- There are some amusing things we can do to build up a collection of sets from a starting ingredient. For example, let's start with the set $A = \varnothing$, and notice that the set $B = \{A\} = \{\varnothing\}$ is different from the set A itself. In particular, the set A has nothing in it, whereas B does! We could write $B = A \cup \{A\}$, which is true by equation 4 on the previous slide.

- This is a process we could repeat, so let's define the set $C = B \cup \{B\}$ and observe that C is different from both A and B. In particular, $C = \{ A , \{A\} \} = \{ A , B \}$.

- Clearly this process can be repeated indefinitely, where at each successive step we define the next set Y from the current set X by $Y = X \cup \{X\}$.

- Notice that we're actually starting with nothing (well, starting with the empty set!), and that if we *defined* a function or process $\sigma(x) = x \cup \{x\}$, then we could *define* $0 = \varnothing$ and then $1 = \sigma(0)$, followed by *defining* $2 = \sigma(1)$, $3 = \sigma(2)$, $4 = \sigma(3)$, *et simile*. So using this *successor function* σ allows us to *construct* all the positive integers (a.k.a., the *natural numbers*, denoted $\mathbb{N}$) from just the empty set itself! This process is called an *inductive definition*.*

- Now that we have the natural numbers, it's easy enough to construct the rest of the integers, denoted $\mathbb{Z}$, by adding the solutions x to the equation $n + x = 0$ for each $n \in \mathbb{N}$, or more formally, $\mathbb{Z} = \{ x \mid n + x = 0, n \in \mathbb{N} \} \cup \mathbb{N}$.**

- Before we can construct the fractions, a.k.a., rational numbers (denoted Q), we'll need to take a short detour to define coordinate-like animals which live in *Cartesian* spaces, as well as being much more formal and precise about the word '*is*'.

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* We'll see this idea reappear later in the context of *proof by induction*.

** Of course, this presumes that we've defined already the meaning of '+' on $\mathbb{N}$. This can be done inductively by defining the action of '+1' via σ. Extending this to a definition of '+' on $\mathbb{Z}$ is then simply a matter of defining '+1' *via* '+(−1)' for our newly created *negative* numbers, and then proceeding inductively.

- Define the *ordered pair* $(a , b) = \{ a , \{a , b\} \}$.* Notice that $(a , b) \neq (b , a)$. It's easy to extend this definition to *ordered triples* (a , b , c) by defining $(a , b , c) = (a , (b , c))$, and inductively to *ordered n-tuples*, being a list of n objects in a specified order.**
- We define the *Cartesian product* of two sets A and B by $A \times B = \{ (a , b) \mid a \in A , b \in B \}$, and again this can be extended to Cartesian products of an finite number of sets. Notice that $A \times (B \times C) = (A \times B) \times C$, which allows us to write $A \times B \times C$ unambiguously ... this too extends to products of a finite number of sets. A familiar example is of course the plane, $\mathbb{R}^2$, which is the Cartesian product of its two axes of values.***
- Given two sets A and B, we define a *relation* ρ between them to be a subset $\rho \subseteq A \times B$. Notice that this definition permits the situation where some members of A might not be 'related' to any members of B, and *vice versa*. Typically, if $(a , b) \in \rho$ then we'll write $a \sim b$ as an easier way to signify that a is related to b. Since we're dealing with ordered pairs, this also means that there's no reason to suppose that $a \sim b$ implies that $b \sim a$, so relations are directional. Moreover, if $\rho \subseteq A \times A$, there's no reason to suppose that $a \sim a$ for any $a \in A$.
- If ρ is a relation *on* A, i.e., $\rho \subseteq A \times A$, then it is an *equivalence relation* if it also satisfies:

  (i) $a \sim a \ \ \forall\, a \in A$ (*reflexive*)

  (ii) $a \sim b \implies b \sim a \ \ \forall\, a, b \in A$ (*symmetric*)

  (iii) $(a \sim b) \wedge (b \sim c) \implies (a \sim c) \ \ \forall\, a, b, c \in A$ (*transitive*)

Notice that this generalises the notion of 'equals'. Being able to treat two things as being 'the same' if they don't differ in any way that we care about is actually a tremendously powerful idea, allowing us to focus on more significant distinctions.

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* This way we've been able to define it in terms of previously established animals, namely sets.

** Ordered n-tuples are typically used to describe points in n-dimensional spaces. Extending this idea further to describe infinite sequences as ordered ∞–tuples is intuitive, but introduces some logical delicacies, especially if one wants to be able to manipulate such sequences by adding or multiplying.

*** I'm being deliberately sloppy with language here, we will be rather more formal later.. Indeed, if you look more carefully at the way we've defined cartesian products, you'll see that $A \times (B \times C) \neq (A \times B) \times C$. However, since we'd really like to be able to write things like $(a , b , c , d , e) \in \mathbb{R}^5$, and move fluidly between thinking of it as being described by *5* one-dimensional real axes, or as a *3*-dimensional space plus two one-dimensional axes, or as a pair of planes plus a line, then we'll want to make this 'equality' kosher. This we can do by defining the obvious equivalence relation!