CS280, Spring 2009: Final

1. [4 points] Recall that the Fibonacci sequence is defined inductively by taking $f_1 = 1$, $f_2 = 1$, $f_{n+1} = f_{n-1} + f_n$ for $n \geq 2$. Prove that $\gcd(f_n, f_{n+1}) = 1$. You can use any number theory results proved in class. [Hint: induction is relevant too.]

   Solution: Let $P(n)$ be the statement that $\gcd(f_n, f_{n+1}) = 1$. $P(1)$ says that $\gcd(f_1, f_2) = 1$. Since $f_1 = f_2 = 1$, this is obviously true. Suppose that $P(n)$ is true. Note that $f_{n+2} = f_n + f_{n+1}$, by definition. Thus, $\gcd(f_{n_1}, f_{n+2}) = \gcd(f_{n+1}, f_n + f_{n+1})$. We proved in class that $\gcd(a, b) = \gcd(a, b - a)$. Thus, $\gcd(f_{n+1}, f_n + f_{n+1}) = \gcd(f_{n+1}, f_n)$, which is 1, by the inductive hypothesis.

2. [3 points] Prove that if $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then $a - c \equiv b - d \pmod{m}$.

   Solution: If $a \equiv b \pmod{m}$, then there exists k such that $a - b = km$. Similarly, if $c \equiv d \pmod{m}$, there exists k' such that $c - d = k'm$. Thus, $(a - c) - (b - d) = (a - b) - (c - d) = (k - k')m$, so $a - c \equiv b - d \pmod{m}$.

3. [10 points] Twelve letters are to be placed into three (different) mailboxes.

   (a) In how many ways can this be done if the letters are different?

   (b) In how many ways can this be done if the letters are identical?

   (c) How many ways are possible if each mailbox must receive at least one letter and the letters are identical?

   Don't just write down the expression; explain how you got it. (It's OK to reduce it to something we've already done in class.) You don't have to simplify any combinatorial expressions that arise.

   Solution: This is a balls and urns problem. You can view the mailboxes as urns, and the letters are balls.

   (a) If the letters are different, for each letter, you have to choose a mailbox for it to go into. You have three choices for each letter. This means that you have 12^3 choices altogether. (If you call the mailboxes a, b, and c, you can identify a choice with a string over a, b, and c. (It was enough to say that there are 3^{12} ways to put 12 distinct balls into 3 distinct urns.)

   (b) If the letters are identical, you're putting 12 identical balls into 3 distinct urns. There are $C(14, 2) = 91$ ways of doing this.

   (c) Suppose that each mailbox must receive at least one letters and the letters are identical. The easy way to do this is to put one letter in each mailbox. Then you're free to to distribute the remaining letters any way you like. This becomes a balls and urns problem with 9 indistinguishable balls (the letters) and three distinguishable urns, so the answer is $C(11, 2) = 55$. Another (more complicated) way to do this is to take the answer in (b), and subtract the number of ways that a box can receive no letters. There are 13 ways for box a to receive no letters, and likewise for b and c. There is one way for both box a and b to receive no letters (put them all in box c), and likewise there is one way for both b and c and both a and c to receive no letters. Applying the inclusion-exclusion rule, there are $3 \times 13 - 3 = 36$ ways for there to be no letters in some box. Thus, there are $C(14, 2) - 36 = 55$ ways to put the letters in boxes so that each box has at least one letter.

Grading: In part (c), you received full credit if you computed 36 and realized you had to subtract that from your answer in part (b), even if your answer in part (b) was wrong. For some reason, I lot of people subtract 39 +3 instead of 39-3; you lost one point if you did this. If you made a mistake in (b) but your answer in (c) was consistent with it, you typically got full credit for (c), but it depended in part how far off you were in (b).

4. [5 points] Show that every simple undirected graph without loops has at least two vertices of the same degree. [Hint: If the graph has n vertices, what is the largest degree that it can have?]

   Solution: Suppose that a graph has n vertices. Each edge can have at most degree $n-1$. If there are no isolated edges (vertices with degree 0), we are done by the pigeonhole principle: the pigeons are the vertices, the holes are the possible degrees; a vertex goes into a hole labeled k if it has degree k. Since there are are n vertices and $n-1$ holes, two vertices must have the same degree. Now suppose that there are some isolated verticies. If there are two isolated vertices, we are done (they have the same degree). If there is only one, then the remaining $n-1$ vertices each has degree between 1 and $n-2$. Again, you can apply the pigeonhole principle to them to show that there are two vertices with the same degree.

   Grading: You lost one point if you didn't take into account the possibility of isolated vertices.

5. [6 points] A space probe near Neptune communicates with Earth using bit strings. Suppose that in its transmissions it sends a 1 one-quarter of the time and a 0 three-quarters of the time. When a 0 is sent, the probability that it is received correctly is 0.88 and the probability that it is received incorrectly (as a 1) is 0.12. When a 1 is sent, the probability that it is received correctly is 0.8 and the probability that it is received incorrectly (as a 0) is 0.2. This information is summarized in the following table, where the ij entry gives the probability that i is received, conditional on j being sent:

	0 Received	1 Received
0 Sent	0.88	0.12
1 Sent	0.2	0.8

   (a) [3 points] Compute the probability that 0 is received.

   (b) [3 points] Find the probability that 0 was transmitted, given that 0 was received.

   Solution: (a) A 0 is received either if a 0 is sent and received correctly, or if 1 is sent and it is garbled. That is

$$\begin{aligned}
\Pr(0\ \mathit{received}) &= \Pr(0\ \mathit{received} \wedge 0\ \mathit{sent}) + \Pr(0\ \mathit{received} \wedge 1\ \mathit{sent}) \\
&= \Pr(0\ \mathit{received} \mid 0\ \mathit{sent}) \Pr(0\ \mathit{sent}) + \\
&\quad\ \Pr(0\ \mathit{received} \mid 1\ \mathit{sent}) \Pr(1\ \mathit{sent}) \\
&= .88(.75) + .2(.25) = .66 + .05 = .71
\end{aligned}$$

(b) We want $\Pr(0\ \mathit{sent} \mid 0\ \mathit{received})$. By Bayes' Rule, this is

$$\frac{\Pr(0\ \mathit{received} \mid 0\ \mathit{sent}) \Pr(0\ \mathit{sent})}{\Pr(0 \mathit{received} \mid 0\ \mathit{sent}) \Pr(0\ \mathit{sent}) + \Pr(0\ \mathit{received} \mid 1\ \mathit{sent}) \Pr(1\ \mathit{sent})}.$$

The numerator here is .66 and the denominator was computed as .71 in part (a). So the answer is 66/71.

6. [4 points] Prove, using the properties of probability, that if A and B are independent, then so are A and $\overline{B}$.

Solution: If A and B are independent, the $\Pr(A \cap B) = \Pr(A)\Pr(B)$. Since $A = (A \cap B) \cup (A \cap \overline{B})$, and $A \cap B$ and $A \cap \overline{B}$ are disjoint sets, it follows that $\Pr(A) = \Pr(A \cap B) + \Pr(A \cap \overline{B})$. Thus,

$$\Pr(A \cap \overline{B}) = \Pr(A) - \Pr(A \cap B) = \Pr(A) - \Pr(A)\Pr(B) = \Pr(A)(1 - \Pr(B)) = \Pr(A)\Pr(\overline{B}).$$

It follows that A and $\overline{B}$ are independent.

7. [5 points] Suppose that three fair dice are rolled.

   (a) [1 point] Carefully describe the sample space.

   (b) [1 point] Describe the random variable X that is the sum of the numbers that appear. [To get credit for this problem, you must demonstrate that you understand the definition of a random variable.]

   (c) [3 points] What is the expected sum of the numbers that appear? [I am looking for an exact expression here, completely simplified. There's an easy way to do this, which requires minimal calculation.]

Solution:

(a) The sample space consists of the 216 elements of the form (i, j, k) such that $1 \leq i, j, k \leq 6$.

(b) A random variable is a function from the sample space to a real number. In this case, $X(i, j, k) = i + j + k)$.

(c) Let X_n be the number on the nth die, for $n = 1, 2, 3$. Then $X = X_1 + X_2 + X_3$. It is easy to see that $E(X_1) = E(X_2) = E(X_3) = 3.5$ (There are six possible numbers, $1, \ldots, 6$, and they each have probability $1/6$, so the expected value is $1/6(1 + \cdots + 6) = 21/6 = 3.5$.) Since expectation is linear, $E(X_1 + X_2 + X_3) = E(X_1) + E(X_2) + E(X_3) = 10.5$.

Grading: In part (b), you had to make it clear that you understood that a random variable was a function. Writing some form of functional notation was enough to do that. Just writing $i+j+k$ without the functional notation lost .5. In part (c), we were mainly looking for an understanding that expectation was linear.

8. [6 points] Suppose that everyone in a class has a webpage, and on their webpages, they can link to other people's webpages. Let $L(x,y)$ be the statement "x links to y's website". Express the following statements using first-order logic formulas:

   (a) Everybody links to either David's or Sara's website.

   (b) Everyone links to someone's website.

   (c) David doesn't link to anyone's website.

   (d) No one links to his or her own website.

   (e) The "links-to" relation is symmetric.

   (f) The "links-to" relation is transitive.

   Solution:

   (a) $\forall x(L(x, David) \vee L(x, Sara))$.

   (b) $\forall x \exists y L(x,y)$.

   (c) $\forall x \neg L(David, x)$.

   (d) $\forall x \neg L(x,x)$.

   (e) $\forall x,y(L(x,y) \Leftrightarrow L(y,x))$.

   (f) $\forall x,y,z((L(x,y) \wedge L(y,z)) \Rightarrow L(x,z))$.

   Grading: Each part was worth 1 point. If you made a minor error, usually switching the order or mixing up $\forall$ and $\exists$, you lost .5; any other mistakes cost you the full point. Most people did fairly well on this question. The most likely errors were syntax errors (putting nots/ors etc. as arguments in the $L(x,y)$; it's not OK, for example, to write $L(\forall x, y)$), using $\exists$ instead of $\forall$ on y in (e) and y and z in (f), and mixing up the reflexive and symmetric properties in part (e).

9. [4 points] Are the formulas $p \Rightarrow (q \Rightarrow r)$ and $(p \Rightarrow q) \Rightarrow r$ equivalent? (If you think they are, then show that their truth tables are identical. If you think they are not, give a truth assignment that shows they are different.)

   Solution: These formulas are not equivalent. This is probably easiest to see if we recall that $A \Rightarrow B$ is equivalent to $\neg A \vee B$. Thus, $p \Rightarrow (q \Rightarrow r)$ is equivalent to $\neg p \vee \neg q \vee r$, and $(p \Rightarrow q) \Rightarrow r$ is equivalent to $\neg(p \Rightarrow q) \vee r$, which is equivalent to $\neg(\neg p \vee q) \vee r$, which is equivalent to $(p \wedge \neg q) \vee r$. Consider the truth assignment that makes p false, q true (or false; it doesn't matter), and r false. Under this truth assignment, $p \Rightarrow (q \Rightarrow r)$ is true, but $(p \Rightarrow q) \Rightarrow r$ is false. (You could also consider the truth assignment that makes p true, q false, and r false; it gives the same result.)

   Grading: If you said "no" and gave a correct counter example (FTF or FFF), you got full credit. If you said "no" but gave multiple counter examples, one of them being wrong, you got 3.5, if you said "no" and gave a bad counter example you got a 3, if you said "yes" but you wrote out the full truth table and your table was wrong but consistent, you got a 2.5. Most people did one of these things. A few people said "yes" and did not give sensible reasonings and thus got a 0. The biggest problem (a lot of people did this) was incorrectly defining the $\Rightarrow$ truth table (saying $F \Rightarrow T$ and/or $F \Rightarrow F$ was false).

10. [2 points] Give an example of a domain where the formula $\exists x(x+2=0)$ is true, and give an example of a domain where it is false.

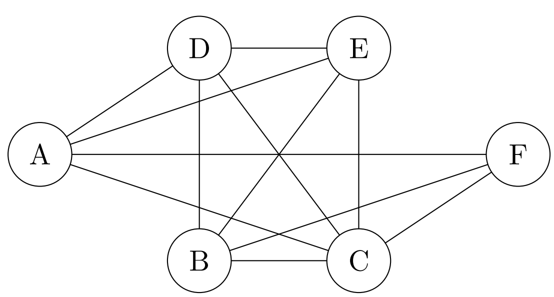
    Solution: The formula is true if the domain is the integers, and false if the domain is the natural numbers. There were lots of possible answers here, but for full credit, the domain had to include 0 and 2 (so that you could interpret these symbols in the formula. You lost .5 if your domain didn't include 0 and 2. (Note that, in general, a domain must be non-empty.

11. [3 points] Is it possible for an insect to crawl along the edges of a cube so as to travel along each edge exactly once? Explain why or why not.

    Solution: If this could be done, there would be an Eulerian path in the cube. But each of the six vertices in the cube has degree three (which is odd), so it can't be done. You got one point for the right answer, and the other two for the right reason (which typically had to involve Eulerian paths).

12. [4 points] Exams need to be scheduled for the following six courses: Anthropology 101 (A), Biology 110(B), Chemistry 112 (C), Development Sociology 211 (D), Economics 113 (E), and Food Science 159 (F). There are no students taking both A and B, no students taking both D and F, and no students taking both E and F, but there are students taking every other combination of courses. We want to schedule the exams so that no student has a conflict. What is the fewest number of slots needed to schedule the exam? You must answer this question by converting the problem to a graph coloring problem. Explain the conversion and then give the answer based on the how you converted the problem. (You get 0 points for just giving the answer without converting it to a graph coloring problem.)

    Solution: Consider the graph with six vertices, labeled A, B, C, D, E, and F. Put an edge between two vertices if there are students taking both courses. Thus, there are edges between every pair of nodes except for the three pairs of courses with no students in common:

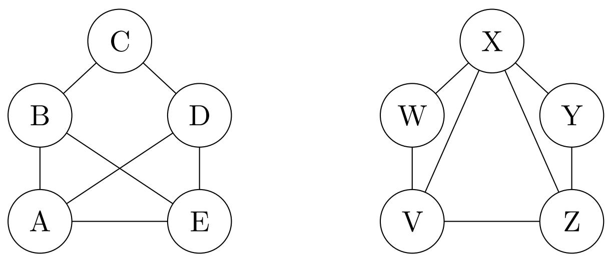


    This graph can be colored using four colors. You need at least four colors, because the subgraph consisting of the vertices labeled B, C, D, and E is complete, so these vertices must all get different colors. You can then give A the same color as B, and guve E and

F the same color. Since you can color the graph with four colors, you'll need four exam slots.

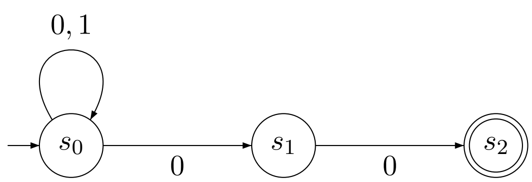
Grading: The four points were split up as follows: 2 points for the correct graph and explanation (definition of nodes, edges), 1 point for showing how coloring was relevant, 1 point for the correct answer +1 mentioning coloring or similar words (such as chromatic numbers). Minor mistakes typically cost .5-1 point.

13. [3 points] Are the following two graphs isomorphic? If you think they are, give the isomorphism that demonstrates this. If not, explain why not.



Solution: The two graphs are not isomorphic. In the second graph, the vertex X has degree 4; there is no vertex in the first graph that has degree 4, so the graphs cannot be isomorphic.

14. [6 points] Here is the graph of a nondeterministic finite automaton.



(a) Explain clearly why it is nondeterministic, and not deterministic.

(b) Give a formal description of the automaton as a tuple.

(c) What language is accepted by the automaton?

Solution:

(a) It is nondeterministic because there are two edges labeled 0 going out of s_0, no edges labeled 1 going out of s_1 or s_2, and no edge labeled 0 going out of s_2. (It was enough to say any one of these things to get full credit.)

(b) The automaton is $M = (S, I, f, s_0, F)$, where $S = \{s_0, s_1, s_2\}$, $F = \{s_2\}$, $I = \{0, 1\}$, and f is given as follows:

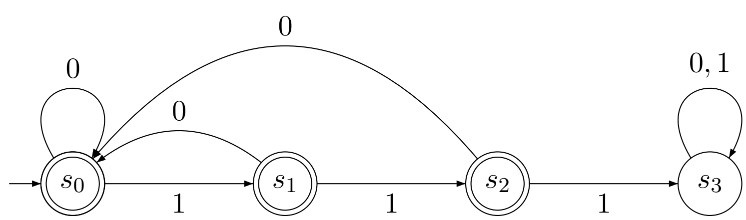
- $f(s_0, 0) = \{s_0, s_1\}$
- $f(s_0, 1) = \{s_0\}$
- $f(s_1, 0) = \{s_2\}$
- $f(s_1, 1) = \emptyset$.
- $f(s_2, 0) = f(s_2, 1) = \emptyset$.

(c) The automaton accepts the language Σ^*00, where $\Sigma = \{0,1\}$ (it's also OK to write $(0 \cup 1)^*00$).

Grading: Each part was worth 2 points. Most students got full credit for (a). For full credit in part (b), you to give the f component of the tuple, and at least three of the four remaining components (S, I, s_0, or F). You did not lose points for any small "typo errors" or for having the elements of the tuple in a different order. If you missed two of S, I, s_0, or F, you lost a half of a point. If you missed all of them, you lost a point. You also had to fully specify the transition function f. No points were taken off for not specifying transitions that went to the empty set. If you simply wrote "f" without specifying the transition function, you lost 1 point. If you wrote out the "general case" of a tuple description without putting down any specifics for this particular case, you got 1/2. In part (c), one common mistakes were saying that the language accepted any string with two consecutive zeros in it", for which you lost one point. Another common mistake was defining the language as 0^*1^*00 (or other variants) instead. This received 0 points. Some of you correctly described the language in words, but incorrectly wrote the language description. This received 1 point. If you wrote $(1 \cap 0)^*00$ instead of $(1 \cup 0)^*00$ lost 1/2 point.

15. [3 points] Give (the graph of) an automaton that accepts the language consisting of strings in $\{0,1\}^*$ that do *not* have three consecutive 1's. That is, accept a string unless 111 is a substring of it. Explain the intutive meaning of each of the states in your automaton.

Solution: Here is the automaton:



Intuitively, in s_0 you're starting out or you've just seen a 0; in s_1 you've just seen a 1; in s_2 you've just seen two consecutive 1's; in s_3 you have seen 111.

Grading: Generally, you lost points based upon how difficult it would be to turn your automaton into a correct solutions to the question. Common errors: If the automaton excluded the empty string, you lost 0.5 points. If the automaton did not allow a set of strings that should be accepted (eg, 10011011011..), then you lost between 0.5 and 1 points. If the automaton did not allow "longer strings" (in other words, the set of strings that should be accepted using loops in the DFA), then you lost between 0.5 and 1 point. If the automaton did allow some strings of the form $(0^*1^*)^*111(0^*1^*)^*$ to be accepted, then you lost between 1 and 1.5 points.

16. [4 points] Prove carefully that if A and B are languages such that $A \subseteq B$, then $A^* \subseteq B^*$. (Hint: induction is useful here too.)

Solution: We first prove by induction on n that $A^n \subseteq B^n$. Since $A^0 = B^0 = \{\lambda\}$, the base case is immediate. Suppose that $A^n \subseteq B^n$. We want to show that $A^{n+1} \subseteq B^{n+1}$.

By definition, $x \in A^{n+1}$ if there exist $y \in A^n$ and $z \in A$ such that $x = yz$. Since $A^n \subseteq B^n$ by the induction hypothesis, it follows that $y \in B^n$. Since $A \subseteq B$ by assumption, it follows that $z \in B$. Hence $x = yz \in B^{n+1}$. This completes the induction proof. Thus, $A^* = \cup_{n=0}^{\infty} A^n \subseteq \cup_{n=0}^{\infty} B^n = B^*$.

Grading: You lost .5–1 point if you didn't deal correctly with the base case (which involved realizing that the base case was 0, not 1, and $A^0 = \{\lambda\}$. (Note that $\{\lambda\}$ is not the same as the empty set! It's the set which contains one element, which is the empty string λ.) You also lost .5 if you did the induction, and then just concluded that $A^* \subseteq B^*$ without mentioning that $A^* = \cup_{n=0}^{\infty} A^n$, and similarly for B^*.

17. [3 points] Prove that $\{0^n 1 0^n : n = 0, 1, 2, 3, \ldots\}$ is not a regular language. (You can use any result proved in class.)

    Solution: Suppose that this language is accepted by some deterministic finite automaton with N states. Consider the string $x = 0^n 1 0^n$. Since x is in the language and $|x| \geq N$, by the Pumping Lemma, there exist strings u, v, and w such that $x = uvw$, $|v| \geq 1$, $|uv| \leq N$, and M accepts $uv^i w$ for all $i > 0$. Since $|uv| \leq N$, it must be the case that uv is a string of 0's, and that w contains the 1 in $0^N 1 0^N$. Thus, if $i > 1$, $uv^i w$ has more than N 0s to the left of the 1 and only N 0s to the right of the 1, and thus is not in the language. This contradicts the assumption that the language is accepted by M (since M accepts a string not in the language).

    Grading: Just mentioning the pumping lemma in some vaguely correct way typically got you 1 point. For full credit you had to argue that no matter how v was chosen, $uv^i w$ should not be accepted by M. This was easy if you started with the string $0^N 1 0^N$ as I did and recalled that $|uv| \leq N$, but if you started with an arbitrary string of length at least N, then you had to work a bit harder (typically, you had to consider three cases: v is completely to the left of the 1, v includes the 1, and v is completely to the right of the 1.