Nondeterministic Finite Automata and Regular Expressions

CS 2800: Discrete Structures, Spring 2015

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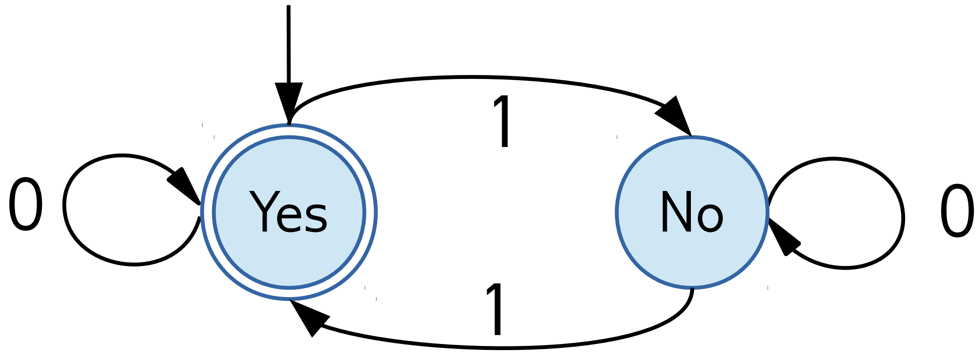
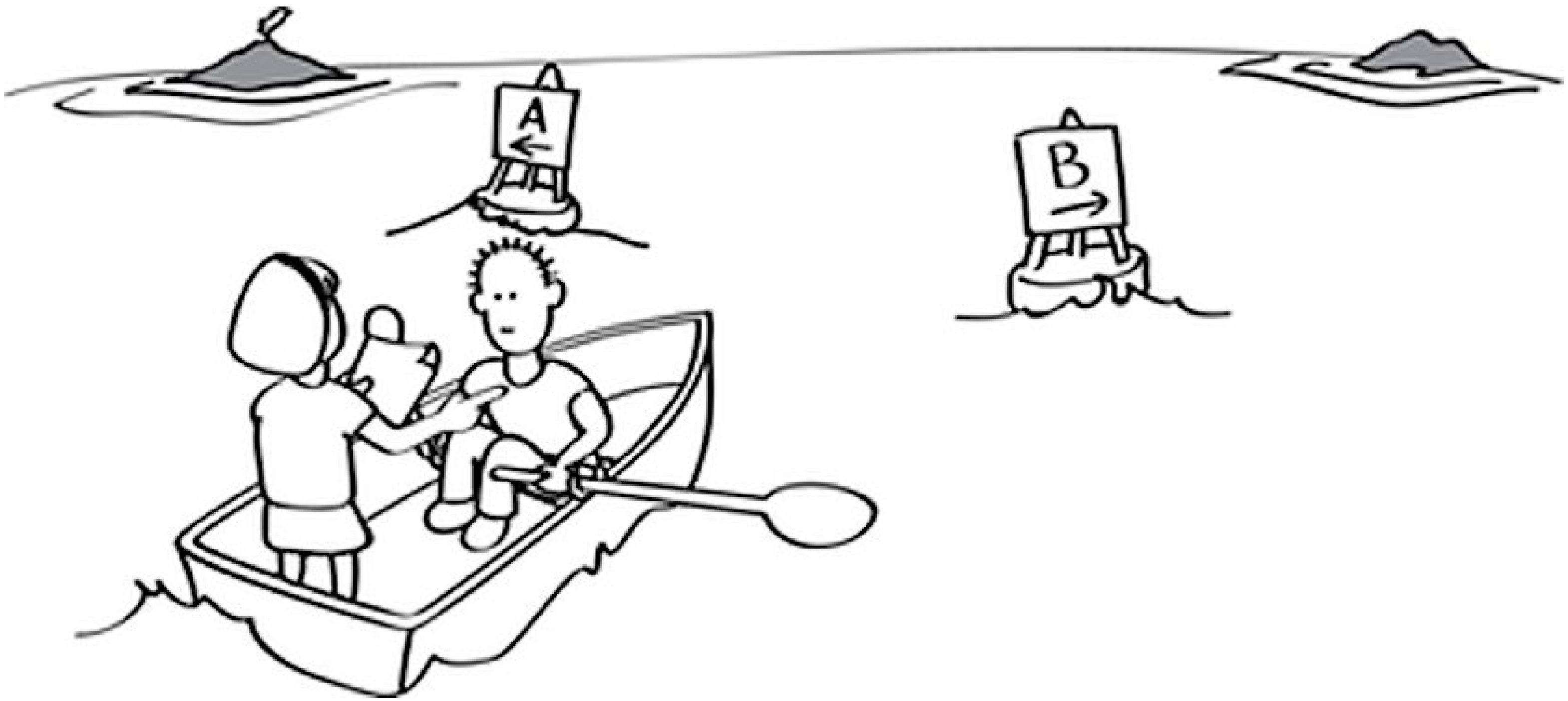
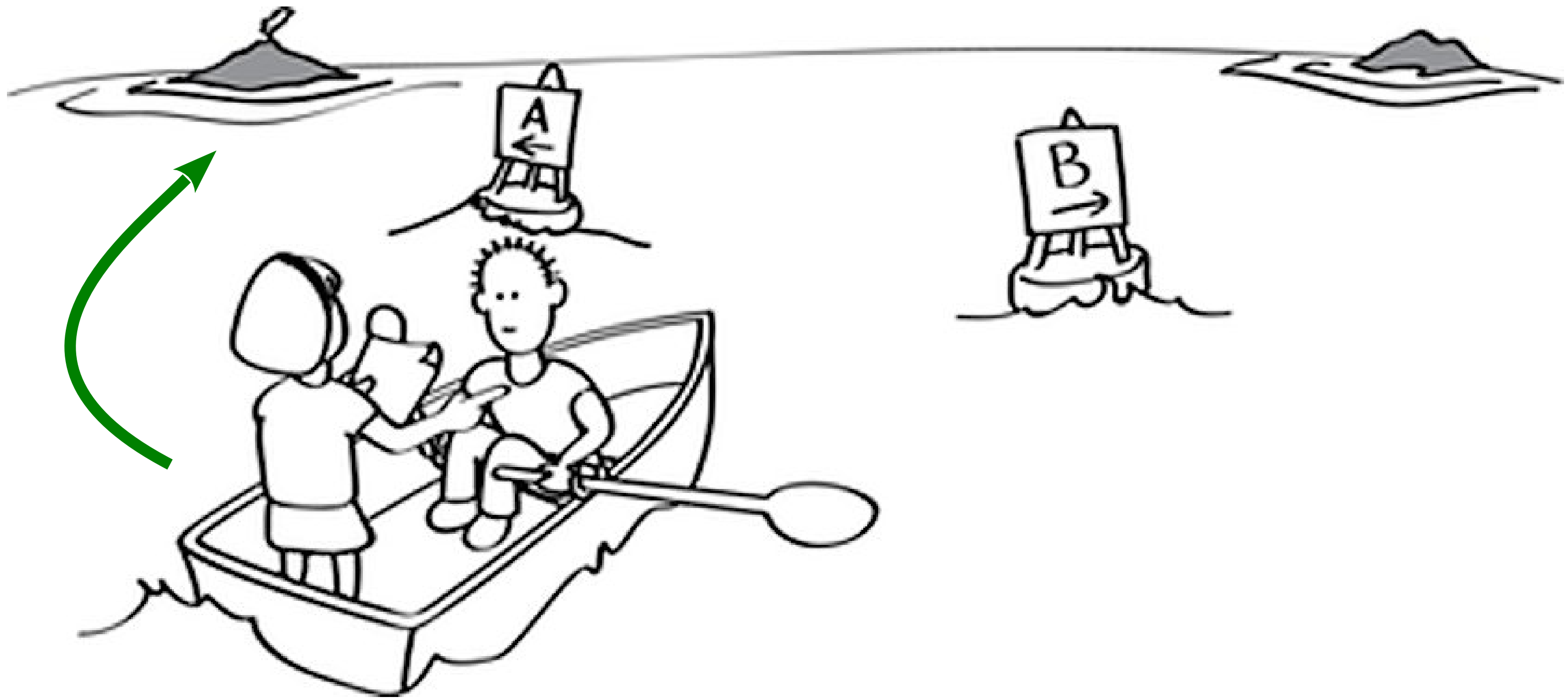
Recap: Deterministic Finite Automaton

- A DFA is a 5-tuple $M = (Q, \Sigma, \delta, q_0, F)$
  - Q is a finite set of states
  - Σ is a finite input alphabet (e.g. $\{0, 1\}$)
  - δ is a transition function $\delta : Q \times \Sigma \to Q$
  - $q_0 \in Q$ is the start/initial state
  - $F \subseteq Q$ is the set of final/accepting states




A
B

P = NP!!!



csunplugged.com, clubpenguin.wikia.com

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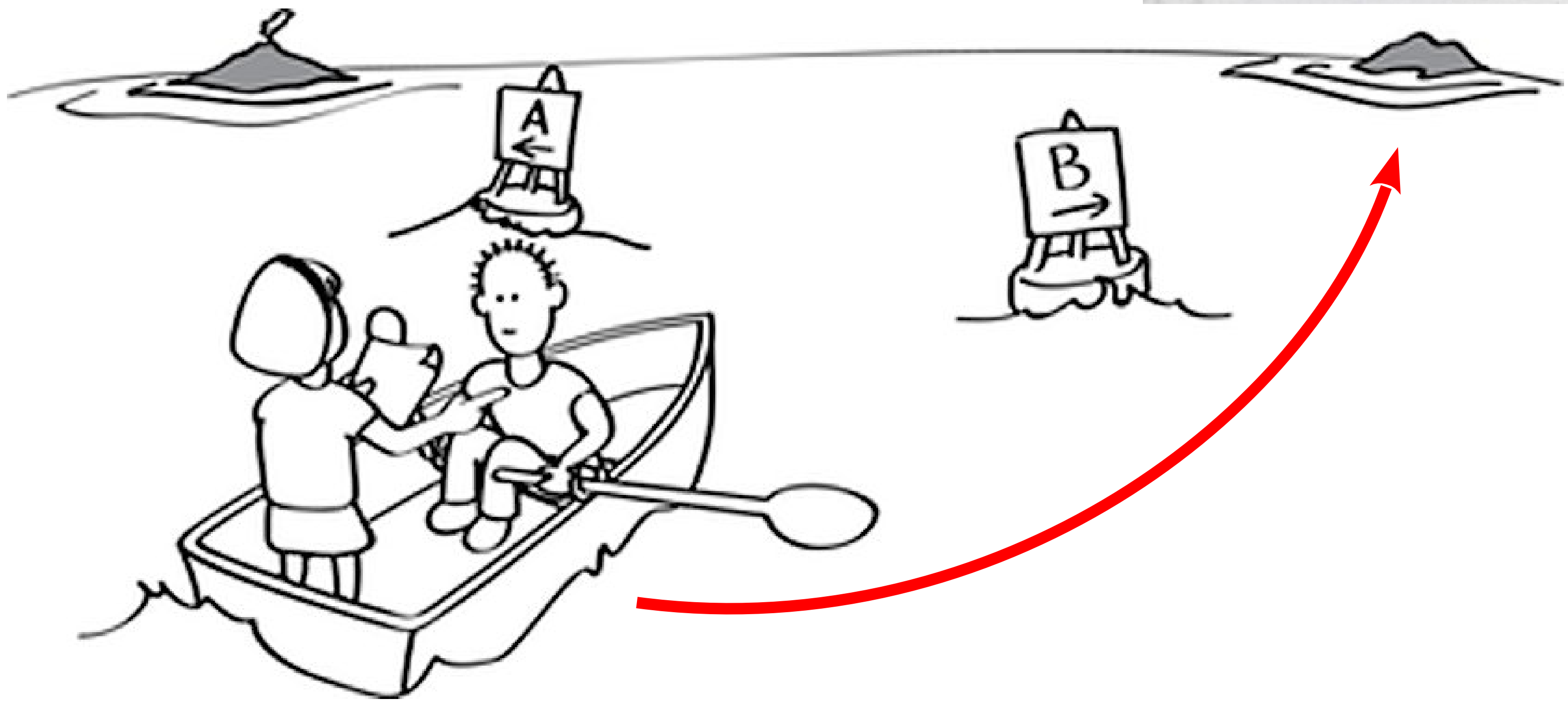

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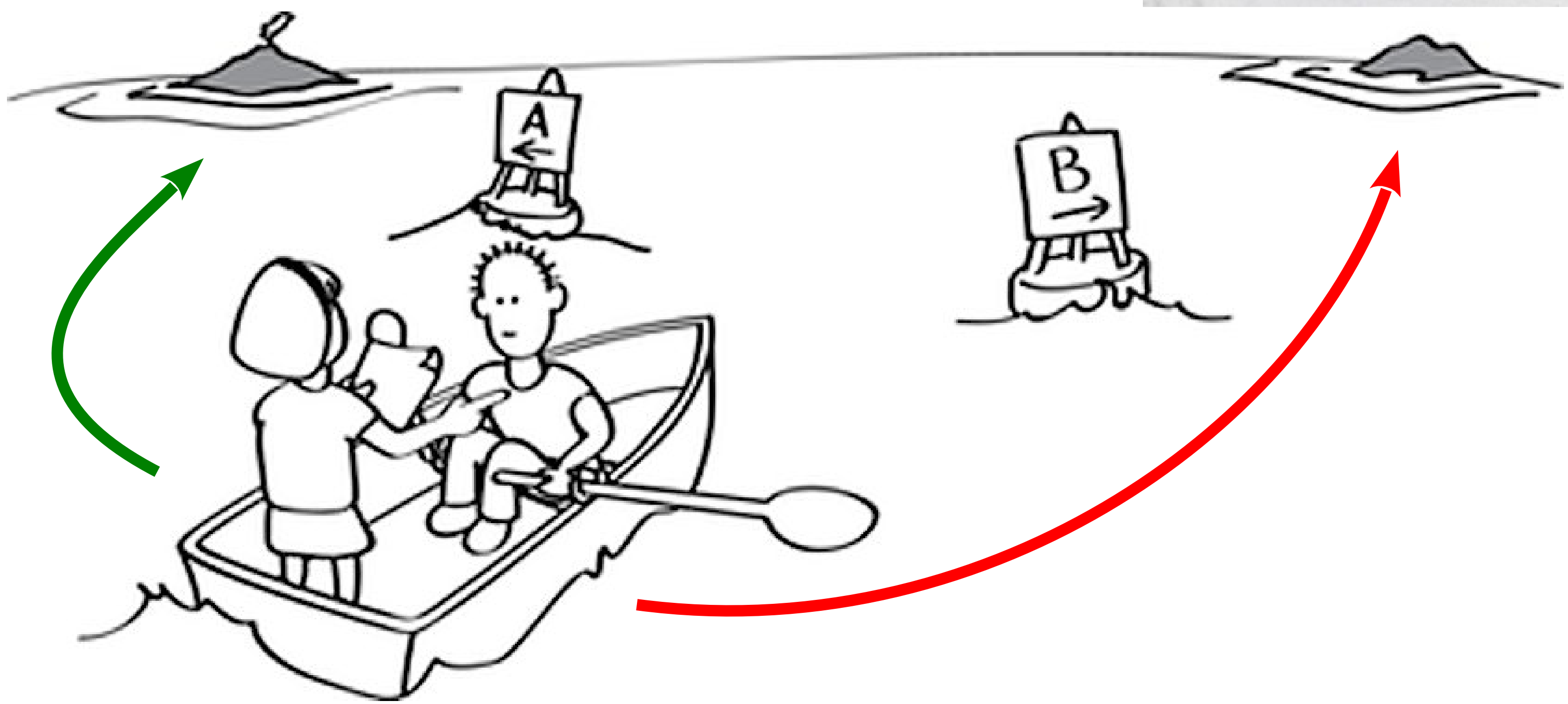
A NON-deterministic finite automaton lets you try all possible choices in parallel. If *ANY* choice leads you to the treasure, the pirate can't harm you!

P = NP!!!



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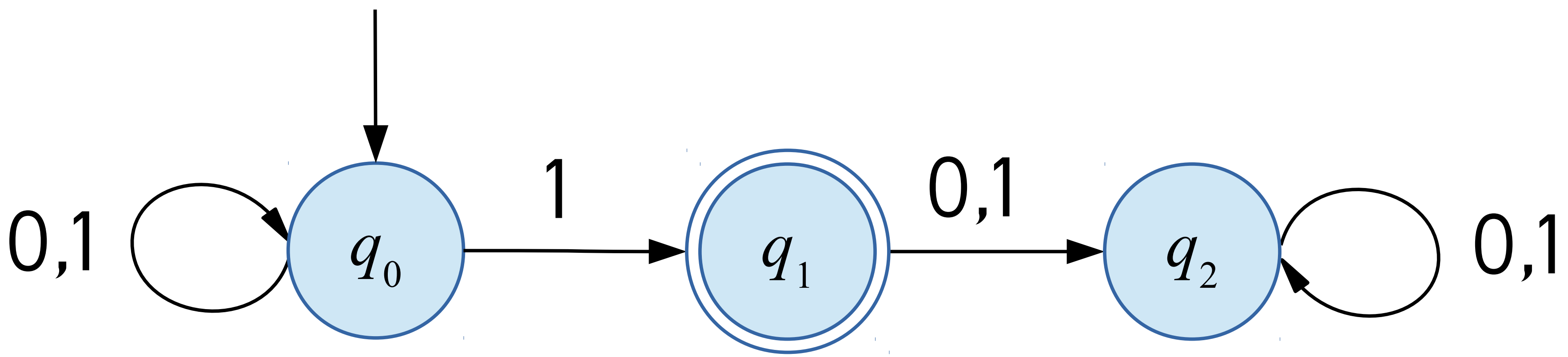
A ***non***-deterministic finite automaton



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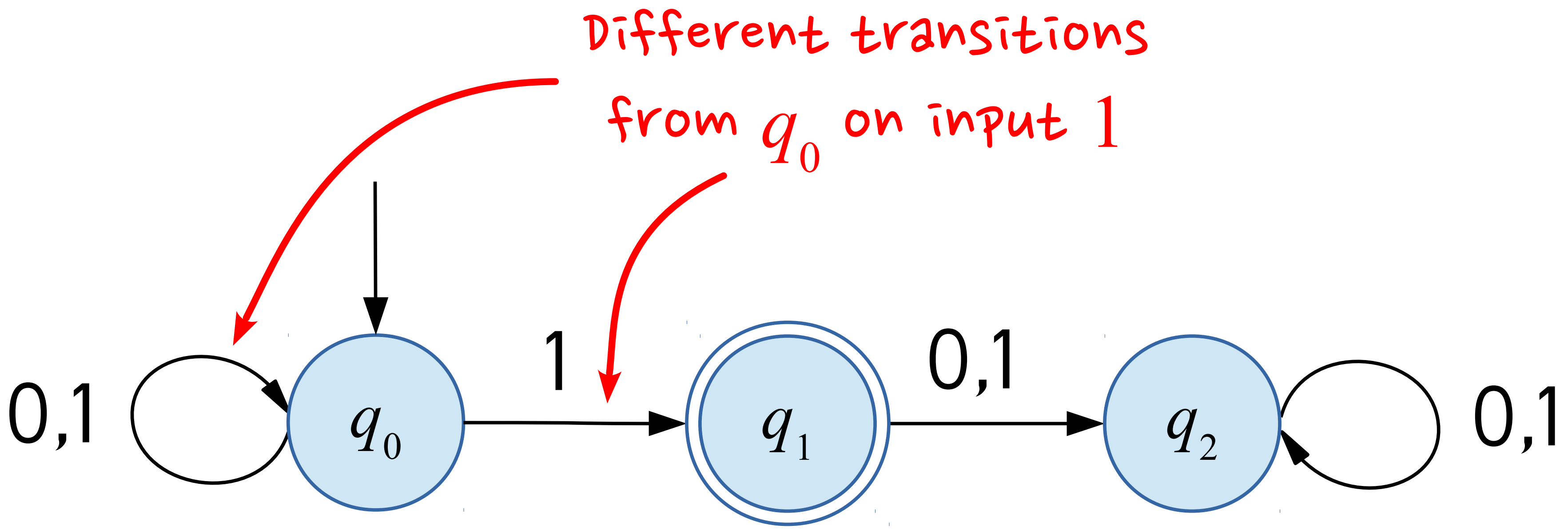

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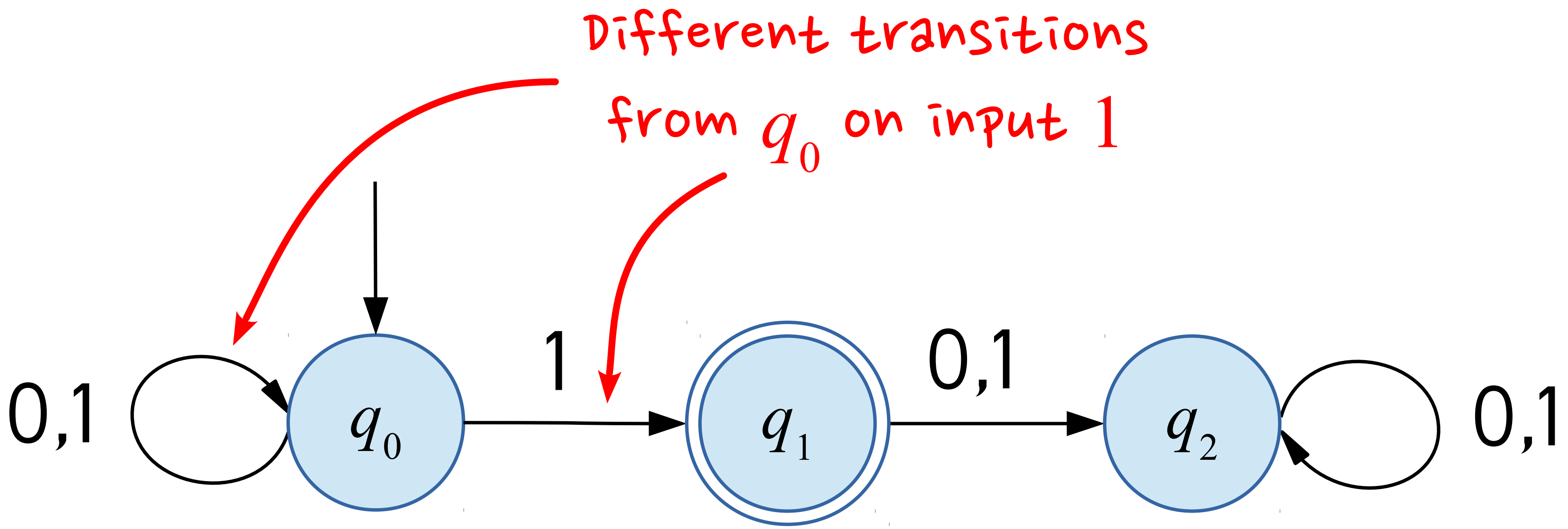

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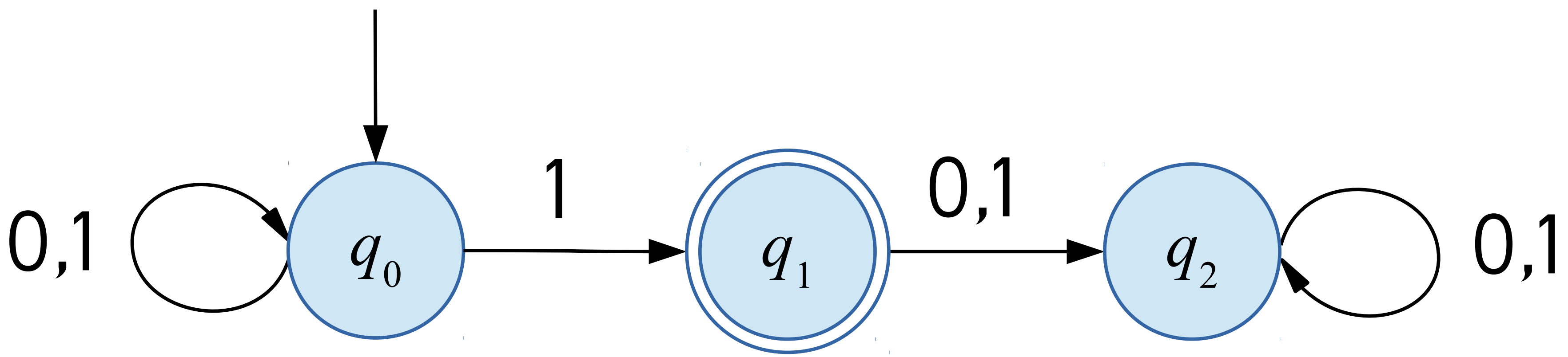

What language does this automaton accept?

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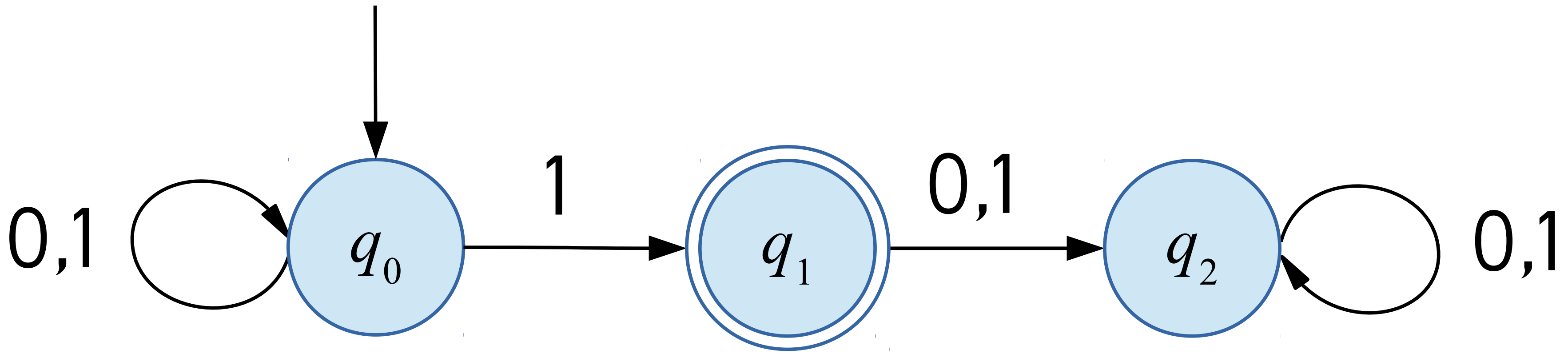
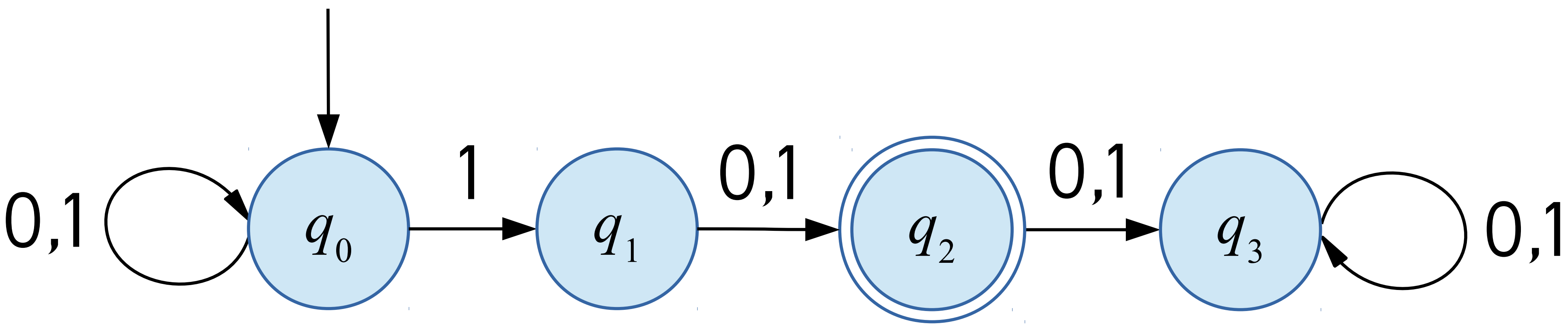


Answer: All strings ending with 1

Another NFA



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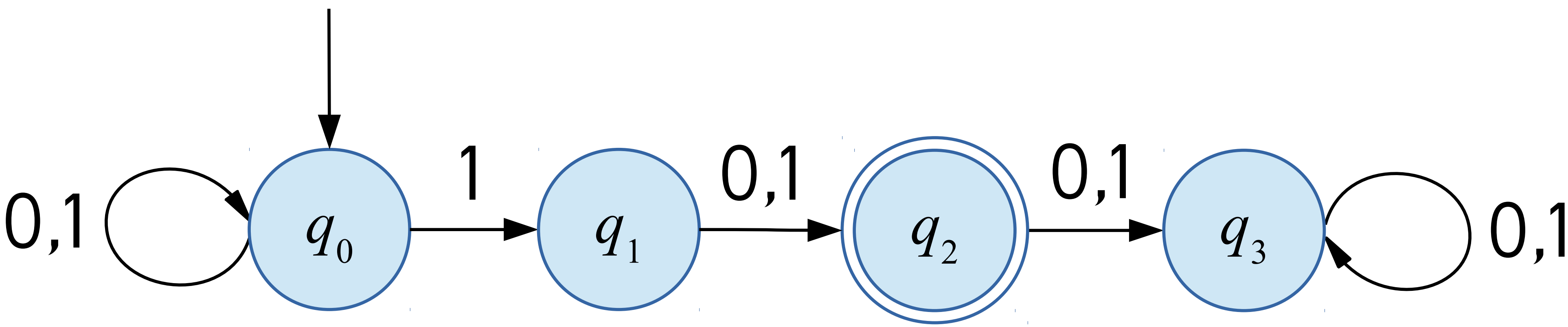

Answer: All strings with 1 in the penultimate place

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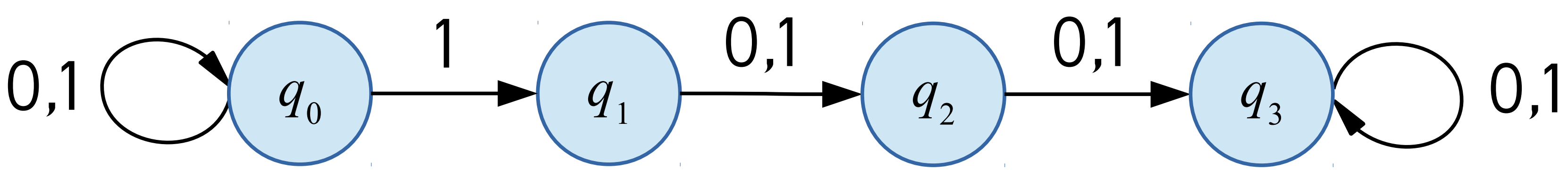

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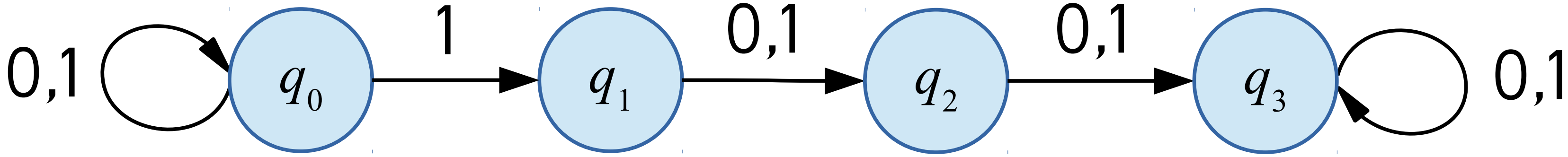

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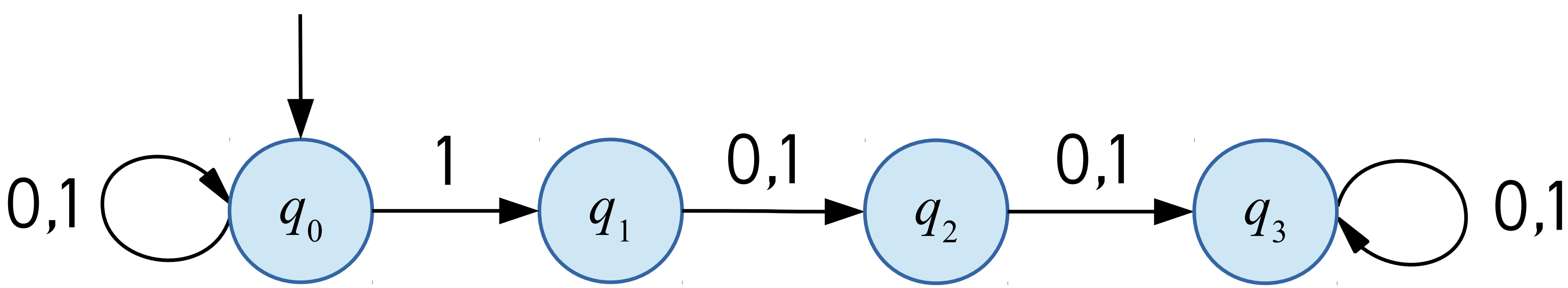

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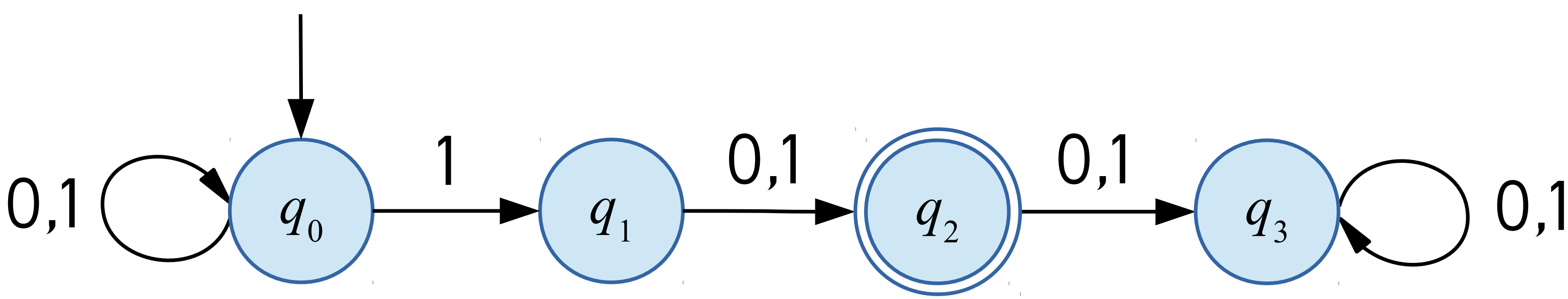

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  - A convenient shortcut for our “hell/black-hole” state
  - <u>Class convention</u>: Draw all possible transitions for DFA. Not required for NFA (missing transitions lead to hell).

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- Every NFA can be simulated by a DFA
  - ... i.e. they accept exactly the same language
  - Exponential blowup: if the NFA has n states, the DFA can require up to 2^n states

Thought for the Day #1

Find an NFA with n states that can't be simulated by a DFA with less than 2^n states

Every NFA can be simulated by a DFA

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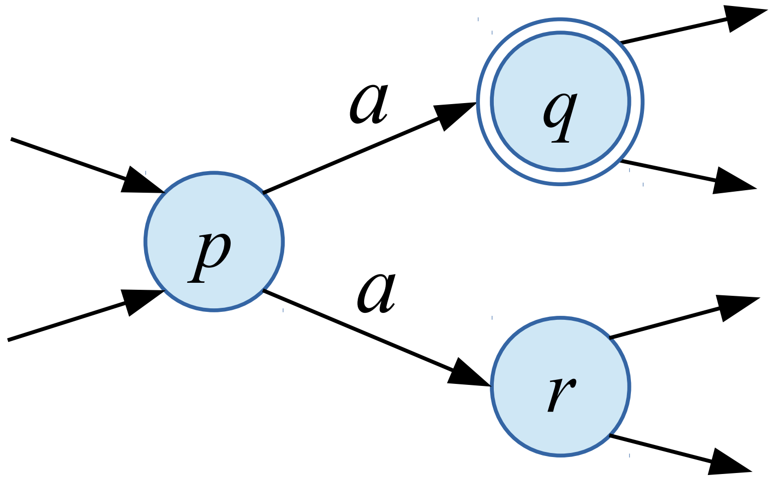
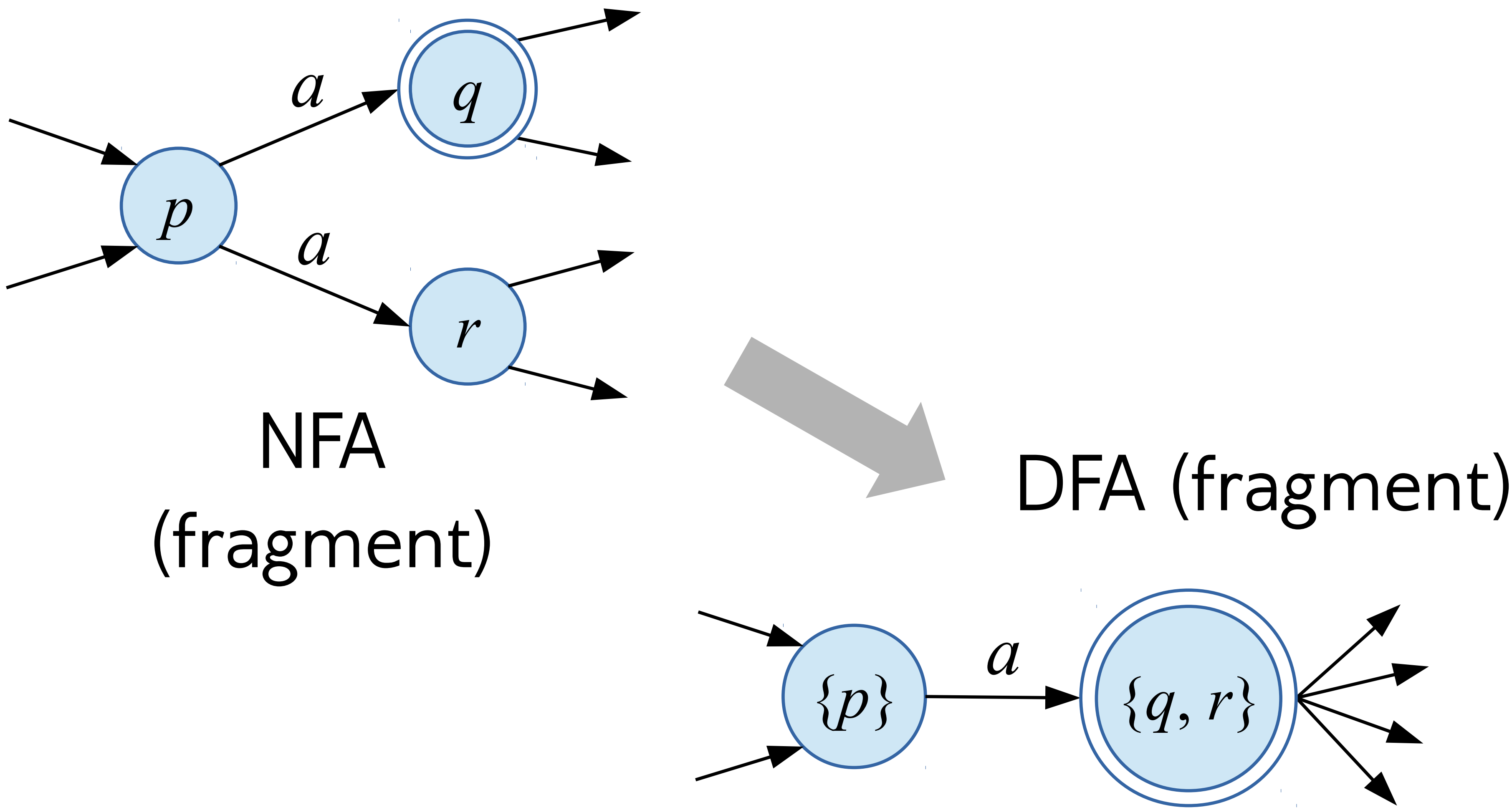


NFA
(fragment)

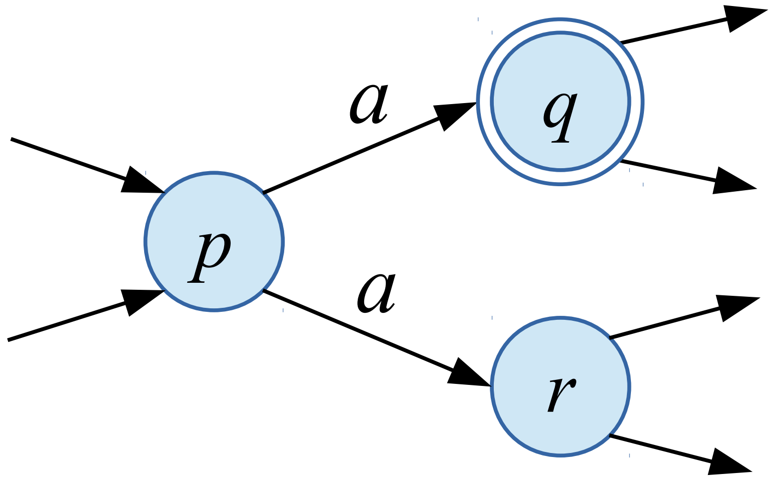
Every NFA can be simulated by a DFA



Every NFA can be simulated by a DFA

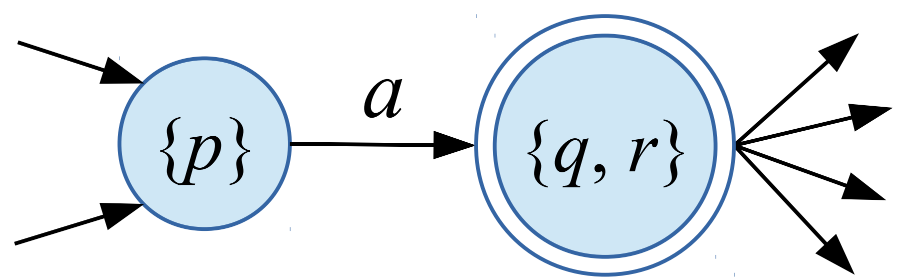
"Subset construction"

Main idea: ONE state of DFA tracks current states of ALL evolving paths of NFA



NFA (fragment)

DFA (fragment)



(This is merely illustrative – formal construction on following slides)

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	NFA	Equivalent DFA
Specification		
States		
Alphabet		
Transition Function		
Initial State		
Accepting States		

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Initial State	q_0	$q'_0 = \{q_0\}$
Accepting States	F	$F' = \{S \in 2^Q \mid S \cap F \neq \varnothing\}$

Why NFAs?

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- They can be way more compact than DFAs

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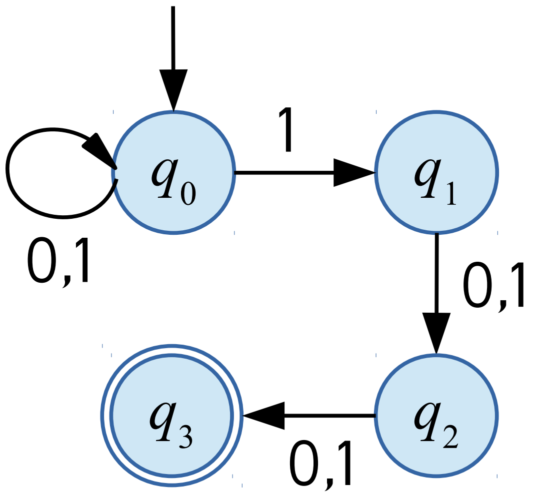

NFA

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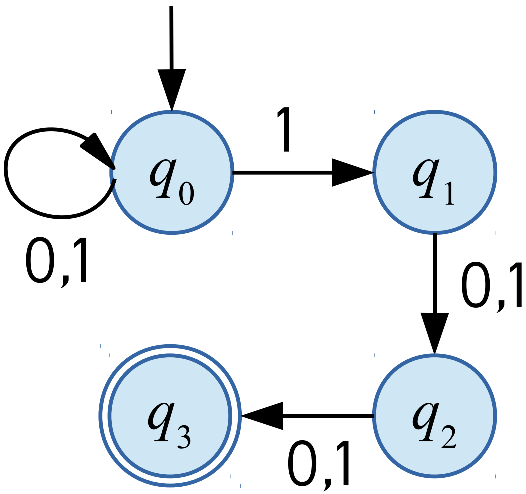

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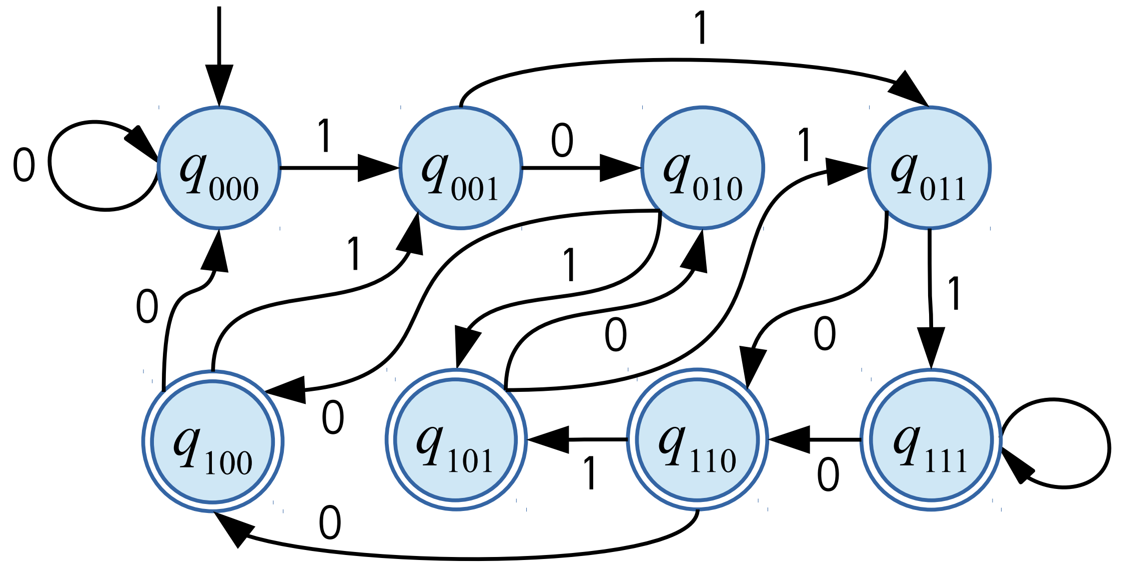

Equivalent minimal DFA

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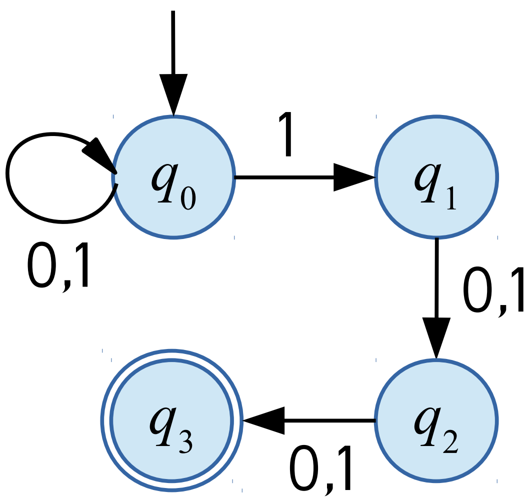

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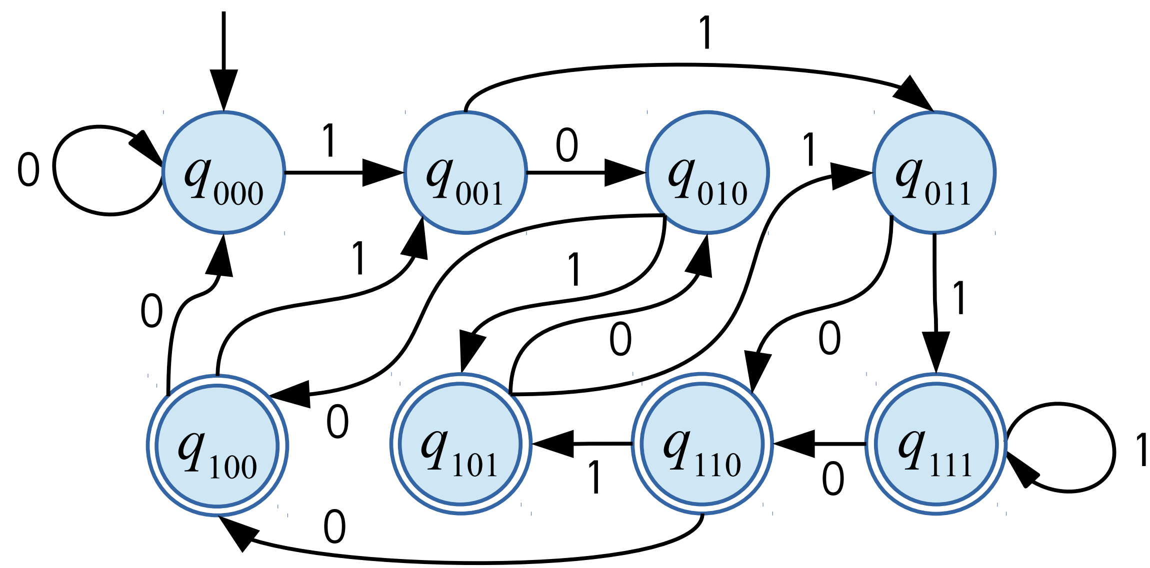

Equivalent minimal DFA

- It's easier to directly convert regular expressions (“wildcard search” on steroids) to NFAs

WHENEVER I LEARN A NEW SKILL I CONCOCT ELABORATE FANTASY SCENARIOS WHERE IT LETS ME SAVE THE DAY.
OH NO! THE KILLER MUST HAVE FOLLOWED HER ON VACATION!
BUT TO FIND THEM WE'D HAVE TO SEARCH THROUGH 200 MB OF EMAILS LOOKING FOR SOMETHING FORMATTED LIKE AN ADDRESS!
IT'S HOPELESS!
EVERYBODY STAND BACK.
I KNOW REGULAR EXPRESSIONS.
PERL!
TAP TAP

XKCD

Playing with regexes

- http://regex101.com/
- http://rubular.com/
- http://www.google.com/search?q=online+regex+tester

Regular expressions (“regex”-es) are defined by *structural induction*

(start with simple base expressions, construct more complicated ones recursively)

Empty set

$$L(\emptyset) = \emptyset$$

Empty string

$$L(\boldsymbol{\varepsilon}) = \{\varepsilon\}$$

Literal character

$$L(\mathbf{x}) = \{x\}$$

e.g. $L(\mathbf{1}) = \{1\}, L(\mathbf{2}) = \{2\}, L(\mathbf{a}) = \{a\}$

Concatenation

$$L(AB) = \{ab \mid a \in L(A), \mathrm{b} \in L(B)\}$$

e.g. $L(\mathbf{12}) = \{12\}, L(\mathbf{aabb}) = \{aabb\}$

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Kleene star

$$L(A^*) =$$
$$\{\varepsilon\} \cup \{\, x_1 x_2 \ldots x_n \mid n \in \mathbf{N},\, x_i \in L(A) \,\}$$

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