $n = 81$ *and* $n = 16023$. *Hint: you can solve this problem with 4 multiplications of single digit numbers. Please fully evaluate all expressions for this question (e.g. write 15 instead of* $3 \cdot 5$*).*

   Solution We use the fact that $8^{\phi(15)} = 1 \mod 15$. $\phi(15) = (3-1)(5-1) = 8$ [multiplication #1], so we can reduce all of the exponents mod 8. We then use repeated squaring to compute 8^{2^k}:

$$\begin{aligned} &[8]^1 = [8] \\ &[8]^2 = [64] = [4] && \text{[multiplication \#2]} \\ &[8]^4 = [4]^2 = [16] = [1] && \text{[multiplication \#3]} \end{aligned}$$

We can then use these to compute the powers of [8]:

$$[8]^5 = [8]^4[8] = [1][8] = [8]$$
$$[8]^{81} = [8]^1 = [8]$$
$$[8]^{16023} = [8]^7 = [8]^4[8]^2[8] = [1][4][8] = [32] = [2] \qquad \text{[multiplication \#4]}$$

4. *(a) Recall Bézout's identity from the homework: for any integers n and m, there exist integers s and t such that $gcd(n, m) = sn + tm$. Use this to show that if $gcd(k, m) = 1$ then $[k]$ is a unit of $\mathbb{Z}_m$.*

   Solution By Bézout's identity, since $gcd(k, m) = 1$, we know that $1 = sk + tm$ for some s and t. Reducing this equation mod m, we find $[1] = [s][k] + [t][m] = [s][k] + [0] = [s][k]$. Therefore, $[k]$ has an inverse, $[s]$, and is thus a unit.

   (b) Use part (a) to show that if p is prime, then $\phi(p) = p - 1$.

   Solution $\phi(p)$ is the number of units mod p. If p is prime, then every number k between 1 and p has $gcd(k, p) = 1$. Therefore, all $p - 1$ non-zero elements of $\mathbb{Z}_p$ are units, so $\phi(p) = p - 1$.

   (c) Use Euler's theorem to compute $3^{38} \mod 37$ (note: 37 is prime).

   Solution Since 37 is prime, $\phi(37) = 36$. Therefore, $3^{38} \equiv 3^2 \mod 37$ since $38 \equiv 2 \mod 36$. Thus $3^{38} \mod 37 = 9$.

5. *(a) What are the units of $\mathbb{Z}$ mod 12?*

   Solution A unit in a set of numbers is a number that has an inverse. In the set $\mathbb{Z}_{12} = \{[0], [1], [2], [3], [4], [5], [6], [7], [8]$ the units are [1], [5], [7], and [11]. In general, $[n]$ is a unit mod m if n and m are relatively prime.

   (b) What are their inverses?

   Solution $[1]^{-1} = [1]$, $[5]^{-1} = [5]$, $[7]^{-1} = [7]$, and $[11]^{-1} = [11]$. This is because $[1] \cdot [1] = [1]$, $[5] \cdot [5] = [25] = [1]$, $[7] \cdot [7] = [49] = [1]$ and $[11] \cdot [11] = [121] = 1$.

   (c) What is $\phi(12)$?

   Solution By definition of ϕ, $\phi(12)$ is the number of units mod 12. Since there are 4 units, $\phi(12) = 4$.

6. *Suppose we pick a bit string of length 4 at random, all bit strings equally likely. Consider the following events:*

   E_1: the string begins with 1.

   E_2: the string ends with 1.

   E_3: the string has exactly two 1's.

   (a) Find $\Pr(E_1)$, $\Pr(E_2)$, and $\Pr(E_3)$.

   Solution $\frac{1}{2}$, $\frac{1}{2}$, $\frac{3}{8}$

   (b) Find $\Pr(E_1 \mid E_3)$.

Solution $\frac{1}{2}$

(c) Find $\Pr(E_2 \mid E_1 \cap E_3)$.

Solution $\frac{1}{3}$

(d) Are E_1 *and* E_2 *independent? Justify your answer.*

Solution Yes, $\Pr(E_1 \cap E_2) = \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2} = \Pr(E_1) \cdot \Pr(E_2)$

(e) Are E_2 *and* E_3 *independent? Justify your answer.*

Solution Yes, $\Pr(E_2 \cap E_3) = \frac{3}{16} = \frac{1}{2} \cdot \frac{3}{8} = \Pr(E_2) \cdot \Pr(E_3)$

7. *Give the* formal mathematical definition *of each of the following terms. Your definitions should be valid for a finite or countably infinite sample space* S.

(a) Probability distribution on S.

Solution A function $\Pr : S \to \mathbb{R}$ such that for all $s \in S$, $\Pr(s) \geq 0$ and $\sum_{s \in S} \Pr(s) = 1$.

(b) Event.

Solution A subset of S.

(c) Probability of an event E*, given a probability distribution* Pr *on* S.

Solution $\Pr(E) = \sum_{s \in E} \Pr(s)$.

(d) Conditional probability of E *given* F.

Solution $\Pr(E \mid F) = \Pr(E \cap F) / \Pr(F)$, undefined if $\Pr(F) = 0$.

(e) Real-valued random variable X *on* S.

Solution A function $X : S \to \mathbb{R}$.

(f) Expectation of a random variable X.

Solution $E(X) = \sum_{s \in S} X(s) \cdot \Pr(s)$.

(g) Variance of a random variable X.

Solution $V(X) = E((X - E(X))^2) = E(X^2) - E(X)^2$.

8. *Give an expression describing the probability of the following events. Evaluate the expression if it is easy to do so.*

   (a) *A fair coin is flipped 100 times giving exactly 50 heads.*

   Solution $\binom{100}{50} 2^{-100}$

   (b) *A fair coin is flipped 100 times giving at most 50 heads.*

   Solution $\sum_{n=0}^{50} \binom{100}{n} 2^{-100}$ or $\frac{1}{2} + \frac{1}{2}\binom{100}{50} 2^{-100}$. (Our original solutions said 1/2, which was incorrect. Student answers of 1/2 received full credit.)

   (c) *A roll of two fair dice yields a sum of 7.*

   Solution 1/6

   (d) *The conditional probability that the last card dealt in a 5-card poker hand yields a straight (five cards in sequence, irrespective of suit) given that the first four cards dealt were* 5♠, 6♡, 7♢, 8♣.

   Solution 1/6

   (e) *A controversial bill before the Senate is supported by 50 of 55 Democrats who will vote for the bill and opposed by 40 of 45 Republicans who will vote against the bill. Of the remaining 10 undecided Senators, each Democrat will vote in favor with probability 1/2 and each Republican against with probability 9/10, independent of the other votes. The bill requires 51 votes to pass. What is the probability that it passes?*

   Solution $1 - .9^5 \cdot .5^5$

9. *The following questions refer to successive rolls of a fair six-sided die, where each roll yields an integer* m *in the range* $1 \leq m \leq 6$ *independently with uniform probability.*

   (a) *What is the expected number of times that 2 is rolled in 12 rolls?*

   Solution 2

   (b) *What is the expected value of each roll?*

   Solution $3\frac{1}{2}$

   (c) *What is the expected sum of four rolls?*

   Solution 14

   (d) *What is the expected number of rolls before seeing the first 6?*

Solution 6

10. *The following questions refer to three independent flips of a fair coin.*

    (a) *Give an example of three events that are mutually independent.*

    Solution first flip heads, second flip heads, third flip heads

    (b) *Give an example of three events that are pairwise independent but not mutually independent.*

    Solution first flip heads, second flip tails, first and second flip the same

    (c) *Give an example of three events, no pair of which are independent.*

    Solution three heads, three tails, two heads and one tail

11. *Give an expression describing the probability of the following events. You do not need to evaluate the expression.*

(a) *You obtain at least three consecutive heads in four flips of a fair coin.*

Solution 3/16

(b) *In a poker game, you are dealt a five-card hand containing four aces.*

Solution $48/\binom{52}{5}$

(c) *The conditional probability that the first flip of two flips of a fair coin comes up heads, given that at least one of them does.*

Solution 2/3

(d) *A 13-card bridge hand contains all cards of the same suit.*

Solution $4/\binom{52}{13}$

(e) *A randomly chosen number between 0 and 99 (inclusive) is divisible by 4.*

Solution 1/4

12. *Let $S = \{a, b\}$, and the function P given by the following table:*

A	$P(A)$
$\emptyset$	*0*
$\{a\}$	1/4
$\{b\}$	3/4
$\{a, b\}$	1

(a) **(1 point)** *What are the events of S?*

Solution We helpfully listed them for you when defining P! They are all possible subsets of S, i.e. $\emptyset$, $\{a\}$, $\{b\}$, and $\{a, b\}$.

(b) **(3 points)** *Prove that (S, P) is a probability space.*

Solution We're given that S is a set, and we've just shown that the domain of P covers all possible subsets of that set. The *only* correct way to wrap up the argument is to show that all three of Kolmogorov's axioms hold. Let's take them one by one.

Does Axiom 1 hold? The probability of every event is non-negative – we listed all the events above, and their probabilities are explicitly given in the table. Hence Axiom 1 holds.

Does Axiom 2 hold? $P(S) = P(\{a, b\}) = 1$, again from the table. So Axiom 2 holds.

Does Axiom 3 hold? This is a little tedious but feasible since S is so small. Let's first consider non-empty subsets:

- $P(\{a\} \cup \{b\}) = P(\{a, b\}) = 1$, $P(\{a\}) + P(\{b\}) = 1/4 + 3/4 = 1$. Yep.

...and that's it! For this set, this is the only possible way to break it into disjoint non-empty subsets (note that for the third axiom, the order of the subsets does not matter).

What happens if we include empty subsets? Well, let's take any number of instances of the empty set and add them to any other countable set of disjoint non-empty subsets $A_1, A_2, \ldots$:

$$\begin{aligned} P(A_1 \cup A_2 \cup \cdots \cup \emptyset \cup \emptyset \cup \ldots) &= P(A_1 \cup A_2 \cup \ldots) \qquad \text{(by definition, } X \cup \emptyset = X) \\ &= P(A_1 \cup A_2 \cup \ldots) + 0 + 0 + \ldots \\ &= P(A_1) + P(A_2) + \cdots + 0 + 0 + \ldots \qquad \text{(we just proved this)} \\ &= P(A_1) + P(A_2) + \cdots + P(\emptyset) + P(\emptyset) + \ldots \qquad \text{(from the table)} \end{aligned}$$

This covers all possible countable sets of disjoint subsets of A. So we have proved Axiom 3 holds. (We were a little lenient when grading this part – e.g. we gave you credit even if you didn't handle repetitions of the empty set explicitly.)

Since all three of Kolmogorov's axioms hold, (S, P) is a probability space.

(c) **(1 point)** *Are $\{a\}$ and $\{b\}$ independent? Explain.*

Solution The only way this could be true is if $P(\{a\} \cap \{b\}) = P(\{a\}) \cdot P(\{b\})$. But this is not the case: $P(\{a\} \cap \{b\}) = P(\emptyset) = 0$, and $P(\{a\}) \cdot P(\{b\}) = 1/4 \times 3/4 \neq 0$. So the events are not independent. Note that this is the only possible correct answer, derived from the definition of independence.

13. **(2 points)** *In an infamous criminal case, a mother was accused of murdering her two infant sons. A well-known statistician testified that the chance that* both *deaths were natural was infinitesimal. He proposed the following calculation:*

$$P(D_1 \cap D_2 \mid I) = P(D_1 \mid I) \cdot P(D_2 \mid I)$$

where D_1 and D_2 are the events that the two children respectively died, and I is the event that the mother is innocent.

Since natural infant death is rare in the family's demographic, both probabilities on the right hand side are tiny: about 1/8543. Plugging in the values, we obtain $P(D_1 \cap D_2 \mid I) \approx$ 1/73,000,000. Based on this, the mother was found guilty and imprisoned.

Four years later, the ruling was overturned on grounds of faulty statistics. There are **two** *significant errors in the reasoning above. Briefly and clearly identify both.*

Solution The first error is that the two deaths are presumed independent (conditioned on innocence). This is unjustified: genetic factors etc. can increase the chances of multiple deaths in the same family.

The second error is trickier but more damning: the conditional of interest is $P(I \mid D_1 \cap D_2)$, not $P(D_1 \cap D_2 \mid I)$. If this is written out via Bayes' Theorem and the Theorem of Total Probability,

$$P(I \mid D_1 \cap D_2) = \frac{P(D_1 \cap D_2 \mid I)P(I)}{P(D_1 \cap D_2 \mid I)P(I) + P(D_1 \cap D_2 \mid I')P(I')},$$

the even smaller chance that a mother would actually murder both her sons makes $P(I \mid D_1 \cap D_2) > 1/2$ (e.g. try plugging in $P(I') = 1$ in a billion. This type of error is known as the "prosecutor's fallacy".

Some of you suggested that the error was in not accounting for someone else being the murderer. In other words, you were saying the problem lies in the statement "Since natural infant death is rare in the family's demographic, both probabilities on the right hand side are tiny." It is true that natural infant death is only one of the ways in which the deaths could happen while the mother was innocent — an axe murderer could also be stalking the neighborhood. While I hope axe murderers aren't common enough

that this significantly changes the probability, it is true that there is something slightly fishy here, so we gave you credit. (We hadn't intended this to be the error, btw. We realized later that our wording was slightly off.)

The error is *not* that 1 in 73 million is still not 0. If we required *100 percent* certainty (if such a thing exists), no legal case would ever get settled.

This is an actual case, btw. See `http://en.wikipedia.org/wiki/Sally_Clark`.

14. **(3 points)** *Let E be a herd of 100 elephants. The herd contains 10 adult males, 60 adult females and 30 babies. It is known*[1] *that the adult elephants have an average surface area of $17m^2$, and the babies have an average surface area of $4m^2$. A biologist, unaware of these statistics, picks an elephant uniformly at random from E and measures its surface area (after temporarily and painlessly tranquilizing it). If the measured surface area is represented as a random variable, what are its (a) domain, (b) codomain, and (c) expectation (show your calculations)?*

*(*Note: *There are many correct answers for (a) and (b). Pick any one.)*

(a) Domain: E, or $\{adult, baby\}$, or $\{adult\ male, adult\ female, baby\}$, or other reasonable variation.

(b) Codomain: $\mathbb{R}$, or $\mathbb{R}^+$, or $\{17, 4\}$, or $\{17m^2, 4m^2\}$, or other reasonable variation.

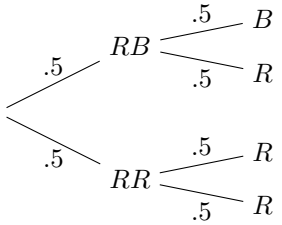
(c) Expectation: the numerical answer (in m^2) is $17 \times 0.7 + 4 \times 0.3 = 13.1$. There are many ways to calculate this, depending on how you choose your domain. E.g. let's choose the domain E, call the random variable A, and for notational convenience (without loss of generality) assume the adult elephants are numbered 1 to 70, and the young 'uns 71-100. Then you might write:

$$\begin{aligned}
\textit{Expectation of } A &= \sum_{i=1}^{100} A(elephant_i)P(elephant_i) \\
&= \sum_{i=1}^{70} A_{adult}P(elephant_i) + \sum_{i=71}^{100} A_{baby}P(elephant_i) \\
&= \sum_{i=1}^{70} 17 \cdot \frac{1}{100} + \sum_{i=71}^{100} 4 \cdot \frac{1}{100} \\
&= 17 \cdot \frac{70}{100} + 4 \cdot \frac{30}{100}
\end{aligned}$$

15. *There are two cards: one is red on both sides and the other is black on one side and red on the other. One of the two cards is chosen at random and a side is picked at random and shown to you..*

(a) Draw a probability tree to describe the situation.

Solution



Here RR refers to the card that is red on both sides, while RB refers to the card that is red on one side and black on the other. R and B refer to seeing the red or black side of the chosen card.

(b) What is the sample space, and what is hte probability of each outcome.

[1] K. P. Sreekumar and G. Nirmalan, "Estimation of the total surface area in Indian elephants (Elephas maximus indicus)", Veterinary Research Communications, 1990;14(1):5-17.

Solution The way I've modeled it here, sample space contains the 3 outcomes (RB, R), (RB, B), and (RR, R). (RB, R) and (RB, B) have probability $1/4$ while (RR, R) has probability $1/2$.

(c) Suppose that the side shown to you is red. What is the probabiity that the other side of the card is red?

Solution Let R be the event where the side seen is red: $R = \{(RB, R), (RR, R)\}$. Let RR be the event that the red/red card was chosen. Then we are interested in $Pr(RR|R)$. By definition, $Pr(RR) = Pr(RR \cap R)/Pr(R)$. Now $Pr(R) = 3/4$, and $RR \cap R = \{(RR, R)\}$, which has probability $1/2$. Thus $Pr(RR|R) = 4/6 = 2/3$.

16. **(0 points)** *How would you (humanely) measure the surface area of an elephant?*

Solution There were many excellent answers, but a favorite was: "very carefully".