CS280, Spring 2009: Final

This exam is out of 75; the amount for each question is as marked. Clearly explain your reasoning. (Remember, we can't read your mind! Besides, it gives us a chance to give you partial credit.) Don't forget to put your name, student number, and net id on each blue book you use. You don't have to do the questions in order; just make sure you mark the question number clearly. No calculators are allowed (and they shouldn't be needed).

Exams should be graded by tomorrow, and will be available for you to pick from Randy Hess, in Upson 5147, starting Monday, if you're still around. Grades should be posted by tomorrow too (or shortly thereafter).

Good luck!

1. [4 points] Recall that the Fibonacci sequence is defined inductively by taking $f_1 = 1$, $f_2 = 1$, $f_{n+1} = f_{n-1} + f_n$ for $n \geq 2$. Prove that $\gcd(f_n, f_{n+1}) = 1$. You can use any number theory results proved in class. [Hint: induction is relevant too.]

2. [3 points] Prove that if $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then $a - c \equiv b - d \pmod{m}$.

3. [10 points] Twelve letters are to be placed into three (different) mailboxes.

   (a) In how many ways can this be done if the letters are different?

   (b) In how many ways can this be done if the letters are identical?

   (c) How many ways are possible if each mailbox must receive at least one letter and the letters are identical?

   Don't just write down the expression; explain how you got it. (It's OK to reduce it to something we've already done in class.) You don't have to simplify any combinatorial expressions that arise.

4. [5 points] Show that every simple undirected graph without loops has at least two vertices of the same degree. [Hint: If the graph has n vertices, what is the largest degree that it can have?]

5. [6 points] A space probe near Neptune communicates with Earth using bit strings. Suppose that in its transmissions it sends a 1 one-quarter of the time and a 0 three-quarters of the time. When a 0 is sent, the probability that it is received correctly is 0.88 and the probability that it is received incorrectly (as a 1) is 0.12. When a 1 is sent, the probability that it is received correctly is 0.8 and the probability that it is received incorrectly (as a 0) is 0.2. This information is summarized in the following table, where the ij entry gives the probability that i is received, conditional on j being sent:

	0 Received	1 Received
0 Sent	0.88	0.12
1 Sent	0.2	0.8

(a) [3 points] Compute the probability that 0 is received.

(b) [3 points] Find the probability that 0 was transmitted, given that 0 was received.

6. [4 points] Prove, using the properties of probability, that if A and B are independent, then so are A and $\overline{B}$.

7. [5 points] Suppose that three fair dice are rolled.

   (a) [1 point] Carefully describe the sample space.

   (b) [1 point] Describe the random variable X that is the sum of the numbers that appear. [To get credit for this problem, you must demonstrate that you understand the definition of a random variable.]

   (c) [3 points] What is the expected sum of the numbers that appear? [I am looking for an exact expression here, completely simplified. There's an easy way to do this, which requires minimal calculation.]

8. [6 points] Suppose that everyone in a class has a webpage, and on their webpages, they can link to other people's webpages. Let $L(x, y)$ be the statement "x links to y's website". Express the following statements using first-order logic formulas:

   (a) Everybody links to either David's or Sara's website.

   (b) Everyone links to someone's website.

   (c) David doesn't link to anyone's website.

   (d) No one links to his or her own website.

   (e) The "links-to" relation is symmetric.

   (f) The "links-to" relation is transitive.

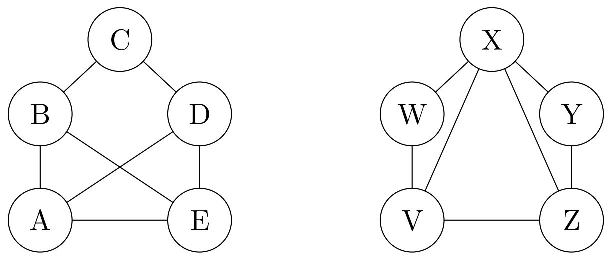
9. [4 points] Are the formulas $p \Rightarrow (q \Rightarrow r)$ and $(p \Rightarrow q) \Rightarrow r$ equivalent? (If you think they are, then show that their truth tables are identical. If you think they are not, give a truth assignment that shows they are different.)

10. [2 points] Give an example of a domain where the formula $\exists x(x + 2 = 0)$ is true, and give an example of a domain where it is false.

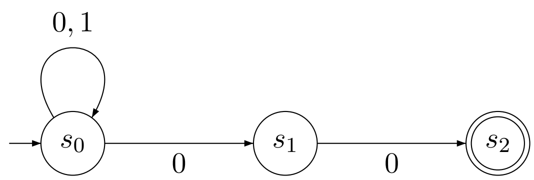
11. [3 points] Is it possible for an insect to crawl along the edges of a cube so as to travel along each edge exactly once? Explain why or why not.

12. [4 points] Exams need to be scheduled for the following six courses: Anthropology 101 (A), Biology 110(B), Chemistry 112 (C), Development Sociology 211 (D), Economics 113 (E), and Food Science 159 (F). There are no students taking both A and B, no students taking both D and F, and no students taking both E and F, but there are students taking every other combination of courses. We want to schedule the exams so that no student has a conflict. What is the fewest number of slots needed to schedule the exam? You must answer this question by converting the problem to a graph coloring problem. Explain the conversion and then give the answer based on the how you converted the problem. (You get 0 points for just giving the answer without converting it to a graph coloring problem.)

13. [3 points] Are the following two graphs isomorphic? If you think they are, give the isomorphism that demonstrates this. If not, explain why not.



14. [6 points] Here is the graph of a nondeterministic finite automaton.



   (a) Explain clearly why it is nondeterministic, and not deterministic.

   (b) Give a formal description of the automaton as a tuple.

   (c) What language is accepted by the automaton?

15. [3 points] Give (the graph of) an automaton that accepts the language consisting of strings in $\{0, 1\}^*$ that do *not* have three consecutive 1's. That is, accept a string unless 111 is a substring of it. Explain the intutive meaning of each of the states in your automaton.

16. [4 points] Prove carefully that if A and B are languages such that $A \subseteq B$, then $A^* \subseteq B^*$. (Hint: induction is useful here too.)

17. [3 points] Prove that $\{0^n 1 0^n : n = 0, 1, 2, 3, \ldots\}$ is not a regular language. (You can use any result proved in class.)