Mathematical Induction

CS 2800: Discrete Structures, Spring 2015

Sid Chaudhuri

Prime factorizability

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  - 2, 3, 5, 7, 11, 13, 17, ...

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- Claim: Every natural number ≥ 2 can be expressed as a finite product of prime numbers
  - E.g. $3 = 3$
    $15 = 3 \times 5$
    $16 = 2 \times 2 \times 2 \times 2$

17

Is it prime?

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↓

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Yes!

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(And there was much rejoicing)

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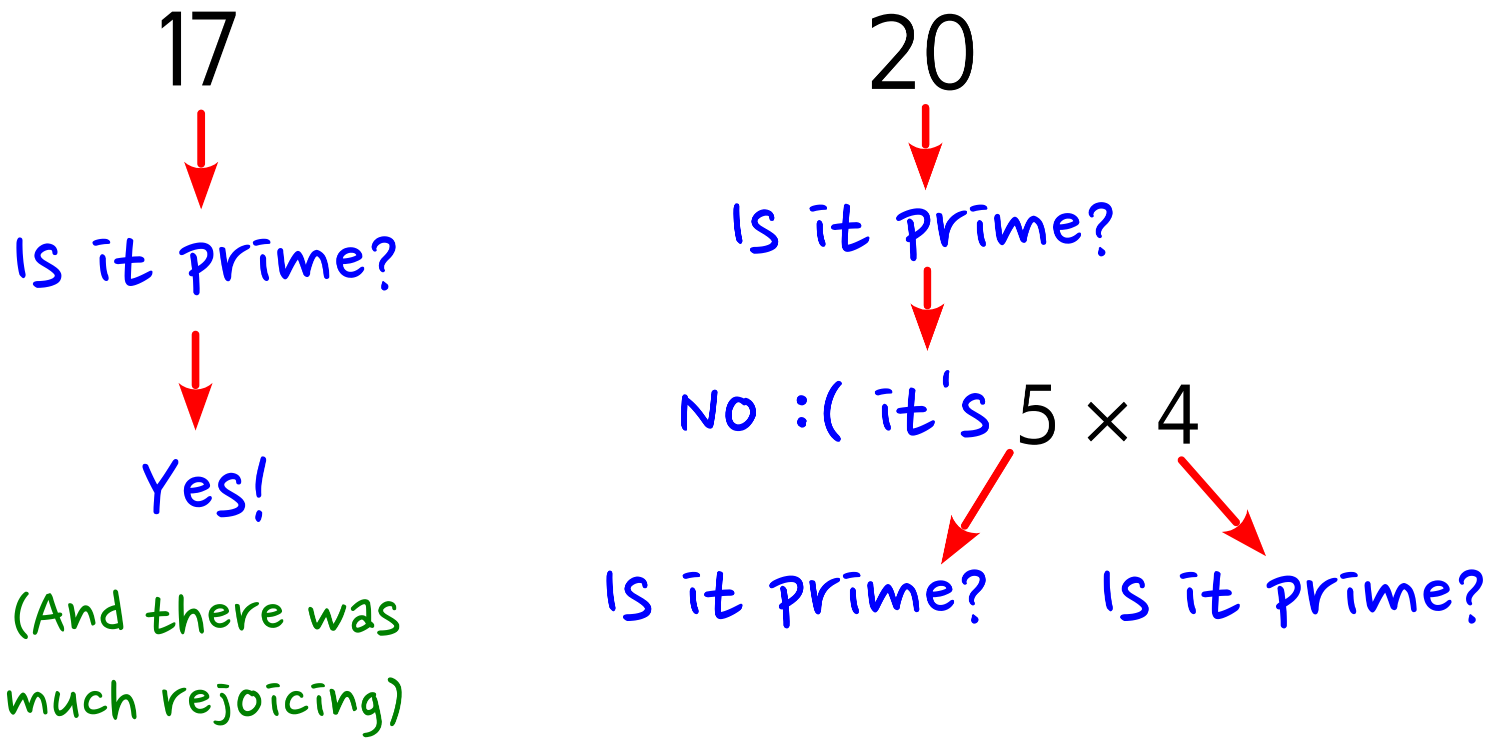

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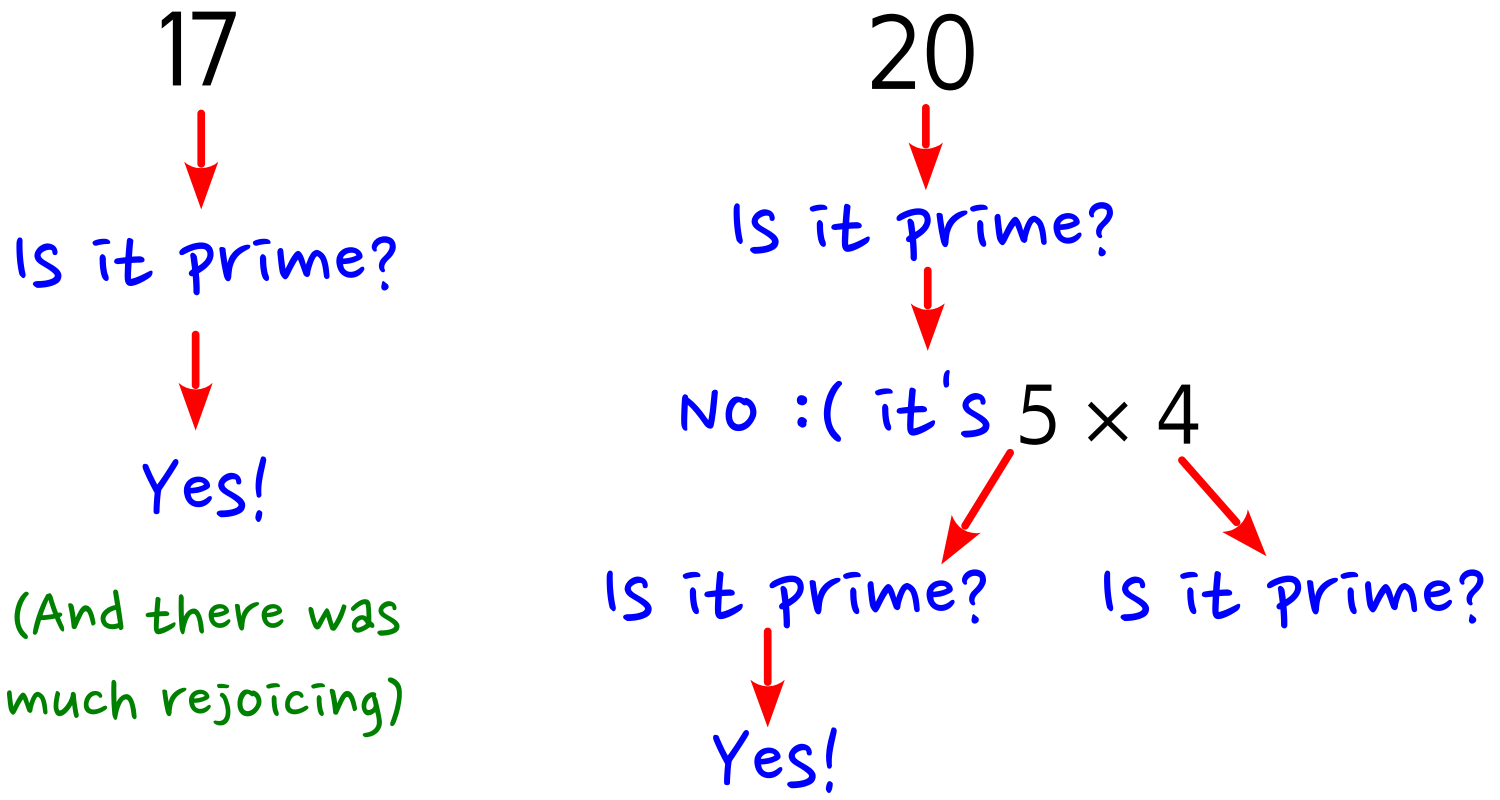
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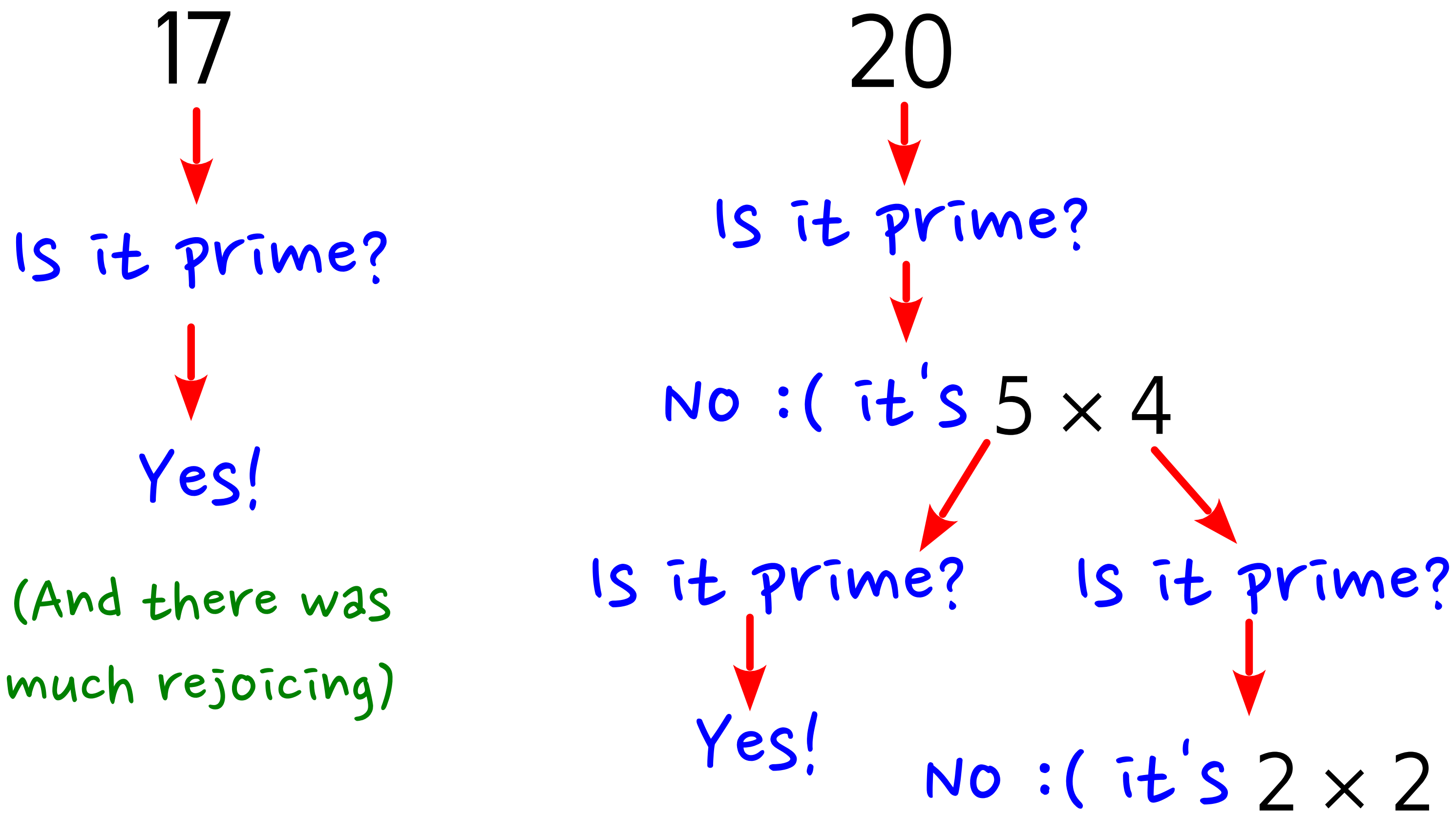
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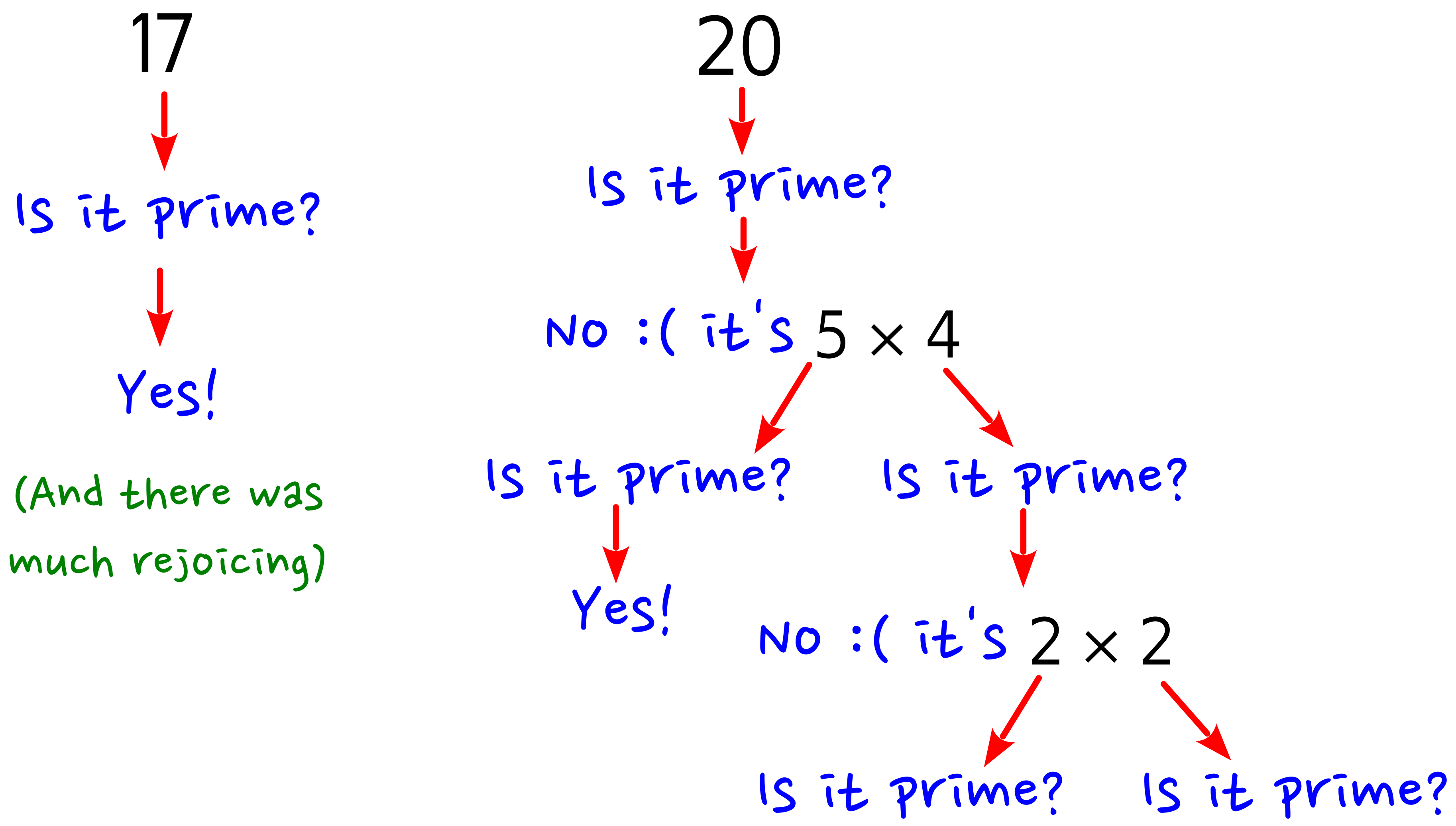
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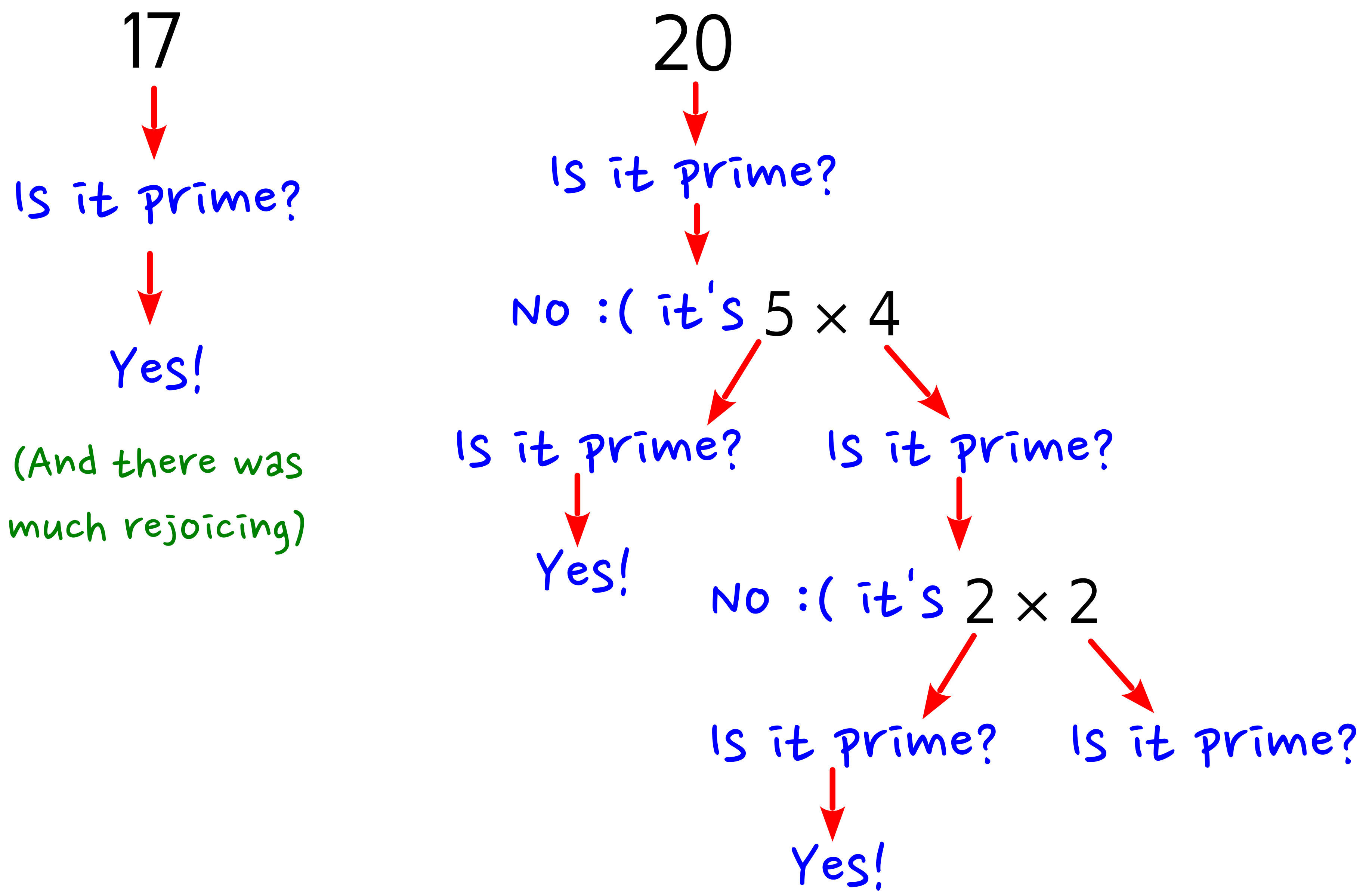
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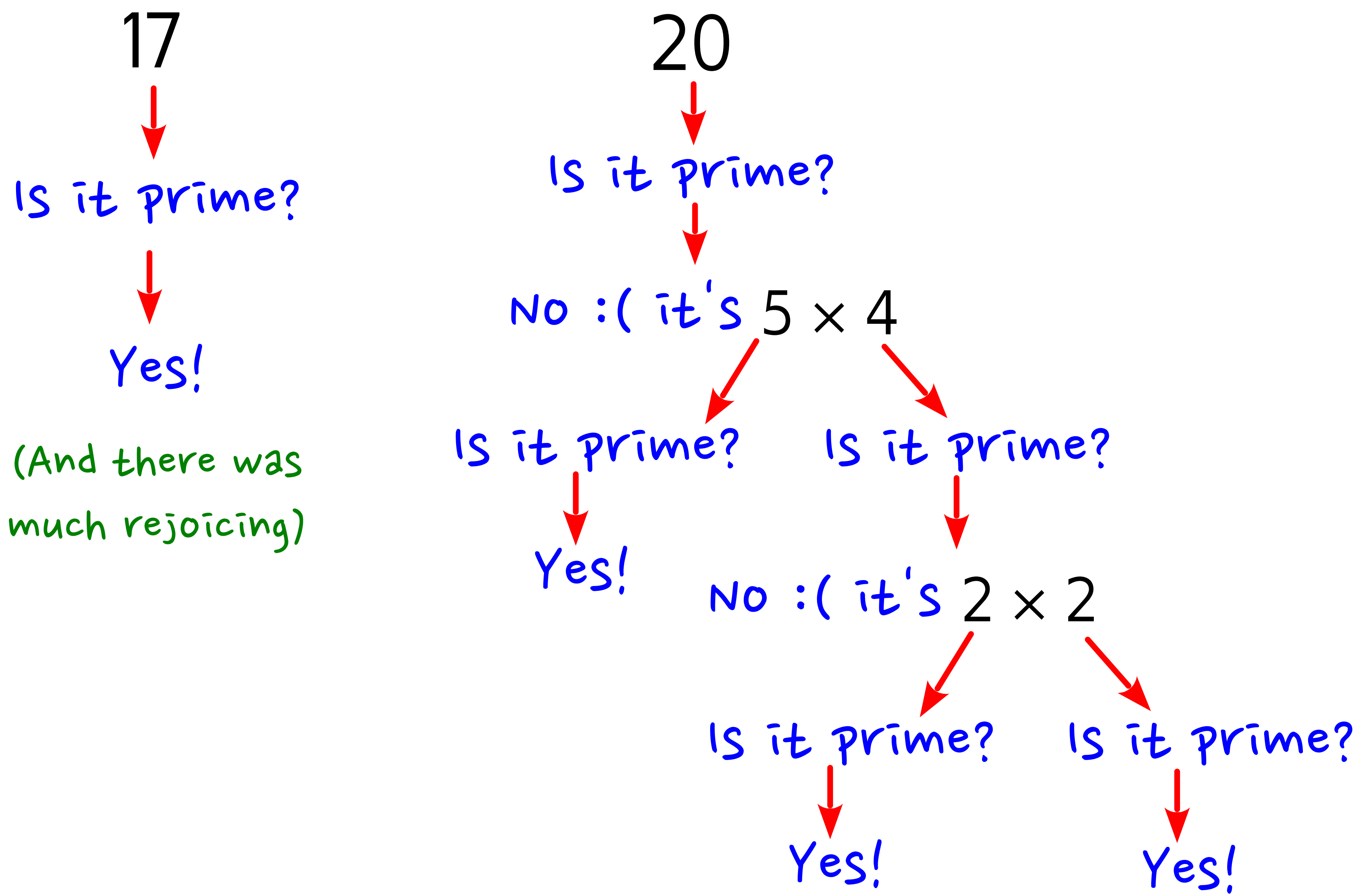
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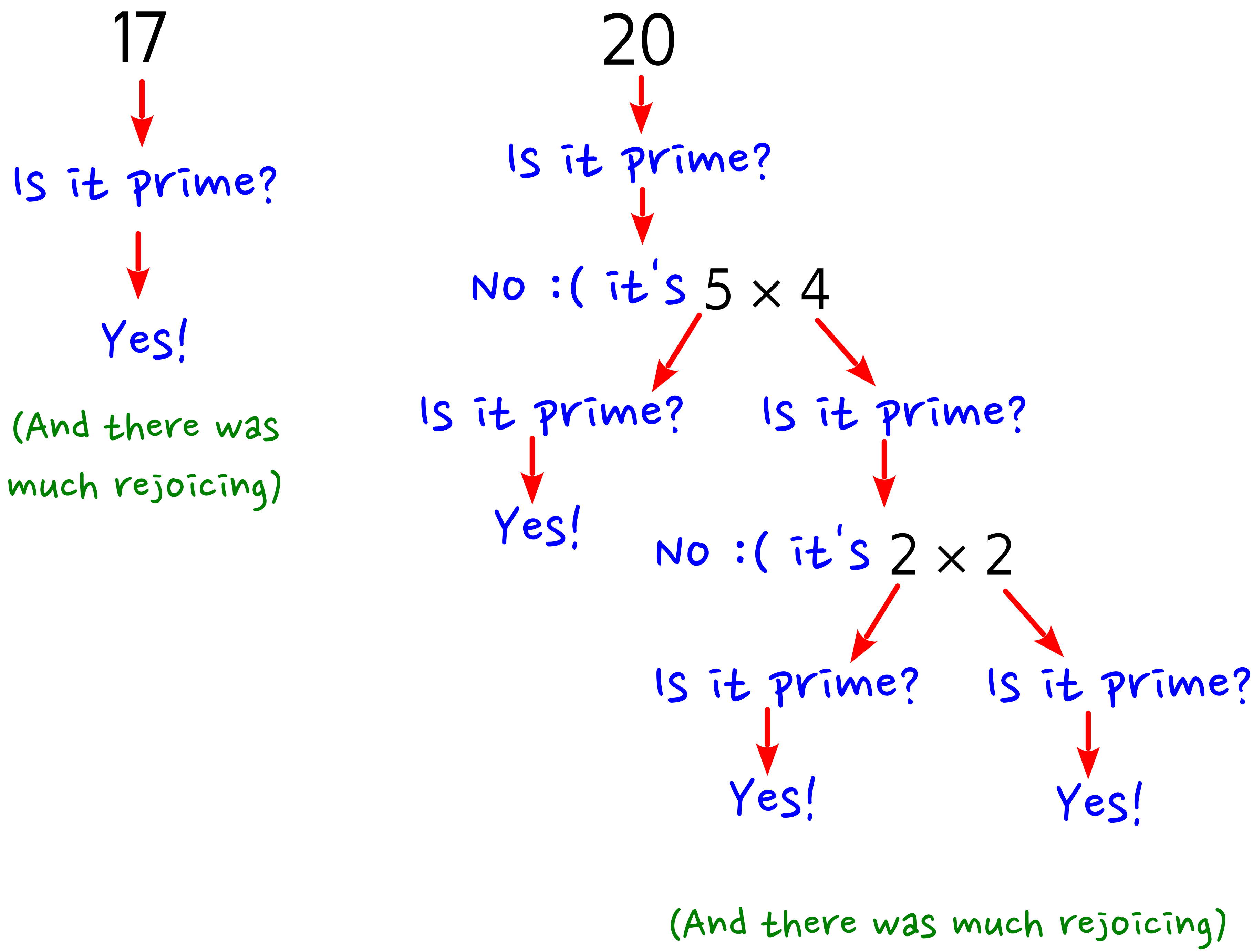
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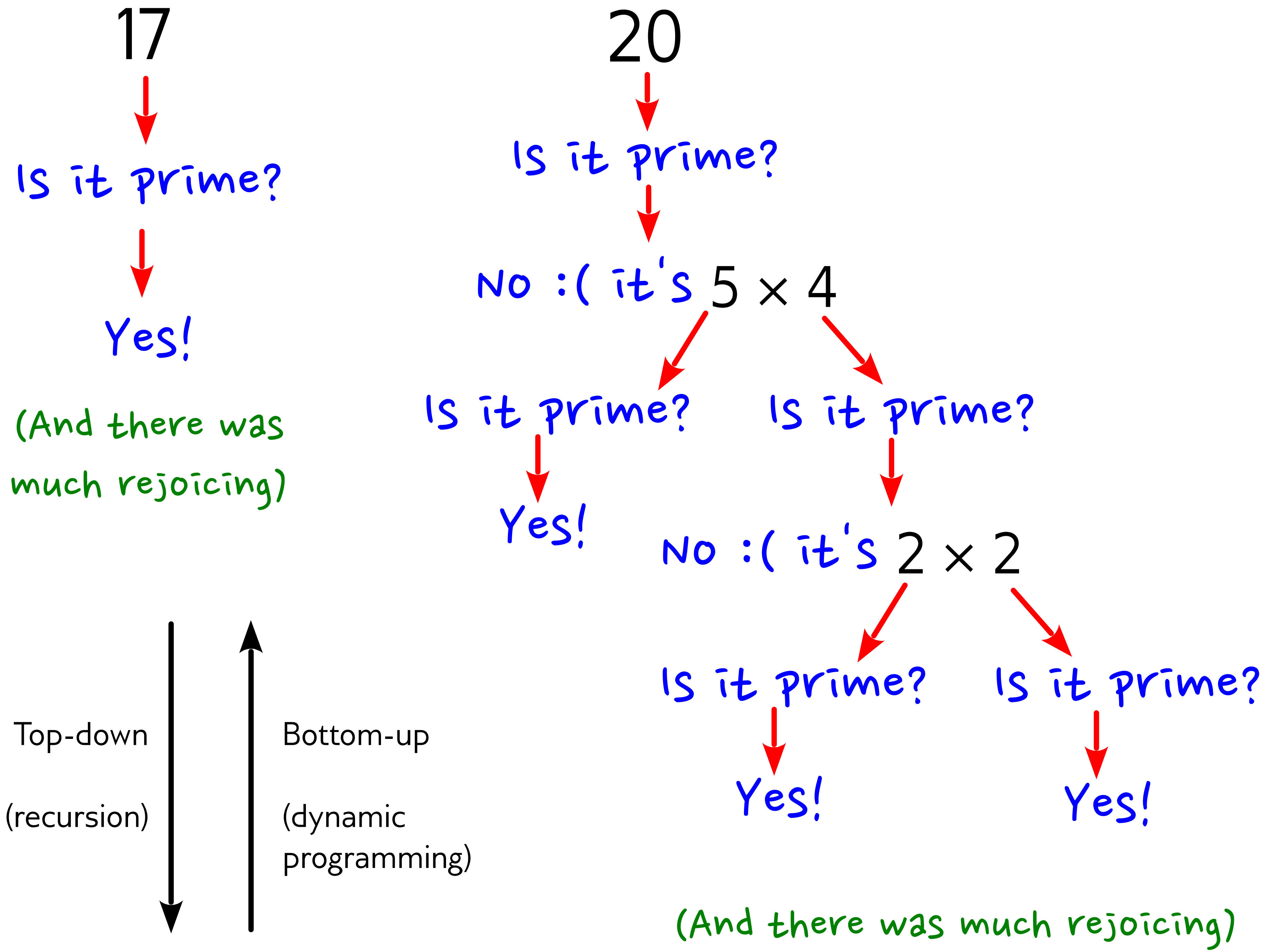
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Top-down
(recursion)
Bottom-up
(dynamic programming)

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which n? we'll assume
n is arbitrary, but ≥ 2

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Prime factorizability

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  - (We've assumed the claim is true for all natural numbers from 2 to n)
  - Consider $n+1$
    - **If it is prime**, then the claim is trivially true for $n+1$

Prime factorizability

- Proof (contd):
  - (We've assumed the claim is true for all natural numbers from 2 to n)
  - Consider $n + 1$
    - **If it is prime**, then the claim is trivially true for $n + 1$
    - **If it is composite**, then it is (by definition), the product of two natural numbers a and b, both > 1 but $< n + 1$

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    - So $a = p_1 p_2 p_3 \ldots p_m$ and $b = q_1 q_2 q_3 \ldots q_k$, where all p_i, q_j are prime

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    - So $a = p_1 p_2 p_3 \ldots p_m$ and $b = q_1 q_2 q_3 \ldots q_k$, where all p_i, q_j are prime
    - So $n + 1 = p_1 p_2 \ldots p_m q_1 q_2 \ldots q_k$, i.e. the claim is true for $n + 1$

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  - If it's true for 2, it must be true for 3. If it's true for 3, it must be true for 4. If it's true for 4...

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  - Yes!
  - If it's true for 2, it must be true for 3. If it's true for 3, it must be true for 4. If it's true for 4...
  - We've applied the Principle of Mathematical Induction

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Inductive hypothesis

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Inductive hypothesis

- ... then $S(n)$ is true for all natural numbers n

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  - Can start from an integer k which is not 0. The statement is then proved for all integers $\geq k$
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  - Can apply to any countable set (prove!)

Example: Sum of the natural numbers

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    $= (n + 1)(n + 2) / 2$

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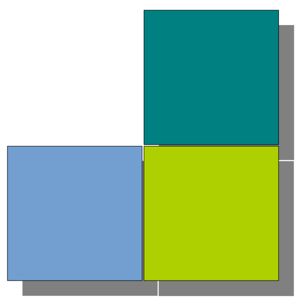
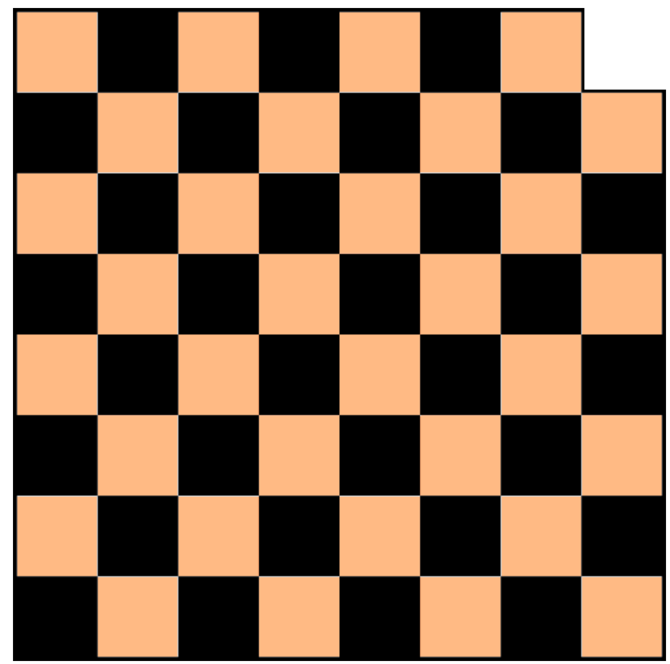
  - Hence $S(n + 1)$ is true
- Hence by induction, $S(n)$ is true for all $n \in \mathbf{N}$

Example: Tiling with triominoes





$S(n)$ = "A $2^n \times 2^n$ chessboard with one corner missing can be tiled with triominoes"

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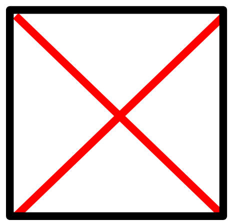

Base case: A 1×1 chessboard with one corner missing is empty, so $S(0)$ is true

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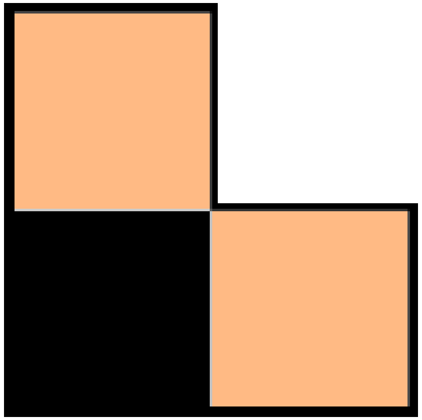

Another base case (although this isn't necessary): A 2×2 chessboard with one corner missing is just a single triomino, so $S(1)$ is true

Example: Tiling with triominoes

- Inductive step:
  - Assume a $2^k \times 2^k$ chessboard with a corner missing can be tiled with triominoes, for all natural numbers $k \leq n$
  - Consider a $2^{n+1} \times 2^{n+1}$ board (with a corner missing)

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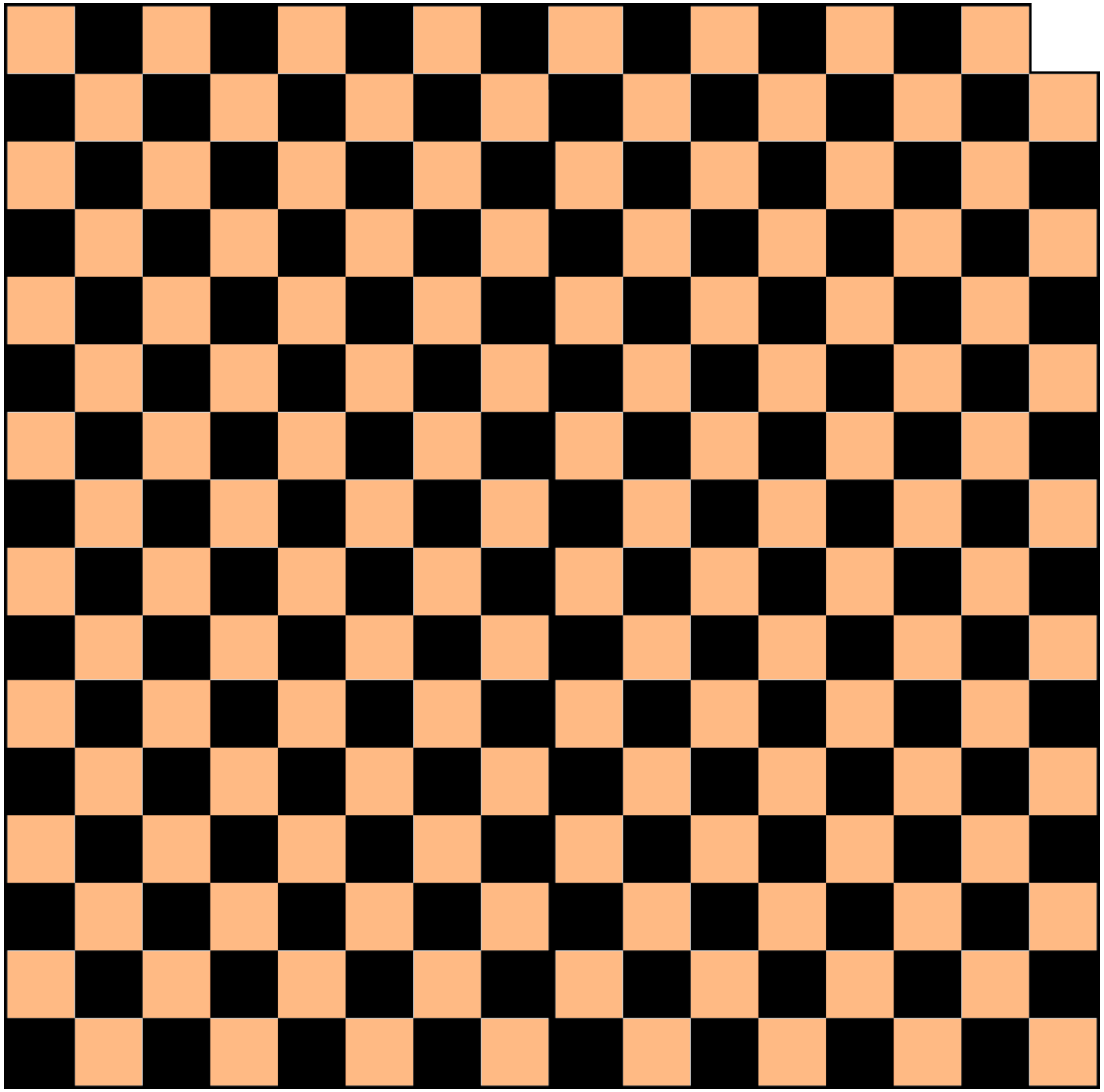

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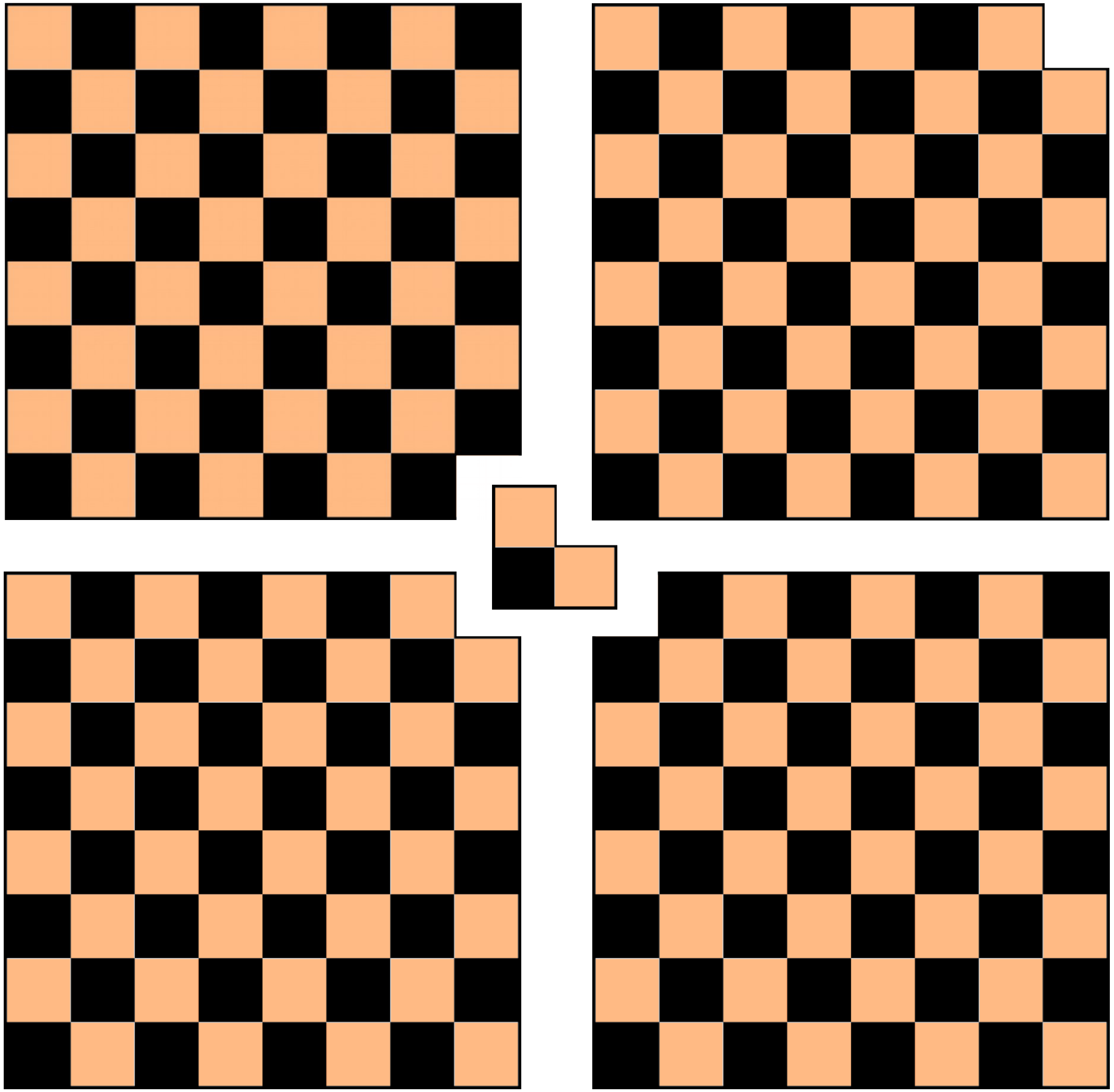

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  - It is just four $2^n \times 2^n$ boards, plus one triomino

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  - From the inductive hypothesis, we know each of these boards can be tiled with triominoes

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  - Hence $S(n+1)$ is true
- Hence by induction, $S(n)$ is true for all $n \in \mathbf{N}$

Example: Fibonacci program

$S(n)$ = "The following program returns the n^{th} Fibonacci number, given input n"

```
int fibonacci(int n)
{
  if (n <= 1)
    return 1;
  else
    return fibonacci(n – 1) + fibonacci(n – 2);
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Example: Fibonacci program

Starting from 0

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Precondition: n ≥ 0

Example: Fibonacci program

- Base cases: `fibonacci(0)` and `fibonacci(1)` return the first two Fibonacci numbers (1 and 1, easily verified), so $S(0)$ and $S(1)$ are true

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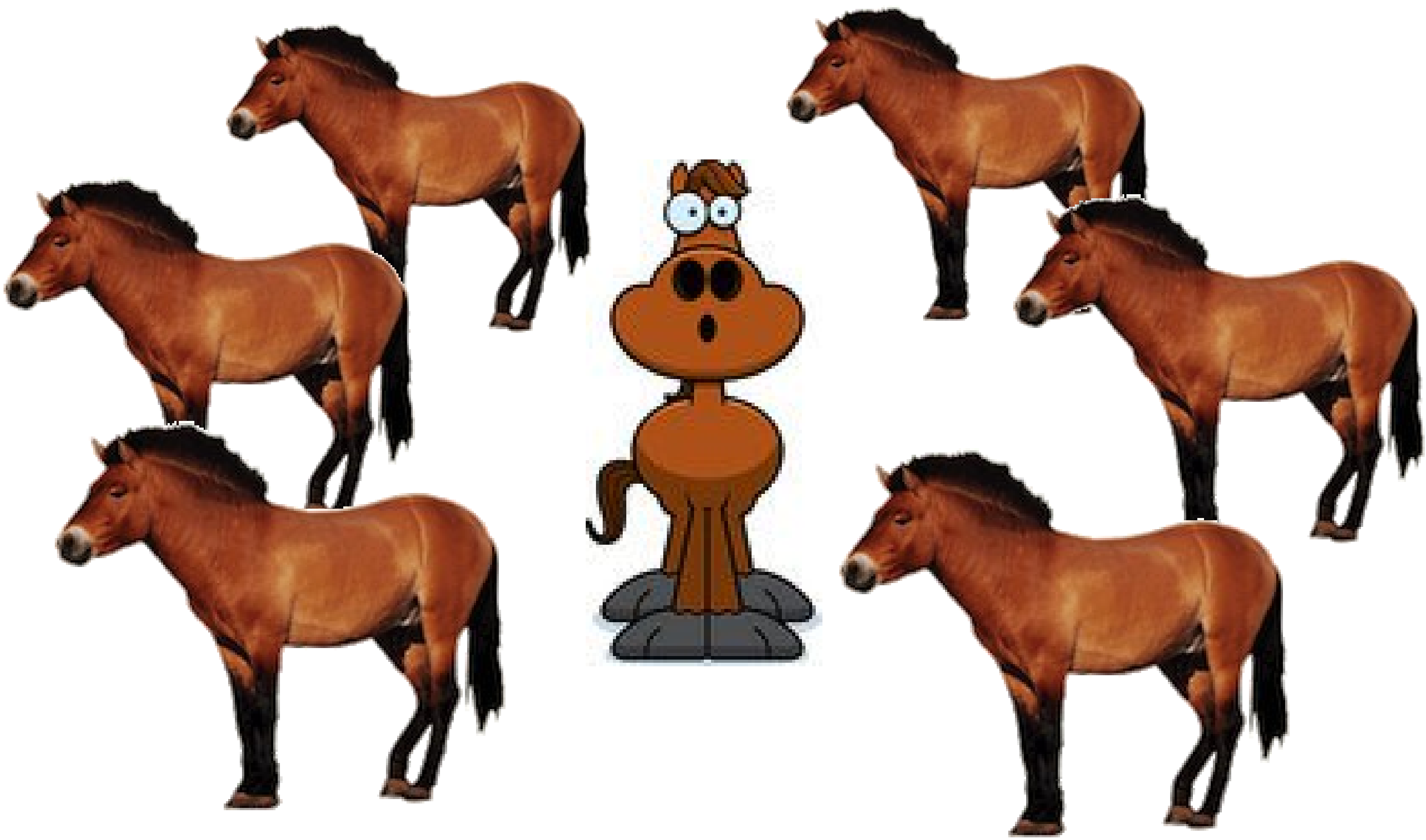
All horses are the same color



www.clipartof.com, www.fg-a.com

All horses are the same color

- $S(n)$ = “Any group of n horses is the same color”

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Doesn't say anything about whether different groups of n horses have different colors or not

All horses are the same color

- $S(n)$ = "Any group of n horses is the same color"
- Base case: A single horse is obviously the same color as itself, so $S(1)$ is true (and $S(0)$ is trivially true)

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