Functions and Inverses

CS 2800: Discrete Structures, Spring 2015

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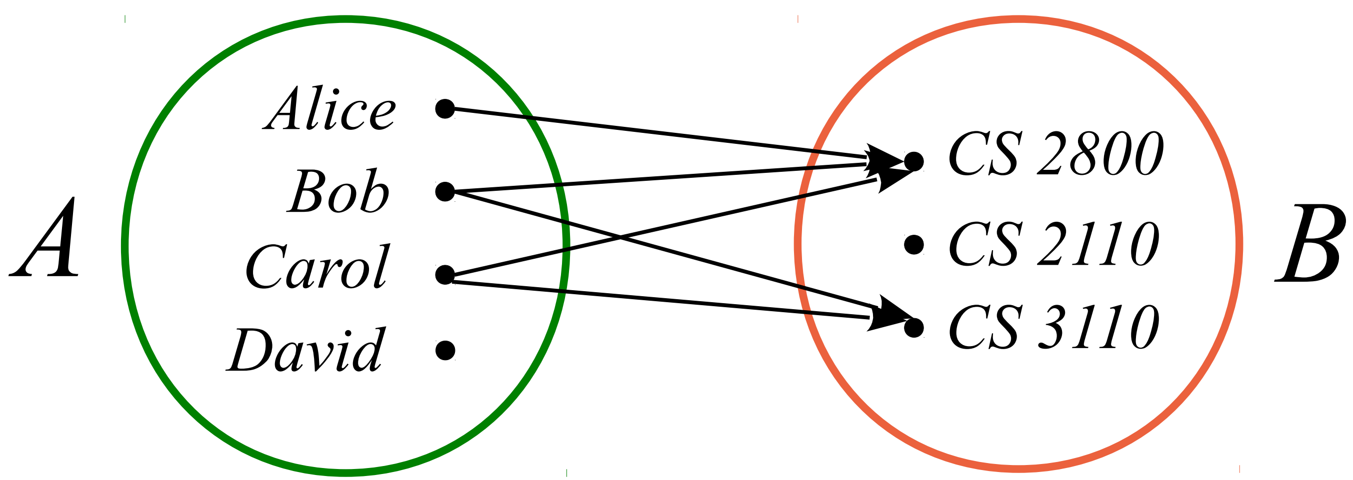
Recap: Relations and Functions

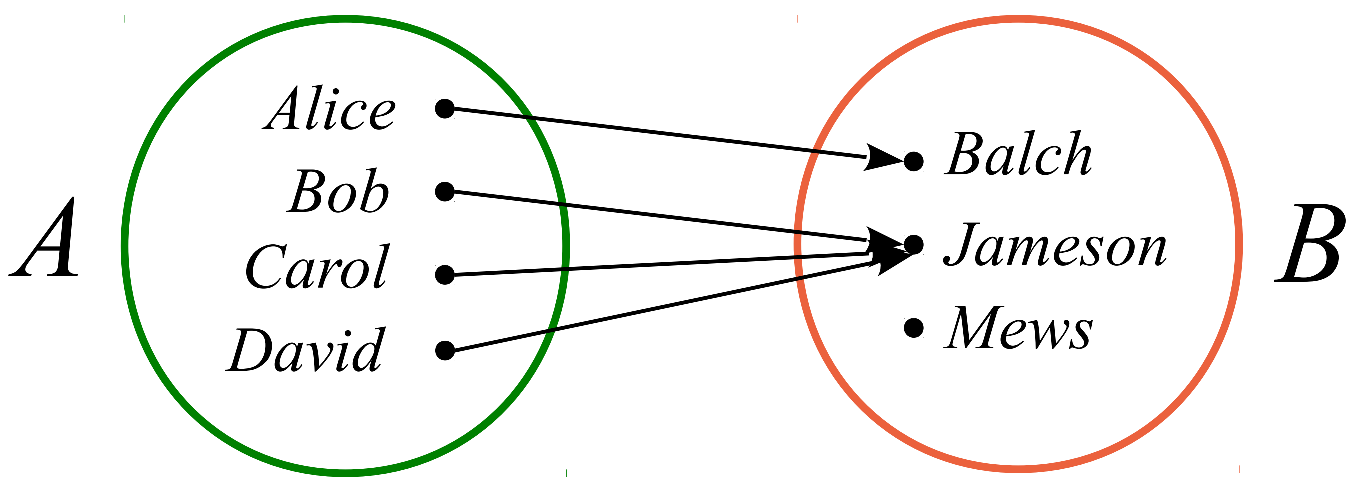
- A **relation** between sets A (the *domain*) and B (the *codomain*) is a set of ordered pairs (a, b) such that $a \in A$, $b \in B$ (i.e. it is a subset of $A \times B$)
  - The relation maps each a to the corresponding b
    - Neither all possible a's, nor all possible b's, need be covered
  - Can be one-one, one-many, many-one, many-many

cartesian product



Recap: Relations and Functions

- A **function** is a relation that maps ***each*** element of A to a ***single*** element of B
  - Can be one-one or many-one
  - All elements of A must be covered, though not necessarily all elements of B
  - Subset of B covered by the function is its *range/image*



Recap: Relations and Functions

- Instead of writing the function f as a set of pairs, we usually specify its domain and codomain as:

$$f : A \longrightarrow B$$

... and the mapping via a rule such as:

$$f(Heads) = 0.5, \quad f(Tails) = 0.5$$

or $f : x \mapsto x^2$

The function f maps x to x^2

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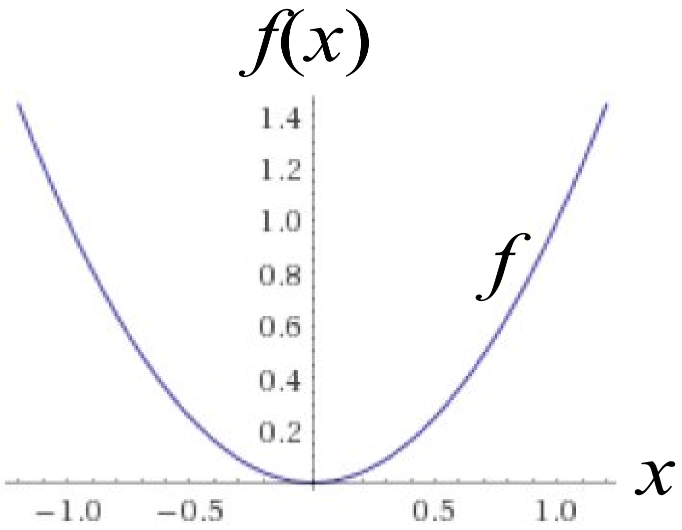
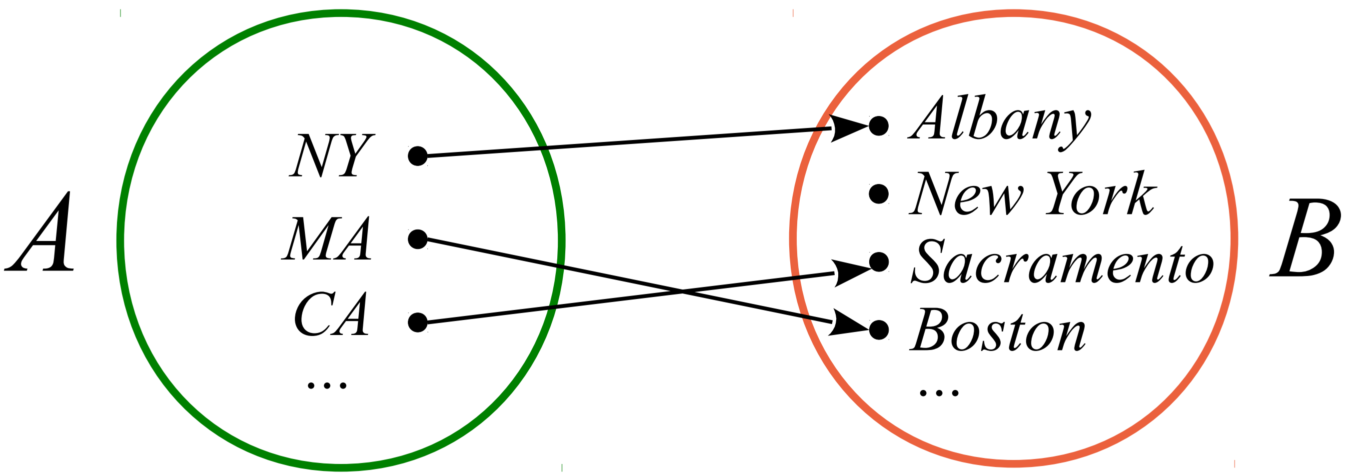
- **Note:** the function is f, not $f(x)$!
  - $f(x)$ is the value assigned by the function f to input x



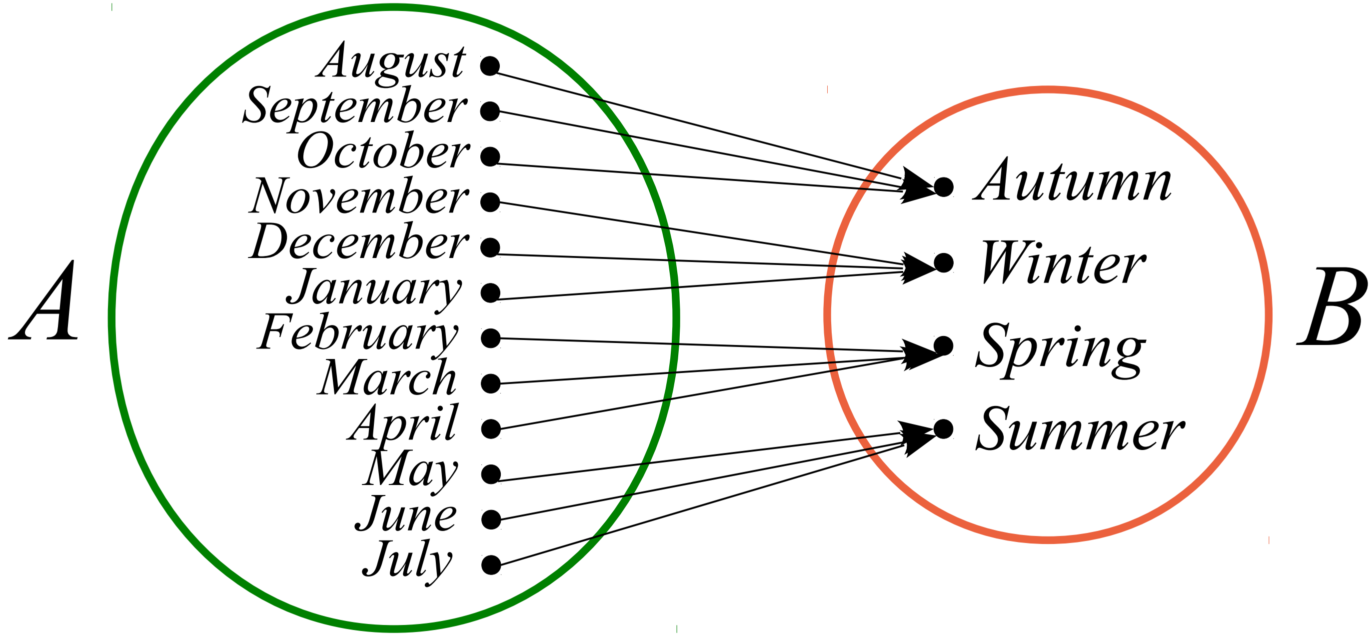
Recap: Injectivity

- A function is **injective** (one-to-one) if every element in the domain has a unique image in the codomain
  - That is, $f(x) = f(y)$ implies $x = y$



Recap: Surjectivity

- A function if **surjective** (onto) if every element of the codomain has a preimage in the domain
  - That is, for every $b \in B$ there is some $a \in A$ such that $f(a) = b$
  - That is, the codomain is equal to the range/image



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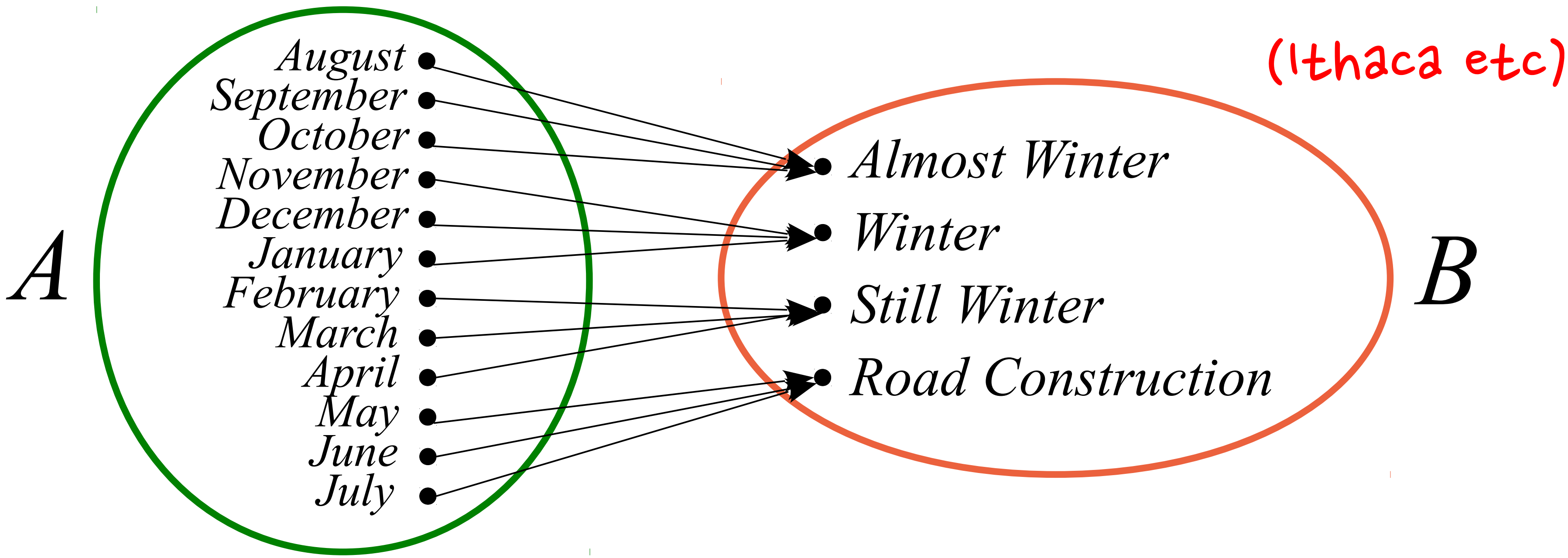
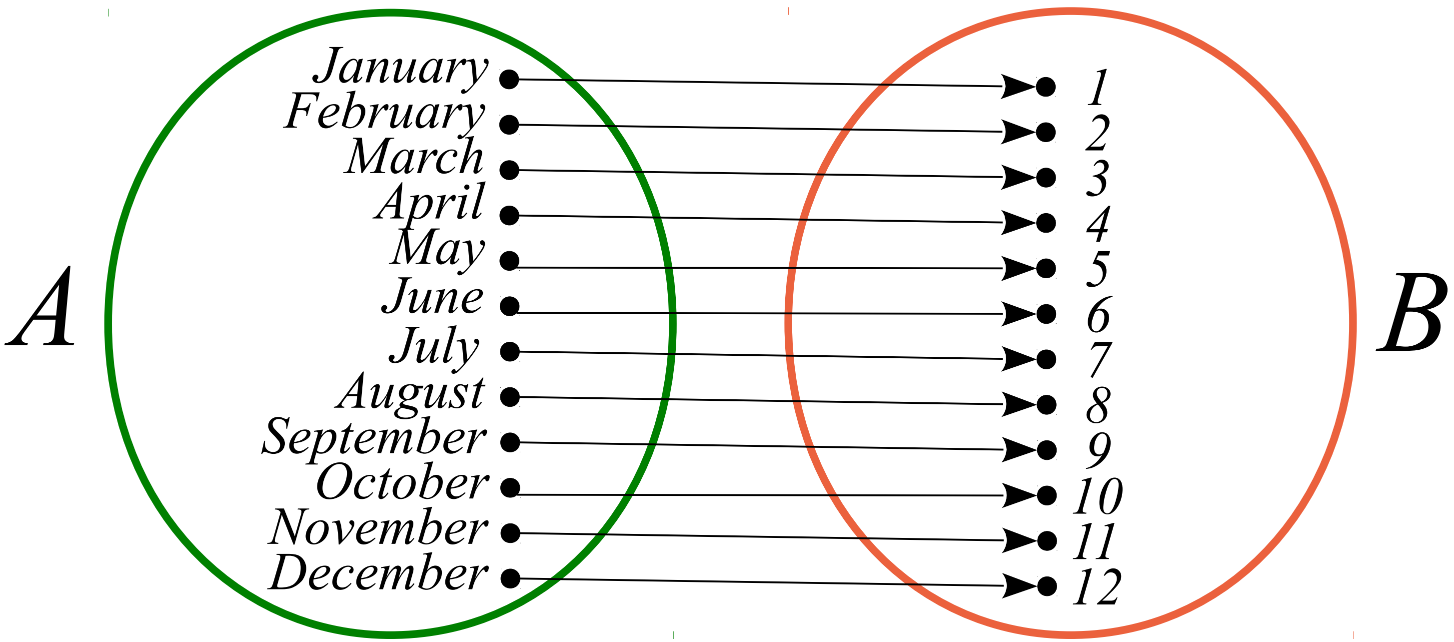


Recap: Bijectivity

- A function is **bijective** if it is both surjective and injective



Composition of Functions

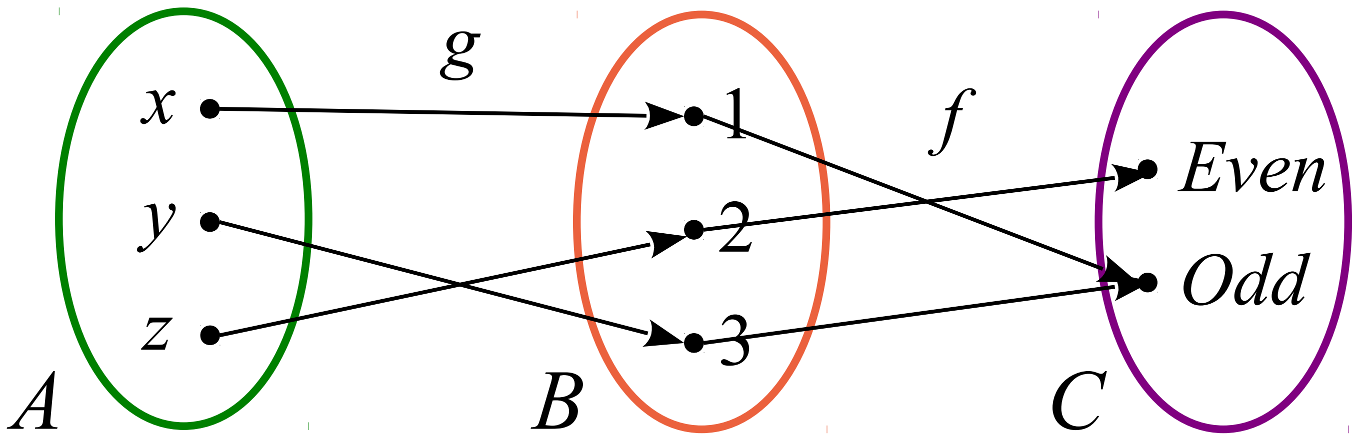
- The **composition** of two functions

$$f : B \longrightarrow C$$

$$g : A \longrightarrow B$$

is the function $f \circ g : A \longrightarrow C$ defined as

$$f \circ g : x \mapsto f(g(x))$$



Composition of Functions

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$$f : B \longrightarrow C$$

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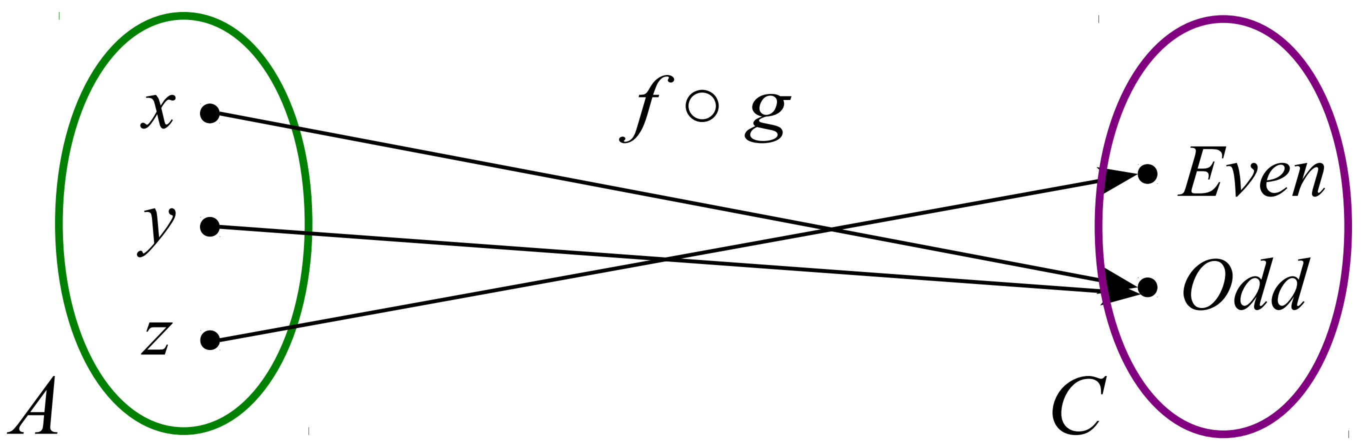
$g \circ f$ is not possible unless $A = C$!

$$f \circ g : x \ \mapsto \ f(\,g\,(x)\,)$$



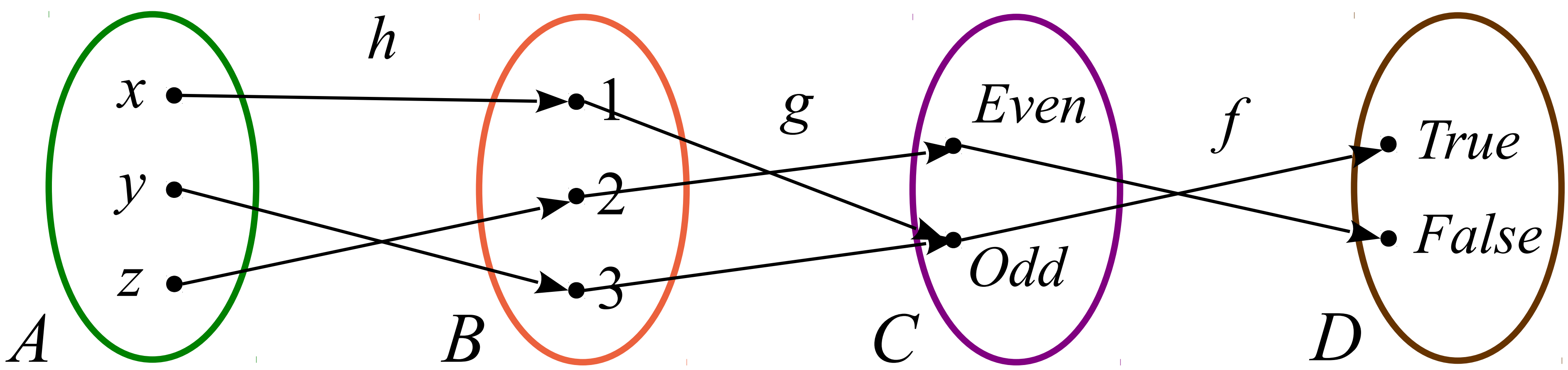
Factoid of the Day #1

Composition is ***associative***

$$(f \circ g) \circ h = f \circ (g \circ h)$$

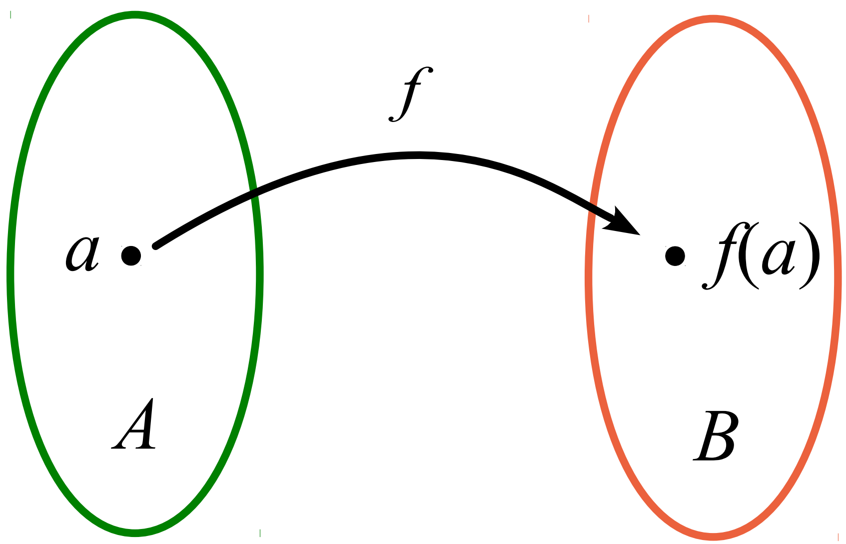
(two functions are equal if for every input, they give the same output)

Prove it!



Left Inverse of a Function

- $g : B \to A$ is a left inverse of $f : A \to B$ if $g\,(\,f\,(a)\,) = a$ for all $a \in A$
  - If you follow the function from the domain to the codomain, the left inverse tells you how to go back to where you started



Left Inverse of a Function

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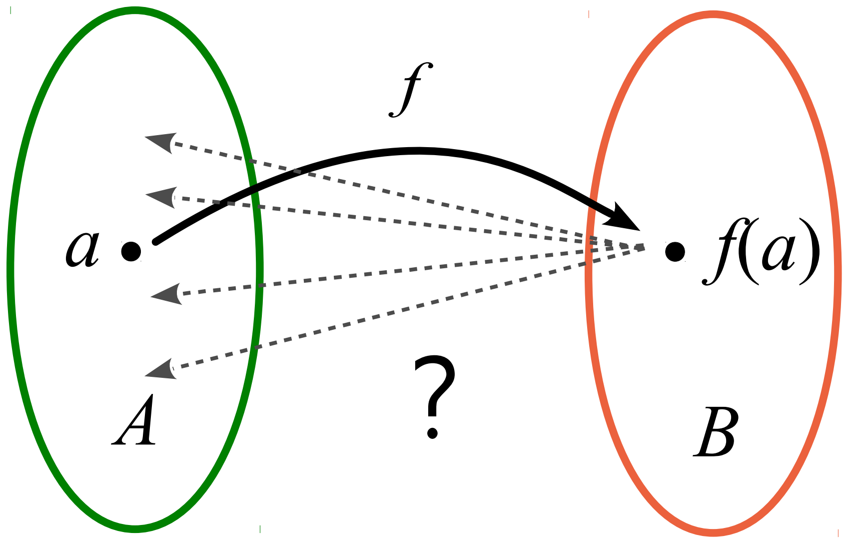

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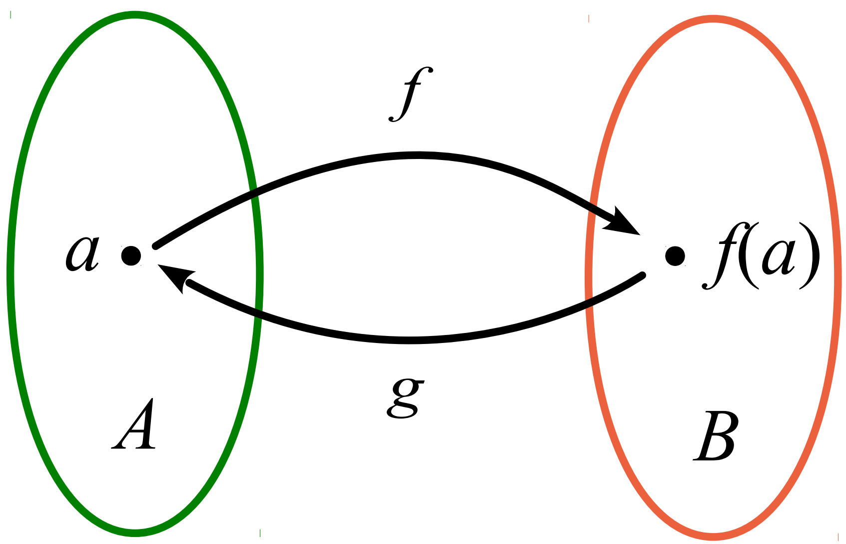
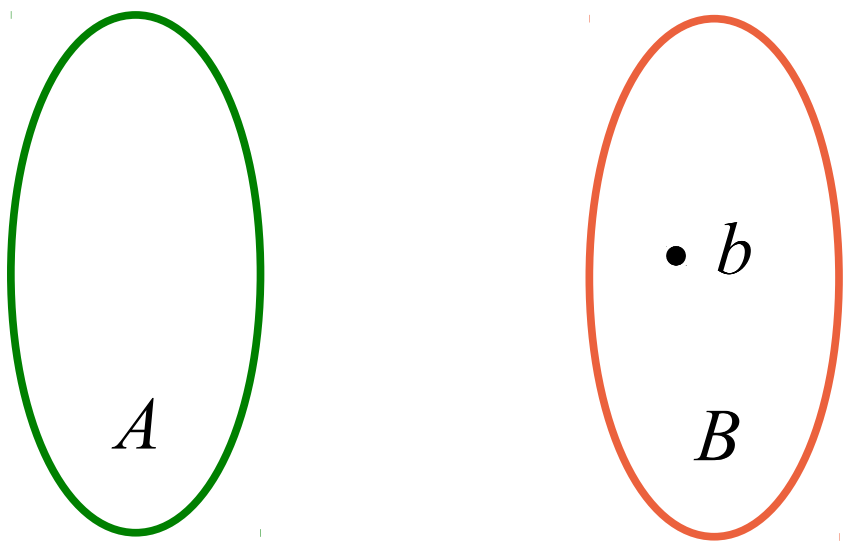


Right Inverse of a Function

- $h : B \rightarrow A$ is a right inverse of $f : A \rightarrow B$ if $f(\,h\,(b)\,) = b$ for all $b \in B$
  - If you're trying to get to a destination in the codomain, the right inverse tells you a possible place to start



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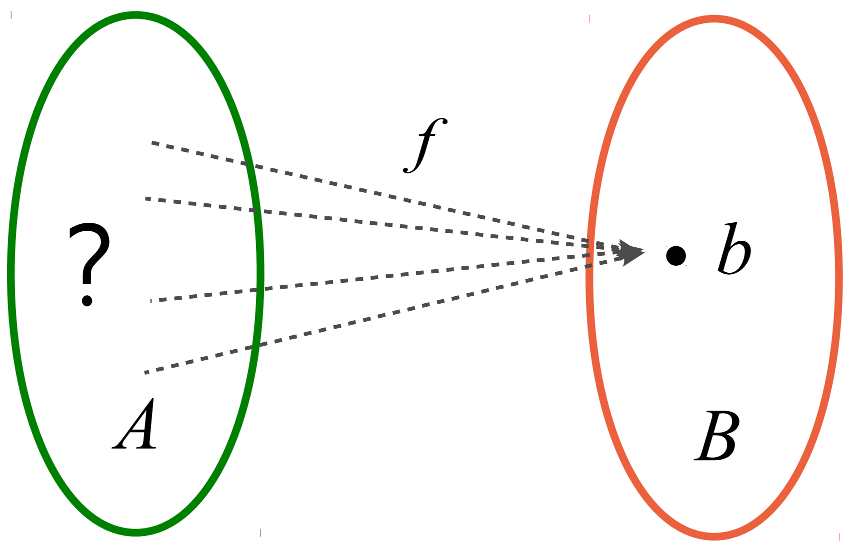

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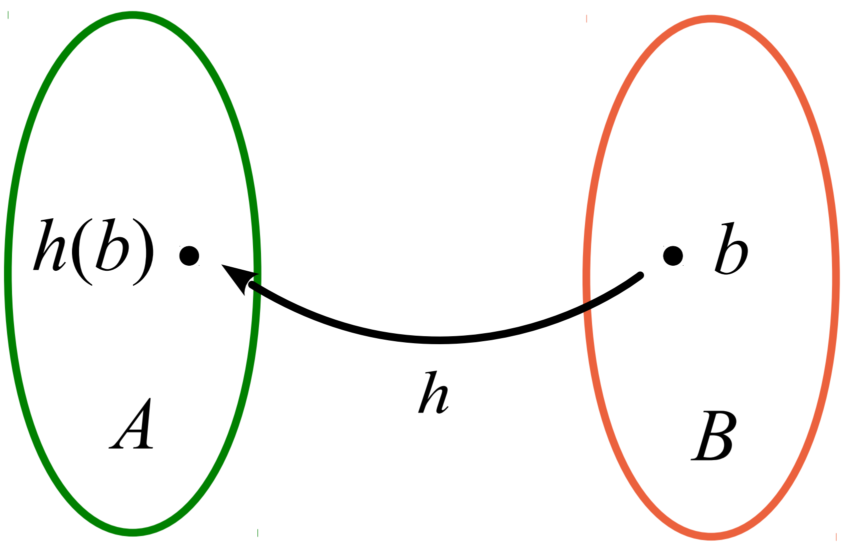

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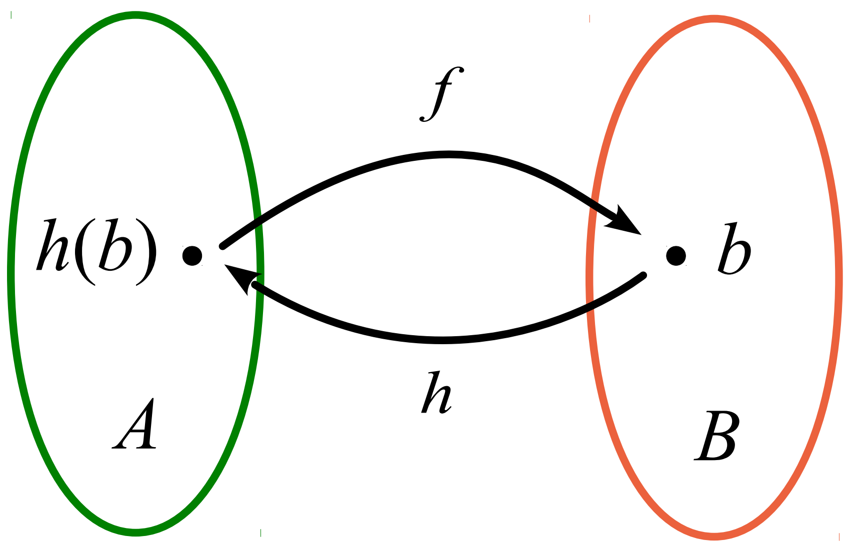

Note the subtle difference!

- The **left inverse** tells you how to *exactly* retrace your steps, *if* you managed to get to a destination
  - "Some places might be unreachable, but I can always put you on the return flight"
- The **right inverse** tells you where you *might* have come from, for *any* possible destination
  - "All places are reachable, but I can't put you on the return flight because I don't know exactly where you came from"

Factoid of the Day #2

Left and right inverses need not exist,
and need not be unique

can you come up with some examples?

Left inverse ⇔ Injective

- **Theorem:** A function is **injective** (one-to-one) iff it has a **left inverse**
- **Proof** (⇐): Assume $f : A \rightarrow B$ has left inverse g
  - If $f(x) = f(y)$ ...
  - ... then $g(f(x)) = g(f(y))$ (*any fn maps equals to equals*)
  - ... i.e. $x = y$ (*since g is a left inverse*)
  - Hence f is injective

Left inverse $\Leftrightarrow$ Injective

- **Theorem:** A function is **injective** (one-to-one) iff it has a **left inverse**
- **Proof** ($\Rightarrow$): Assume $f : A \rightarrow B$ is injective
  - Pick any a_0 in A, and define g as

$$g(b) = \begin{cases} a & \text{if } f(a) = b \\ a_0 & \text{otherwise} \end{cases}$$

  - This is a well-defined function: since f is injective, there can be at most a single a such that $f(a) = b$
  - Also, if $f(a) = b$ then $g(f(a)) = a$, by construction
  - Hence g is a left inverse of f

Right inverse $\Leftrightarrow$ Surjective

- **Theorem:** A function is **surjective** (onto) iff it has a **right inverse**
- **Proof** ($\Leftarrow$): Assume $f : A \rightarrow B$ has right inverse h
  - For any $b \in B$, we can apply h to it to get $h(b)$
  - Since h is a right inverse, $f(h(b)) = b$
  - Therefore every element of B has a preimage in A
  - Hence f is surjective

Right inverse $\Leftrightarrow$ Surjective

- **Theorem:** A function is **surjective** (onto) iff it has a **right inverse**
- **Proof** ($\Rightarrow$): Assume $f : A \rightarrow B$ is surjective
  - For every $b \in B$, there is a non-empty set $A_b \subseteq A$ such that for every $a \in A_b$, $f(a) = b$ (since f is surjective)
  - Define $h : b \mapsto$ an arbitrary element of A_b
  - Again, this is a well-defined function since A_b is non-empty (and assuming the "axiom of choice"!)
  - Also, $f(h(b)) = b$ for all $b \in B$, by construction
  - Hence h is a right inverse of f

Recap: Left and Right Inverses

- A function is ***injective*** (one-to-one) iff it has a ***left inverse***
- A function is ***surjective*** (onto) iff it has a ***right inverse***

Factoid for the Day #3

If a function has both a left inverse and a right inverse, then the two inverses are identical, and this common inverse is unique

(Prove!)

This is called the *two-sided inverse*, or usually just the **inverse** f^{-1} of the function f

http://www.cs.cornell.edu/courses/cs2800/2015sp/handouts/jonpak_function_notes.pdf

Bijection and two-sided inverse

- A function f is bijective iff it has a two-sided inverse
- **Proof** ($\Longrightarrow$): If it is bijective, it has a left inverse (since injective) and a right inverse (since surjective), which must be one and the same by the previous factoid
- **Proof** ($\Longleftarrow$): If it has a two-sided inverse, it is both injective (since there is a left inverse) and surjective (since there is a right inverse). Hence it is bijective.