Combinatorics

Problem: How to count without counting.

- How do you figure out how many things there are with a certain property without actually enumerating all of them.

Sometimes this requires a lot of cleverness and deep mathematical insights.

But there are some standard techniques.

- That's what we'll be studying.

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Answer: 26 choices for the first letter, 26 for the second, 10 choices for the first number, the second number, and the third number:

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Example 2: A traveling salesman wants to do a tour of all 50 state capitals. How many ways can he do this?

Answer: 50 choices for the first place to visit, 49 for the second, . . . : 50! altogether.

There are two general techniques for solving problems. Two of the most important are:

The Sum Rule: If there are $n(A)$ ways to do A and, distinct from them, $n(B)$ ways to do B, then the number of ways to do A *or* B is $n(A) + n(B)$.

- This rule generalizes: there are $n(A) + n(B) + n(C)$ ways to do A *or* B *or* C

The Product Rule: If there are $n(A)$ ways to do A and $n(B)$ ways to do B, then the number of ways to do A *and* B is $n(A) \times n(B)$. This is true if the number of ways of doing A and B are independent; the number of choices for doing B is the same regardless of which choice you made for A.

- Again, this generalizes. There are $n(A) \times n(B) \times n(C)$ ways to do A *and* B *and* C

Some Subtler Examples

Example 3: If there are n Senators on a committee, in how many ways can a subcommittee be formed?

Two approaches:

1. Let N_1 be the number of subcommittees with 1 senator (n), N_2 the number of subcommittees with 2 senator ($n(n-1)/2$), ...
   According to the sum rule:
   $$N = N_1 + N_2 + \cdots + N_n$$
   - It turns out that $N_k = \frac{n!}{k!(n-k)!}$ (n choose k) – proved later.
   - A subtlety: What about N_0? Do we allow subcommittees of size 0? How about size n?
     - The problem is somewhat ambiguous.

   If we allow subcommittees of size 0 and n, then there are 2^n subcommittees altogether.
     - This is just the number of subsets of the set of n Senators: there is a bijection between subsets and subcommittees.

Number of subsets of a set

Claim: $\mathcal{P}(S)$ (the set of subsets of S) has $2^{|S|}$ elements (i.e, a set S has $2^{|S|}$ subsets).

Proof #1: By induction on $|S|$.

Base case: If $|S| = 0$, then $S = \emptyset$. The empty set has one subset (itself).

Inductive Step; Suppose $S = \{a_1, \ldots, a_{n+1}\}$. Let $S' = \{a_1, \ldots, a_n\}$. By the induction hypothesis, $|\mathcal{P}(S')| = 2^n$.

Partition $\mathcal{P}(S)$ into two subsets:
$A=$ the subsets of S that don't contain a_{n+1}.
$B =$ the subsets of S that do contain a_{n+1}.

It's easy to see that $A = \mathcal{P}(S')$: T is a subset of S that doesn't contain a_{n+1} if and only if T is a subset of S'. Thus $|A| = 2^n$.

Claim: $|A|$ and $|B|$, since there is a bijection from A to B.
Proof: Let $f : A \to B$ be defined by $f(T) = T \cup \{a_{n+1}\}$. Clearly if $T \neq T'$, then $f(T) \neq f(T')$, so f is an injection. And if $T' \in B$, then $a_{n+1} \in T$, $T' = T - \{a_{n+1}\} \in A$, and $f(T') = T$, so f is a surjection. Thus, f is a bijection.

Thus, $|A| = |B|$, so $|B| = 2^n$. Since $\mathcal{P}(S) = A \cup B$, by the Sum Rule, $|S| = |A| + |B| = 2 \cdot 2^n = 2^{n+1}$.

Proof #2: Suppose $S = \{a_1, \ldots, a_n\}$. We can identify $\mathcal{P}(S)$ with the set of bitstrings of length n. A bitstring $b_1 ... b_n$, where $b_i \in \{0, 1\}$, corresponds to the subset T where $a_i \in T$ if and only if $b_i = 1$.

Example: If $n = 5$, so $S = \{a_1, a_2, a_3, a_4, a_5\}$, the bitstring 11001 corresponds to the set $\{a_1, a_2, a_5\}$. It's easy to see this correspondence defines a bijection between the bitstrings of length n and the subsets of S.

Why are there 2^n bitstrings?

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Proof #2: Suppose $S = \{a_1, \ldots, a_n\}$. We can identify $\mathcal{P}(S)$ with the set of bitstrings of length n. A bitstring $b_1 \ldots b_n$, where $b_i \in \{0, 1\}$, corresponds to the subset T where $a_i \in T$ if and only if $b_i = 1$.

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Why are there 2^n bitstrings?
That's the product rule: two choices for b_1 (0 or 1), two choices for b_2, ..., two choices for b_n.

$\square$

Back to the senators:

2. Simpler method: Use the product rule, jsut like above.
   - Each senator is either in the subcommittee or out of it: 2 possibilities for each senator:
     - $2 \times 2 \times \cdots \times 2 = 2^n$ choices altogether

General moral: In many combinatorial problems, there's more than one way to analyze the problem.

Question: How many ways can the full committee be split into two sides on an issue?

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Answer: This question is also ambiguous.

- If we care about which way each Senator voted, then the answer is again 2^n: Each subcommittee defines a split + vote (those in the subcommittee vote Yes, those out vote No); and each split + vote defines defines a subcommittee.
- If we don't care about which way each Senator voted, the answer is $2^n/2 = 2^{n-1}$.
  - This is an instance of the Division Rule (coming up).

Coping with Ambiguity

If you think a problem is ambiguous:

1. Explain why
2. Choose one way of resolving the ambiguity
3. Solve the problem according to your interpretation
   - Make sure that your interpretation doesn't render the problem totally trivial

More Examples

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Answer: The product rule again: Each ring gets to “vote” for which pole it’s on.

- Once you’ve decided which rings are on each pole, their order is determined.
- The total number of configurations is 3^n

Example 5: How many distinguishable ways can the letters of “computer” be arranged? How about “discrete”?

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Example 5: How many distinguishable ways can the letters of "computer" be arranged? How about "discrete"?

For computer, it's 8!:

- 8 choices for the first letter, for the second, ...

Question: Is it also 8! for "discrete"? Not quite.

- There are two e's

Suppose we called them e_1, e_2:

- There are two "versions" of each arrangement, depending on which e comes first: discre$_1$te$_2$ is the same as discre$_2$te$_1$.
- Thus, the right answer is $8!/2!$

Division Rule: If there is a k-to-1 correspondence between of objects of type A with objects of type B, and there are $n(A)$ objects of type A, then there are $n(A)/k$ objects of type B.

A *k-to-1 correspondence* is an onto mapping in which every B object is the image of exactly k A objects.

Permutations

A *permutation* of n things taken r at a time, written $P(n, r)$, is an arrangement in a row of r things, taken from a set of n distinct things. Order matters.

Example 6: How many permutations are there of 5 things taken 3 at a time?

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Example 6: How many permutations are there of 5 things taken 3 at a time?

Answer: 5 choices for the first thing, 4 for the second, 3 for the third: $5 \times 4 \times 3 = 60$.

- If the 5 things are a, b, c, d, e, some possible permutations are:

$$\begin{array}{llllll} abc & abd & abe & acb & acd & ace \\ adb & adc & ade & aeb & aec & aed \\ \ldots & & & & & \end{array}$$

In general

$$P(n, r) = \frac{n!}{(n-r)!} = n(n-1)\cdots(n-r+1)$$

Combinations

A *combination* of n things taken r at a time, written $C(n, r)$ or $\binom{n}{r}$ ("n choose r") is any subset of r things from n things. Order makes no difference,

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Example 7: How many ways can we choose 3 things from 5?

Answer: If order mattered, then it would be $5 \times 4 \times 3$. Since order doesn't matter,

$$abc, acb, bac, bca, cab, cba$$

are all the same.

- For way of choosing three elements, there are $3! = 6$ ways of ordering them.

Therefore, the right answer is $(5 \times 4 \times 3)/3! = 10$:

$$\begin{array}{ccccc} abc & abd & abe & acd & ace \\ ade & bcd & bce & bde & cde \end{array}$$

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In general, it's $C(n,r) = \frac{n!}{(n-r)!r!} = n(n-1)\cdots(n-r+1)/r!$.

More Examples

Example 8: How many full houses are there in poker?

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Answer: You need to find a systematic way of counting:

- Choose the denomination for which you have three of a kind: 13 choices.
- Choose the three: $C(4,3) = 4$ choices
- Choose the denomination for which you have two of a kind: 12 choices
- Choose the two: $C(4,2) = 6$ choices.

Altogether, there are:

$$13 \times 4 \times 12 \times 6 = 3744 \text{ choices}$$

0!

It's useful to define $0! = 1$.

Why?

1. Then we can inductively define

$$(n+1)! = (n+1)n!,$$

   and this definition works even taking 0 as the base case instead of 1.
2. A better reason: Things work out right for $P(n,0)$ and $C(n,0)$!

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How many permutations of n things from n are there?

$$P(n,n) = \frac{n!}{(n-n)!} = \frac{n!}{0!} = n!$$

How many ways are there of choosing n out of n?
0 out of n?

$$\binom{n}{n} = \frac{n!}{n!0!} = 1; \quad \binom{n}{0} = \frac{n!}{0!n!} = 1$$

More Questions

Q: How many ways are there of choosing k things from $\{1, \ldots, n\}$ if 1 and 2 can't both be chosen? (Suppose $n, k \geq 2$.)

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Method #1: There are $C(n, k)$ ways of choosing k things from n with no constraints. There are $C(n-2, k-2)$ ways of choosing k things from n where 1 and 2 are definitely chosen:

- This amounts to choosing $k-2$ things from $\{3, \ldots, n\}$: $C(n-2, k-2)$.

Thus, the answer is

$$C(n, k) - C(n-2, k-2)$$

Method #2: There are

- $C(n-2, k-1)$ ways of choosing n from k where 1 is chosen, but 2 isn't from n where 1 is chosen, but 2 isn't;
    - choose $k-1$ things from $\{3, \ldots, n\}$ (which, together with 1, give the choice of k things)
- $C(n-2, k-1)$ ways of choosing k things from n where 2 is chosen, but 1 isn't;
- $C(n-2, k)$ ways of choosing k things from n where neither 1 nor 2 are

So the answer is $2C(n-2, k-1) + C(n-2, k)$.

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Why is
$C(n,k)-C(n-2,k-2)=2C(n-2,k-1)+C(n-2,k)$?

- Stay tuned!

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A: Have to compute how many ways there are of picking k things, two of which are 1 and 2.

$$k(k-1)P(n-2, k-2)$$

Q: How many ways are there to distribute four distinct balls evenly between two distinct boxes (two balls go in each box)?

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$$C(4, 2) = 6$$

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Q: What if the boxes are indistinguishable?

A: $C(4,2)/2 = 3$.

Combinatorial Identities

There all lots of identities that you can form using $C(n,k)$. They seem mysterious at first, but there's usually a good reason for them.

Theorem 1: If $0 \leq k \leq n$, then

$$C(n,k) = C(n, n-k).$$

Proof:

$$C(n,k) = \frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)!(n-(n-k))!} = C(n, n-k)$$

Q: Why should choosing k things out of n be the same as choosing $n-k$ things out of n?

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Q: Why should choosing k things out of n be the same as choosing $n-k$ things out of n?

A: There's a 1-1 correspondence. For every way of choosing k things out of n, look at the things not chosen: that's a way of choosing $n-k$ things out of n.

This is a better way of thinking about Theorem 1 than the combinatorial proof.

Theorem 2: If $0 < k < n$ then

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Proof 1: (Combinatorial) Suppose we want to choose k objects out of $\{1, \ldots, n\}$. Either we choose the last one (n) or we don't.

1. How many ways are there of choosing k without choosing the last one? $C(n-1, k)$.
2. How many ways are there of choosing k including n? This means choosing $k-1$ out of $\{1, \ldots, n-1\}$: $C(n-1, k-1)$.

Proof 2: Algebraic ...

Note: If we define $C(n, k) = 0$ for $k > n$ and $k < 0$, Theorems 1 and 2 still hold.

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This explains why
$C(n, k) - C(n-2, k-2) = 2C(n-2, k-1) + C(n-2, k)$

$$\begin{aligned}
&C(n,k) = C(n-1,k) + C(n-1,k-1)\\
&= C(n-2,k) + C(n-2,k-1) + C(n-2,k-1) + C(n-2,k-2)\\
&= C(n-2,k) + 2C(n-2,k-1), +C(n-2,k-2)
\end{aligned}$$

Pascal's Triangle

Starting with $n = 0$, the nth row has $n + 1$ elements:

$$C(n, 0), \dots, C(n, n)$$

Note how Pascal's Triangle illustrates Theorems 1 and 2.

Theorem 3: For all $n \geq 0$:

$$\Sigma_{k=0}^{n} \binom{n}{k} = 2^n$$

Proof 1: $\binom{n}{k}$ tells you all the way of choosing a subset of size k from a set of size n. This means that the LHS is *all* the ways of choosing a subset from a set of size n. The product rule says that this is 2^n.

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Proof 2: By induction. Let $P(n)$ be the statement of the theorem.

Basis: $\Sigma_{k=0}^{0} \binom{0}{k} = \binom{0}{0} = 1 = 2^0$. Thus $P(0)$ is true.

Inductive step: How do we express $\Sigma_{k=0}^{n} C(n,k)$ in terms of $n-1$, so that we can apply the inductive hypothesis?

- Use Theorem 2!

More combinatorial identities

Theorem 4: For n a non-negative integer

$$\Sigma_{k=0}^{n} k \binom{n}{k} = n2^{n-1}$$

Proof 1:

$$\begin{aligned}
& \Sigma_{k=0}^{n} k \tbinom{n}{k} \\
= \quad & \Sigma_{k=1}^{n} k \tfrac{n!}{(n-k)!k!} \\
= \quad & \Sigma_{k=1}^{n} \tfrac{n!}{(n-k)!(k-1)!} \\
= \quad & n\Sigma_{k=1}^{n} \tfrac{(n-1)!}{(n-k)!(k-1)!} \\
= \quad & n\Sigma_{j=0}^{n-1} \tfrac{(n-1)!}{(n-1-j)!j!} \qquad [\text{Let } j = k-1] \\
= \quad & n\Sigma_{j=0}^{n-1} \tbinom{n-1}{j} \\
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\end{aligned}$$

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= \quad & n\Sigma_{j=0}^{n-1} \tbinom{n-1}{j} \\
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\end{aligned}$$

Proof 2: LHS tells you all the ways of picking a subset of k elements out of n (a subcommittee) and designating one of its members as special (subcomittee chairman).

What's another way of doing this? Pick the chairman first, and then the rest of the subcommittee!

The Binomial Theorem

We want to compute $(x + y)^n$.
Some examples:

$$(x + y)^0 = 1$$

$$(x + y)^1 = x + y$$

$$(x + y)^2 = x^2 + 2xy + y^2$$

$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

$$(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

The pattern of the coefficients is just like that in the corresponding row of Pascal's triangle!

Binomial Theorem:

$$(x+y)^n = \Sigma_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

Proof 1: By induction on n. $P(n)$ is the statement of the theorem.

Basis: $P(1)$ is obviously OK. (So is $P(0)$.)

Inductive step:

$$\begin{array}{rl} & (x+y)^{n+1} \\ = & (x+y)(x+y)^n \\ = & (x+y)\Sigma_{k=0}^{n}\binom{n}{k}x^{n-k}y^k \\ = & \Sigma_{k=0}^{n}\binom{n}{k}x^{n-k+1}y^k + \Sigma_{k=0}^{n}\binom{n}{k}x^{n-k}y^{k+1} \\ = & \ldots \qquad\qquad \text{[Lots of missing steps]} \\ = & y^{n+1} + \Sigma_{k=0}^{n}(\binom{n}{k} + \binom{n}{k-1})x^{n-k+1}y^k \\ = & y^{n+1} + \Sigma_{k=0}^{n}(\binom{n+1}{k}x^{n+1-k}y^k \\ = & \Sigma_{k=0}^{n+1}\binom{n+1}{k}x^{n+1-k}y^k \end{array}$$

Binomial Theorem:

$$(x+y)^n = \Sigma_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

Proof 2: What is the coefficient of the $x^{n-k}y^k$ term in $(x+y)^n$?

Using the Binomial Theorem

Q: What is $(x+2)^4$?

A:

$$\begin{aligned}
& (x+2)^4 \\
= \;& x^4 + C(4,1)x^3(2) + C(4,2)x^2 2^2 + C(4,3)x 2^3 + 2^4 \\
= \;& x^4 + 8x^3 + 24x^2 + 32x + 16
\end{aligned}$$

Q: What is $(1.02)^7$ to 4 decimal places?

A:

$$\begin{aligned}
& (1+.02)^7 \\
= \;& 1^7 + C(7,1)1^6(.02) + C(7,2)1^5(.0004) + C(7,3)(.000008) + \cdots \\
= \;& 1 + .14 + .0084 + .00028 + \cdots \\
\approx \;& 1.14868 \\
\approx \;& 1.1487
\end{aligned}$$

Note that we have to go to 5 decimal places to compute the answer to 4 decimal places.

Balls and Urns

"Balls and urns" problems are paradigmatic. Many problems can be recast as balls and urns problems, once we figure out which are the balls and which are the urns.

How many ways are there of putting b balls into u urns?

- That depends whether the balls are distinguishable and whether the urns are distinguishable

How many ways are there of putting 5 balls into 2 urns?

- If both balls and urns are distinguishable: $2^5 = 32$
  - Choose the subset of balls that goes into the first urn
  - Alternatively, for each ball, decide which urn it goes in
  - This assumes that it's OK to have 0 balls in an urn.

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  - Choose the subset of balls that goes into the first urn
  - Alternatively, for each ball, decide which urn it goes in
  - This assumes that it's OK to have 0 balls in an urn.
- If urns are distinguishable but balls aren't: 6
  - Decide how many balls go into the first urn: $0, 1, \ldots, 5$

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- If both balls and urns are distinguishable: $2^5 = 32$
  - Choose the subset of balls that goes into the first urn
  - Alternatively, for each ball, decide which urn it goes in
  - This assumes that it's OK to have 0 balls in an urn.
- If urns are distinguishable but balls aren't: 6
  - Decide how many balls go into the first urn: 0, 1, ..., 5
- If balls are distinguishable but urns aren't: $2^5/2 = 16$
- If balls and urns are indistinguishable: $6/2 = 3$

What if we had 6 balls and 2 urns?

- If balls and urns are distinguishable: 2^6
- If urns are distinguishable and balls aren't: 7

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$C(u + b - 1, b)$

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First view the urns as distinguishable: u^b

For every solution, look at all $u!$ permutations of the urns. That should count as one solution.

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This can't be right! It's not an integer (e.g. $7^3/7!$).

What's wrong?

The situation is even worse when we have indistinguishable balls in indistinguishable urns.

Reducing Problems to Balls and Urns

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- $C(67, 65) = 67 \times 33 = 2211$

Q3: How many ways can 8 electrons be assigned to 4 energy states?

A: The electrons are the balls; they're indistinguishable. The energy states are the urns; they're distinguishable.

- $C(11, 8) = (11 \times 10 \times 9)/6 = 165$

Inclusion-Exclusion Rule

Remember the Sum Rule:

The Sum Rule: If there are $n(A)$ ways to do A and, distinct from them, $n(B)$ ways to do B, then the number of ways to do A *or* B is $n(A) + n(B)$.
What if the ways of doing A and B aren't distinct?

Example: If 112 students take CS280, 85 students take CS220, and 45 students take both, how many take either CS280 or CS220.

A = students taking CS280
B = students taking CS220

$$|A \cup B| = |A| + |B| - |A \cap B| = 112 + 85 - 45 = 152$$

This is best seen using a Venn diagram:

What happens with three sets?

$$\begin{array}{l}|A \cup B \cup C| = \\ |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| \\ \qquad\qquad +|A \cap B \cap C|\end{array}$$

Example: If there are 300 engineering majors, 112 take CS280, 85 take CS 220, 95 take AEP 356, 45 take both CS280 and CS 220, 30 take both CS 280 and AEP 356, 25 take both CS 220 and AEP 356, and 5 take all 3, how many don't take any of these 3 courses?

A = students taking CS 280
B = students taking CS 220
C = students taking AEP 356

$$\begin{array}{rl} & |A \cup B \cup C| \\ = & |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| \\ & \qquad +|A \cap B \cap C| \\ = & 112 + 85 + 95 - 45 - 30 - 25 + 5 \\ = & 197 \end{array}$$

We are interested in $\overline{A \cup B \cup C} = 300 - 197 = 103$.

The General Rule

More generally,

$$|\cup_{k=1}^{n} A_k| = \sum_{k=1}^{n} \sum_{\{I | I \subset \{1,\ldots,n\}, |I|=k\}} (-1)^{k-1} |\cap_{i \in I} A_i|$$

Why is this true? Suppose $a \in \cup_{k=1}^{n} A_k$, and is in exactly m sets. a gets counted once on the LHS. How many times does it get counted on the RHS?

- a appears in m sets (1-way intersection)
- a appears in $C(m, 2)$ 2-way intersections
- a appears in $C(m, 3)$ 3-way intersections
- ...

Thus, on the RHS, a gets counted

$$\sum_{k=1}^{m} (-1)^{k-1} C(m, k) = 1 \text{ times.}$$

Why is $\sum_{k=1}^{m}(-1)^{k-1}C(m,k) = 1$?

- That certainly doesn't seem obvious!

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By the binomial theorem:

$$\begin{aligned} 0 = (-1+1)^m &= \sum_{k=0}^{m}(-1)^k 1^{m-k} C(m,k) \\ &= 1 + \sum_{k=1}^{m}(-1)^k C(m,k) \end{aligned}$$

Thus, $\sum_{k=1}^{m}(-1)^k C(m,k) = 11$, so

$$\sum_{k=1}^{m}(-1)^{k-1}C(m,k) = 1.$$

Sometimes math is amazing :-)

The Pigeonhole Principle

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This seems obvious. How can it be used in combinatorial anlysis?

Q1: If you have only blue socks and brown socks in your drawer, how many do you have to pull out before you're sure to have a matching pair.

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Q1: If you have only blue socks and brown socks in your drawer, how many do you have to pull out before you're sure to have a matching pair.

A: The socks are the pigeons and the holes are the colors. There are two holes. With three pigeons, there have to be at least two in one hole.

- What happens if we also have black socks?

A more surprising use of the pigeonhole principle

Q2: Alice and Bob play the following game: Bob gets to pick any 10 integers from 1 to 40. Alice has to find two different sets of three numbers that have the same sum. Prove that Alice always wins.

A: So what are the pigeons and what are the holes?

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The pigeons are the possible sets of three numbers. There are $C(10, 3) = 120$ of them.

The holes are the possible sums. The sum is at least 6, and at most $38 + 39 + 40 = 117$. So there are 112 hols.

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- There are more pigeons that holes!

Therefore, no matter which set of 10 numbers Bob picks, Alice can find two subsets of size three that have the same sum!

One more

Q3: Show that for every integer n there is a multiple of n that has only 0s and 1s in its decimal expansion:

For example.

- For 2, $2 \times 5 = 10$
- For 3, $3 \times 37 = 111$
- For 4, $4 \times 25 = 100$

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Hint: Given n, consider the numbers 1, 11, 111, up to $1 \ldots 1$ (with $n + 1$ 1s).
These numbers are the pigeons, the holes are the numbers mod n.

Two numbers must go in the same hole. Subtract the smaller from the larger. The difference has just 0s and 1s, and is 0 mod n!

- In fact, the difference has the form $1 \ldots 10 \ldots 0$.