Logic and Quantifiers

CS 2800: Discrete Structures, Spring 2015

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Propositional Logic

- Primitive/atomic symbols/statements
  - $P \equiv$ “It's sunny”
  - $Q \equiv$ “I'm out running”
  - $R \equiv$ “Mary had a little lamb”
  - $S \equiv$ “Colorless green ideas sleep furiously”

  …

Propositional Logic

- Logical connectives
  - and ($\wedge$)
    - "P and Q", or "$P \wedge Q$" (*it's sunny and I'm out running*)
  - or ($\vee$)
    - "P or Q", or "$P \vee Q$" (*it's sunny or I'm out running*)
  - not ($\neg$)
    - "not P" or "$\neg P$" (*it's not sunny*)
  - implies ($\Rightarrow$)
    - "if P, then Q" or "$P \Rightarrow Q$" (*if it's sunny, then I'm out running*)
  - is implied by ($\Leftarrow$)
    - "Q, if P" or "$Q \Leftarrow P$" (*I'm out running, if it's sunny*)
  - if and only if ($\Leftrightarrow$)
    - "P if and only if Q" or "$Q \Leftrightarrow P$" (*I'm out running if and only if it's sunny*)

When writing proofs, prefer plain English.

$\Rightarrow$ or $\Leftrightarrow$ can be used judiciously to denote a chain of reasoning.

Propositional Logic

- Inference rules
  - *Modus ponens*: If P is true, and P implies Q, then Q must be true
    - $P, P \Rightarrow Q \models Q$ (entails)
    - Premise 1: P
      Premise 2: $P \Rightarrow Q$
      Conclusion: Q
    - Premise 1: It's sunny
      Premise 2: If it's sunny, then I'm out running
      Conclusion: I'm out running

Propositional Logic

- Inference rules
  - *Modus ponens*: If P is true, and P implies Q, then Q must be true
    - Cornerstone of direct proofs
    - If the first statement in a chain of forward implications is true, modus ponens lets us conclude that the last statement must also be true
    - Premise 1: $S_0 \Rightarrow S_1 \Rightarrow S_2 \Rightarrow \ldots \Rightarrow S_n \Rightarrow S$
      Premise 2: S_0
      Conclusion (after recursively applying modus ponens): S
  - This is not the only possible inference rule
    - e.g. *Modus tollens*: $P \Rightarrow Q, \neg Q \models \neg P$ (proof by contradiction)

First-Order Logic

- Just like propositional logic, but introduces
  - Variables: $x, y, z, \ldots$
  - Predicates: $P(x), Q(x), R(x), \ldots$
  - Quantifiers: ∀ ("for all"), ∃ ("there exists")
- Every x has property $F(x)$: $\forall x, F(x)$
- There is some x with property $F(x)$: $\exists x, F(x)$
- "*All men are mortal*" : $\forall x, Man(x) \Rightarrow Mortal(x)$
  - Painstakingly: "*for every object, if it is a man, then it is mortal*"
- "*Somebody here is asleep*" : $\exists x \in Humans, Here(x) \wedge Asleep(x)$
  - Painstakingly: "*there exists a human being who is here and who is asleep*"

Thought for the Day #1

Is this statement true or false?

"All eleven-legged alligators have
orange and blue spots"

This statement is ***true***!!!

... else there would be an eleven-legged alligator
(which lacks orange spots, or blue spots, or both)

Any statement about each member of an empty set is always true

Negating Quantified Statements

- All dogs have fleas
  - There is a dog which has no fleas
- There is a horse that can add
  - No horse can add
- Every koala can climb
  - Some koala cannot climb

- Everybody in this class likes mathematics
  - Somebody in this class does not like mathematics

Negating Quantified Statements

- There exists a pig that can swim and catch fish
  - No pig can both swim and catch fish
- In every country, there is a city by a river
  - In some country, there is no city by a river
- Old MacDonald had a farm, and on that farm he had a cow
  - Old MacDonald either had no farm, or had a farm without cows

Negating Quantified Statements

- In every mathematics class there is some student who falls asleep during lectures
  - There is a mathematics class in which no student falls asleep during lectures
- “There must be some way out of here” said the joker to the thief
  - The joker did not say to the thief: “There must be some way out of here”
- To every thing there is a season
  - To some thing, there is no season

Negating Quantified Statements

- It is **not** the case that **every** x **has** property $F(x)$ $\Leftrightarrow$ there is **some** x **without** property $F(x)$

$$\neg(\forall x, F(x)) \;\Leftrightarrow\; \exists x, \neg F(x)$$

- It is **not** the case that there is **some** x **with** property $F(x)$ $\Leftrightarrow$ **every** x **lacks** property $F(x)$

$$\neg(\exists x, F(x)) \;\Leftrightarrow\; \forall x, \neg F(x)$$

- *"Flip leftmost quantifier, move negation one step rightwards"*

Caution

- English (and other natural languages) are not structured like standard first order logic
  - Don't try applying these rules to English
- Negation of “*there is a horse that can add*”
  - ... is ***not*** “*all horses cannot add*” (which is read as “*not all horses can add*”)
  - ... though it could be “*all horses lack the ability to add*” or “*no horse can add*”