Proofs

And why not to reason backwards

CS 2800: Discrete Structures, Spring 2015

Sid Chaudhuri

Monty Python and the Holy Grail (Witch Scene)

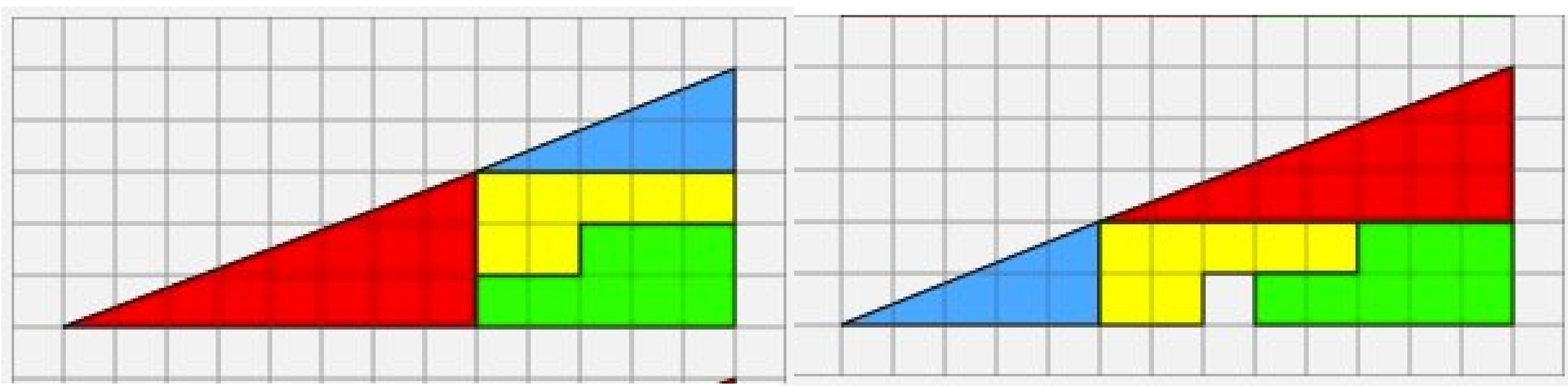
Thought for the Day #1

Identify all the logical flaws in this "proof"

http://youtu.be/X2xIQaimsGg

Faulty Logic

- How do you known she is a witch?
- She looks like one!



Beware of results that "look right"!
A picture is **not** a proof

Logical Implication

- "x is a witch" $\Rightarrow$ "x looks like a witch" (implies that)
- This does **not** mean
  "x looks like a witch" $\Rightarrow$ "x is a witch"

- Circumstantial evidence is not proof!
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- **Circumstantial evidence is not proof!!!**

Logical Implication

- Another example:
  - “It's sunny” $\Rightarrow$ “I will go for a run”
- This does **not** mean
  - “I will go for a run” $\Rightarrow$ “It's sunny”
    (i.e. if I'm out running, then it must be sunny)
  - I might also go for a run on a cloudy day!
- However, it **is true** that
  - If I'm not out running, it cannot be sunny

Logical Implication

- More generally, if $P \Rightarrow Q$
  - It need not be the case that $Q \Rightarrow P$
  - However, it is always the case that $\neg Q \Rightarrow \neg P$

"not"
(logical negation)

If there's one thing you take away from this course, let this be it

Outline of a correct proof

- We need to prove statement S
- Start with a statement S_0 known to be true
- Show that it logically implies S_1
- Show that S_1 logically implies S_2
- ... and so on until you end up implying S
- The proof looks like

Something definitely true

$$S_0 \Rightarrow S_1 \Rightarrow S_2 \Rightarrow \ldots \Rightarrow S_n \Rightarrow S$$

what you want to prove

Note the direction of the chain of implications!

Beware of reasoning backwards!

- This is **<u>not</u>** a proof of statement S

$$S \Rightarrow S_n \Rightarrow \ldots \Rightarrow S_2 \Rightarrow S_1 \Rightarrow S_0$$

what you want to prove

Something definitely true

- A very common error in this course!
- We *will* treat backwards proofs as incorrect

A backwards proof

- Prove that $a + b = a$, whenever $a = b \neq 0$

  Patently absurd, claims 1 + 1 = 1

- *"Proof":*

$$a + b = a$$

$$(a + b)(a - b) = a(a - b)$$

$$a^2 - b^2 = a^2 - ab$$

$$b^2 = ab$$

$$b = a \qquad \text{(dividing by } b \neq 0\text{)}$$

... which is true (given), hence "proved"

What went wrong?

we need implications
in this direction

... but that doesn't
work (division by zero
going from second
line to first)

$$a + b = a$$
$$\Leftarrow (a+b)(a-b) = a(a-b)$$
$$\Leftarrow a^2 - b^2 = a^2 - ab$$
$$\Leftarrow b^2 = ab$$
$$\Leftarrow b = a$$

A backwards proof of a *true* result

- If x and y are positive real numbers, then $(x+y)/2 \ \geq \ \sqrt{(xy)}$
- *"Proof":*

$$(x+y)/2 \ \geq \ \sqrt{(xy)}$$

$$(x+y)^2/4 \ \geq \ xy$$

$$x^2 + 2xy + y^2 \ \geq \ 4xy$$

$$x^2 - 2xy + y^2 \ \geq \ 0$$

$$(x-y)^2 \ \geq \ 0$$

... which is true, hence "proved"

If the direction of implications is not specified, the proof is **assumed to be "forward"**

This proves that
if $(x+y)/2 > \sqrt{(xy)}$,
then $(x-y)^2 > 0$,
not the other way round

You may lose points for writing the proof exactly like this

A correct proof

- If x and y are positive real numbers, then $(x + y)/2 \ \geq\ \sqrt{}(xy)$
- *Proof:*

$$(x - y)^2 \ \geq\ 0 \qquad \textit{(square of a real number is} \geq 0\textit{)}$$

$$\Rightarrow\ x^2 - 2xy + y^2 \ \geq\ 0$$

$$\Rightarrow\ x^2 + 2xy + y^2 \ \geq\ 4xy$$

$$\Rightarrow\ (x + y)^2/4 \ \geq\ xy$$

$$\Rightarrow\ (x + y)/2 \ \geq\ \sqrt{}(xy)$$

Hence proved

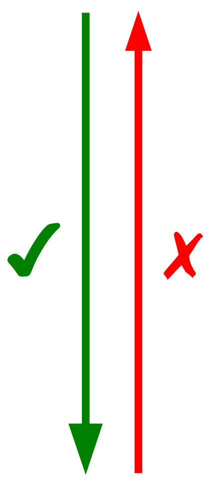
It's ok to figure out the proof "backwards" (often easier, else you're searching for that "magic" place to start), as long as your final chain of reasoning works "forwards"!

Thought for the Day #2

If the statement S to be proved is actually true, can I really construct a chain that works backwards (from S) but not forwards (to S)?

Yes!

- Prove that $a + b \geq a - b$ for $a \geq b > 0$
- *"Proof":*



$$a + b \geq a - b$$

$$(a + b)(a - b) \geq (a - b)(a - b) \qquad \textit{(}a - b \geq 0\textit{)}$$

$$a^2 - b^2 \geq a^2 - 2ab + b^2$$

$$-2b^2 \geq -2ab$$

$$b \leq a \qquad \textit{(dividing by } -2b < 0\textit{)}$$

... which is true (given)

- Division by zero when $a = b$, going in the direction we actually want (upwards)

Life Lesson #0

Avoid backwards proofs. Always write out the direction of implications using $\Longrightarrow$ ("implies"), $\Longleftarrow$ ("is implied by") and $\Longleftrightarrow$ ("if and only if") symbols, and ensure they point the right way.

It's not just for math and CS...

wikipedia.org

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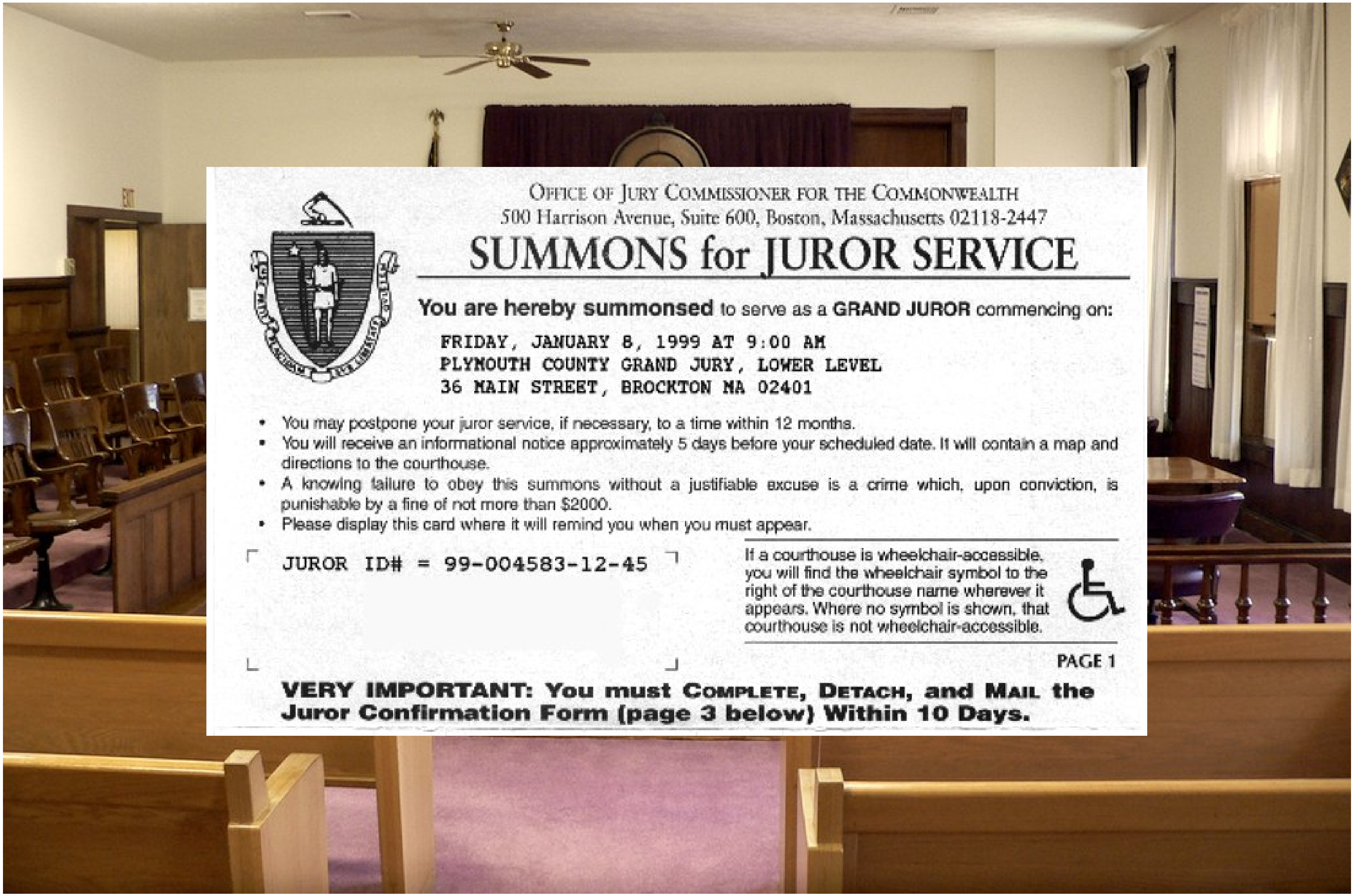


OFFICE OF JURY COMMISSIONER FOR THE COMMONWEALTH
500 Harrison Avenue, Suite 600, Boston, Massachusetts 02118-2447

SUMMONS for JUROR SERVICE

You are hereby summonsed to serve as a **GRAND JUROR** commencing on:

FRIDAY, JANUARY 8, 1999 AT 9:00 AM
PLYMOUTH COUNTY GRAND JURY, LOWER LEVEL
36 MAIN STREET, BROCKTON MA 02401

- You may postpone your juror service, if necessary, to a time within 12 months.
- You will receive an informational notice approximately 5 days before your scheduled date. It will contain a map and directions to the courthouse.
- A knowing failure to obey this summons without a justifiable excuse is a crime which, upon conviction, is punishable by a fine of not more than $2000.
- Please display this card where it will remind you when you must appear.

JUROR ID# = 99-004583-12-45

If a courthouse is wheelchair-accessible, you will find the wheelchair symbol to the right of the courthouse name wherever it appears. Where no symbol is shown, that courthouse is not wheelchair-accessible.

PAGE 1

VERY IMPORTANT: You must COMPLETE, DETACH, and MAIL the Juror Confirmation Form (page 3 below) Within 10 Days.

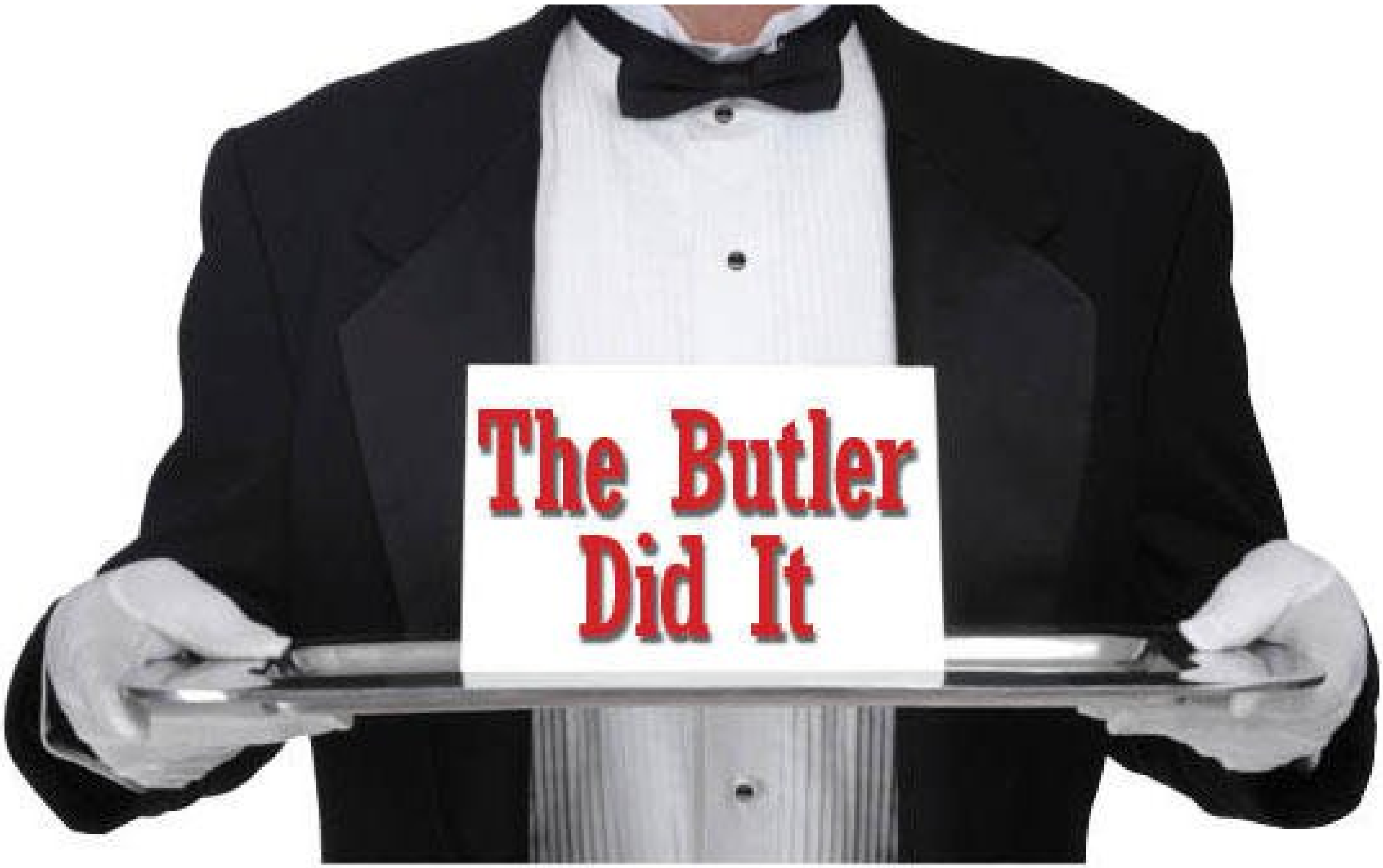
wikipedia.org


POLICE LINE DO NOT CROSS

Observation

- A man is discovered lying dead in his country house with a kitchen knife stuck in his side

Hypothesis



Proposed Proof

- Let's assume the butler did it!

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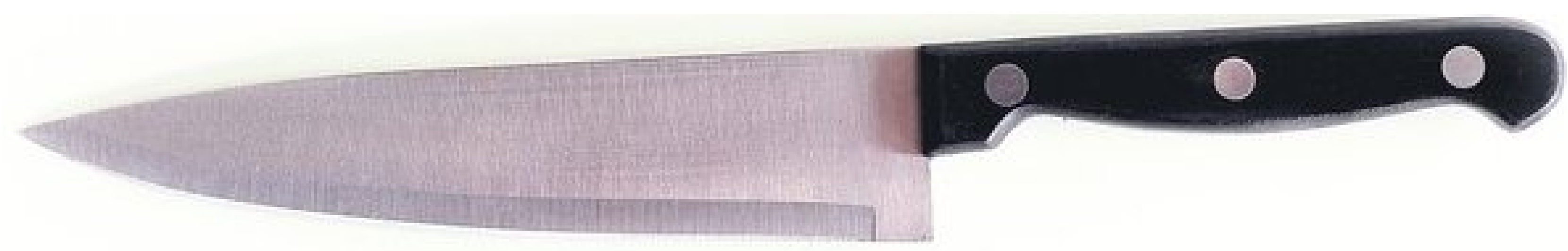
- Let's assume the butler did it!
- He needed to get the weapon, and have a motive
- The cook didn't see a kitchen knife missing during day, so the butler must have obtained it at night



www.stokesentinel.co.uk

Proposed Proof (contd)

- (Let's assume the butler did it!)

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- The parlormaid, who was sneaking back into the house after a liaison with the gardener, saw the butler walking towards the kitchen at 2am

Proposed Proof (contd)

- (Let's assume the butler did it!)
- The parlormaid, who was sneaking back into the house after a liaison with the gardener, saw the butler walking towards the kitchen at 2am
- The chauffeur testified the late master overruled the butler's preference to serve red wine instead of white. The butler took it as a mortal insult.

Does this prove the butler did it?

- No, the proof is backwards
- It shows that *if* the butler did it, *then* two things would be highly probable
  - He would go towards the kitchen at night
  - He would have a motive
- But it does *not* show that the observations conclusively incriminate the butler
- He could have been going to the restroom, and someone else could have had a stronger motive!

Remember

- A solid understanding of logical implications can save innocent lives
- We will revisit this topic in the context of conditional probability
  - Instead of "if A, then definitely B" ($A \Rightarrow B$)
  - … we have "if A, then probably B" ($\mathrm{P}(B \mid A) = \ldots$)