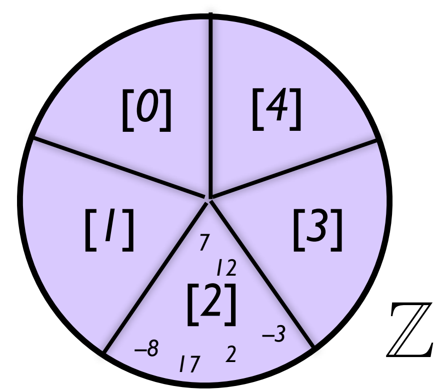
- To make our earlier (boxed) remark more precise, if ρ is an equivalence relation on A, then for any $a \in A$ we define the set $[a] = \{ x \in A \mid x \sim a \}$ to be the *equivalence class* of a and the set $A/\!\sim \; = \{ [a] \mid a \in A \}$ to be the *quotient* of A by the relation.

- Notice that if $[a] \cap [b] \neq \varnothing$ then $[a] = [b]$. To see this, let $x \in [a] \cap [b]$, then $x \in [a]$ and $x \in [b]$, but then $x \sim b$ and $x \sim a$, and so $a \sim x$ (by symmetry). So then $y \in [a] \Longrightarrow y \sim a \Longrightarrow y \sim x$ (transitivity) $\Longrightarrow y \sim b$ (transitivity) $\Longrightarrow y \in [b]$, and *mutatis mutandis*[*] $z \in [b] \Longrightarrow z \in [a]$.

- A consequence of this is that an equivalence relation on a set A *partitions* A into disjoint subsets (the equivalence classes). i.e., each element of A lives in exactly one equivalence class. In terms of the equivalence relation, each member of $[a]$ '*is*' the same as every other member of $[a]$, so is equally qualified to 'represent' the class.

- We describe $A/\!\sim$ as "quotienting (or dividing) A out by ~" since, although in reality $A/\!\sim$ is a set of sets, we can treat it *by abuse of notation*[**] as a set of *representatives*.

  - Consider the example of the set $\mathbb{Z}$ of integers under the relation $x \sim y \iff 5 \mid (y - x)$.[**] It's easy to check that this is an equivalence relation and that there are 5 equivalence classes. We usually denote $\mathbb{Z}/\!\sim$ by $\mathbb{Z}_5 = \{ [0], [1], [2], [3], [4] \}$, and observe that by defining $+$ on $\mathbb{Z}_5$ by $[a] + [b] = [a + b]$ we get equations like $[3] + [4] = [2]$ since $[7] = [2]$. We then typically abuse the notation by writing $\mathbb{Z}_5 = \{ 0, 1, 2, 3, 4 \}$ and $3 + 4 = 2$, occasionally (!!) alerting the casual reader by writing $3 + 4 = 2 \pmod 5$.



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* This phrase means that by swapping the labels, in this case a and b, the same proof works for these new values.

*** This notation means that 5 divides $(y - x)$ exactly, i.e., $(y - x)$ is a (positive or negative or zero) multiple of 5.

** This is a common yet useful trick by mathematicians; it allows an escape from potential drowning in a sea of hieroglyphs, usually engendered by masses of nested definitions. It is also a very common cause of confusion ... typically it helps to have a pad handy on which to scribble the *actual* reality before relaxing into the realms of notationally-pruned clarity. (*Imagine having equivalence classes of constrained sequences of equivalence classes of sets of abstractions of sets!!!! Actually, that's what the real numbers are!)*

- Consider the set $\mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z}$ of ordered pairs of integers. We call this a two-dimensional *integer lattice* (think grid lines on graph paper). Now consider the relation on this defined by $(a, b) \sim (c, d) \iff ad - bc = 0$. It's easy to show that this is an equivalence relation, but what are the equivalence classes?

- Notice that, for example, $(2, 6) \sim (3, 9) \sim (-5, -15) \sim (1, 3)$, so they're all in the same class.

- Suppose we were to define × on $\mathbb{Z}^2/\!\sim$ by $[(a, b)] \times [(c, d)] = [(ac, bd)]$ then, for example, $[(2, 5)] \times [(10, 6)] = [(20, 30)] = [(2, 3)]$. Continuing in the same mold, were we to define + on $\mathbb{Z}^2/\!\sim$ by $[(a, b)] + [(c, d)] = [(ad + bc, bd)]$ then a similar example calculation might be $[(2, 8)] + [(9, 6)] = [(12 + 72, 48)] = [(84, 48)] = [(7, 4)]$.

- It's not hard to see that $\mathbb{Z}^2/\!\sim$ is behaving just like regular fractions, where we notate the equivalence class $[(a, b)]$ conventionally by a / b. Indeed, this is the formal definition of the *rationals*, Q, although by convention we usually start with the set $\mathbb{Z} \times \mathbb{Z}^*$, where $\mathbb{Z}^* = (\mathbb{Z} - \{0\})$ to avoid unpleasant awkwardnesses from having zero denominators!

- We're not going to go into the construction of the real numbers, $\mathbb{R}$, in this course (although you're welcome to ask about it!) since it gets both a bit messy and technical (plus computers tend to balk when asked to compute with infinite precision ... plus this is a *discrete* mathematics course). We'll assume for this course that the reals have now been built* from the rationals, and that they can be represented by potentially infinite decimals.

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* Technically, we start by considering (infinite) sequences of rationals. This of course presumes that we've defined what an infinite sequence is (convergence is irrelevant at this stage). The usual definition of convergence presumes that we have access to any potential limits, and with for example $\sqrt{2}$ not being rational, it's disingenuous to refer to something that's yet to exist! So we define a kind of *moral* convergence, where if the terms are getting arbitrarily closer to one another as one proceeds down the sequence, then we say that the sequence *has the ability to converge* to something (and only one such thing). Then we build equivalence classes of these sequences based on mutual moral convergence, and show that similar sequences (but now of these equivalence classes) actually now have an existing equivalence class to converge to (so nothing more needs to be added), and it's these equivalence classes that are the real numbers.