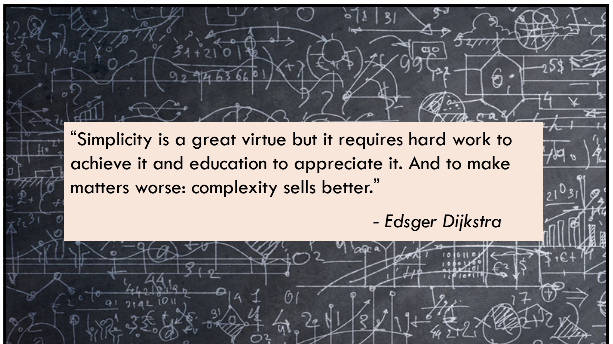


"Simplicity is a great virtue but it requires hard work to achieve it and education to appreciate it. And to make matters worse: complexity sells better."

- Edsger Dijkstra

ASYMPTOTIC COMPLEXITY

Lecture 10
CS2110 – Spring 2018

What Makes a Good Algorithm?

2

Suppose you have two possible algorithms that do the same thing; which is *better*?

What do we mean by *better*?
- Faster?
- Less space?
- Easier to code?
- Easier to maintain?
- Required for homework?

FIRST, Aim for simplicity, ease of understanding, correctness.

SECOND, Worry about efficiency only when it is needed.

How do we measure speed of an algorithm?

Basic Step: one "constant time" operation

3

Constant time operation: its time doesn't depend on the size or length of anything. Always roughly the same. Time is bounded above by some number

Basic step:
- Input/output of a number
- Access value of primitive-type variable, array element, or object field
- assign to variable, array element, or object field
- do one arithmetic or logical operation
- method call (not counting arg evaluation and execution of method body)

Counting Steps

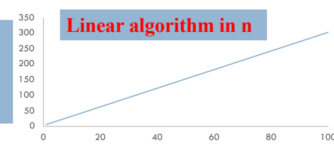
4

```
// Store sum of 1..n in sum
sum= 0;
// inv: sum = sum of 1..(k-1)
for (int k= 1; k <= n; k= k+1){
    sum= sum + k;
}
```

Statement:	# times done
sum= 0;	1
k= 1;	1
k <= n	n+1
k= k+1;	n
sum= sum + k;	n
Total steps:	3n + 3

All basic steps take time 1. There are n loop iterations. Therefore, takes time proportional to n.

Linear algorithm in n



Not all operations are basic steps

5

```
// Store n copies of 'c' in s
s= "";
// inv: s contains k-1 copies of 'c'
for (int k= 1; k <= n; k= k+1){
    s= s + 'c';
}
```

Statement:	# times done
s= "";	1
k= 1;	1
k <= n	n+1
k= k+1;	n
s= s + 'c';	n
Total steps:	3n + 3

Concatenation is not a basic step. For each k, catenation creates and fills k array elements.

String Concatenation

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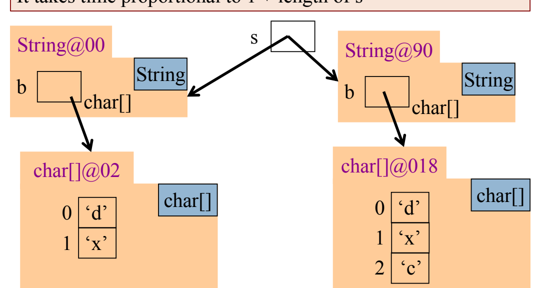
s= s + "c"; is NOT constant time.
It takes time proportional to 1 + length of s



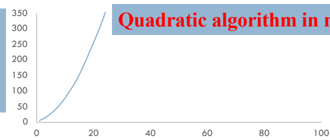
Not all operations are basic steps

7

```
// Store n copies of 'c' in s
s= "";
// inv: s contains k-1 copies of 'c'
for (int k= 1; k <= n; k= k+1){
    s= s + 'c';
}
```

Statement:	# times	# steps
s= "";	1	1
k= 1;	1	1
k <= n	n+1	1
k= k+1;	n	1
s= s + 'c';	n	k
Total steps:	n*(n-1)/2 + 2n + 3	

Concatenation is not a basic step. For each k, catenation creates and fills k array elements.



Quadratic algorithm in n

Linear versus quadractic

8

```
// Store sum of 1..n in sum
sum= 0;
// inv: sum = sum of 1..(k-1)
for (int k= 1; k <= n; k= k+1)
    sum= sum + n
```

Linear algorithm

```
// Store n copies of 'c' in s
s= "";
// inv: s contains k-1 copies of 'c'
for (int k= 1; k = n; k= k+1)
    s= s + 'c';
```

Quadratic algorithm

In comparing the runtimes of these algorithms, the exact number of basic steps is not important. What's important is that
One is linear in n—takes time proportional to n
One is quadratic in n—takes time proportional to n^2

Looking at execution speed

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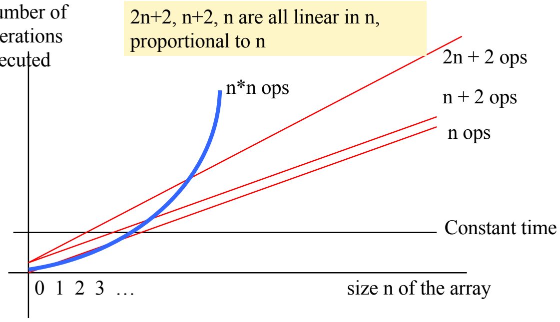
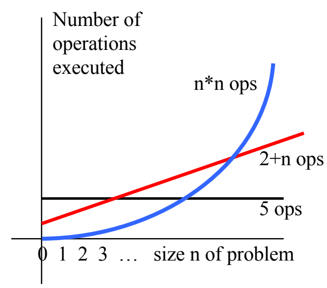
2n+2, n+2, n are all linear in n, proportional to n



What do we want from a definition of "runtime complexity"?

10



1. Distinguish among cases for large n, not small n

2. Distinguish among important cases, like
- n*n basic operations
- n basic operations
- log n basic operations
- 5 basic operations

3. Don't distinguish among trivially different cases.
- 5 or 50 operations
- n, n+2, or 4n operations

"Big O" Notation

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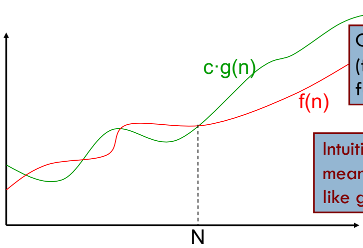
Formal definition: f(n) is O(g(n)) if there exist constants c > 0 and N ≥ 0 such that for all n ≥ N, f(n) ≤ c·g(n)



Get out far enough (for n ≥ N) f(n) is at most c·g(n)

Intuitively, f(n) is O(g(n)) means that f(n) grows like g(n) or slower

Prove that $(2n^2 + n)$ is $O(n^2)$

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Formal definition: f(n) is O(g(n)) if there exist constants c > 0 and N ≥ 0 such that for all n ≥ N, f(n) ≤ c·g(n)

Example: Prove that $(2n^2 + n)$ is $O(n^2)$

Methodology:

Start with f(n) and slowly transform into c · g(n):
- Use = and <= and < steps
- At appropriate point, can choose N to help calculation
- At appropriate point, can choose c to help calculation

Prove that $(2n^2 + n)$ is $O(n^2)$

13

Formal definition: f(n) is O(g(n)) if there exist constants $c > 0$ and $N \geq 0$ such that for all $n \geq N$, $f(n) \leq c \cdot g(n)$

Example: Prove that $(2n^2 + n)$ is $O(n^2)$

f(n)
= <definition of f(n)>
$2n^2 + n$
<= <for $n \geq 1$, $n \leq n^2$>
$2n^2 + n^2$
= <arith>
$3*n^2$
= <definition of $g(n) = n^2$>
3*g(n)

Transform f(n) into $c \cdot g(n)$:
- Use =, <= , < steps
- Choose N to help calc.
- Choose c to help calc

Choose
N = 1 and c = 3

Prove that 100 n + log n is O(n)

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Formal definition: f(n) is O(g(n)) if there exist constants $c > 0$ and $N \geq 0$ such that for all $n \geq N$, $f(n) \leq c \cdot g(n)$

f(n)
= <put in what f(n) is>
100 n + log n
<= <We know $\log n \leq n$ for $n \geq 1$>
100 n + n
= <arith>
101 n
= <g(n) = n>
101 g(n)

Choose
N = 1 and c = 101

O(…) Examples

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Let $f(n) = 3n^2 + 6n - 7$
- f(n) is $O(n^2)$
- f(n) is $O(n^3)$
- f(n) is $O(n^4)$
- …

$p(n) = 4 n \log n + 34 n - 89$
- p(n) is O(n log n)
- p(n) is $O(n^2)$

$h(n) = 20 \cdot 2^n + 40n$

h(n) is $O(2^n)$

a(n) = 34
- a(n) is O(1)

Only the *leading* term (the term that grows most rapidly) matters

If it's $O(n^2)$, it's also $O(n^3)$ etc! However, we always use the smallest one

Do NOT say or write f(n) = O(g(n))

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Formal definition: f(n) is O(g(n)) if there exist constants $c > 0$ and $N \geq 0$ such that for all $n \geq N$, $f(n) \leq c \cdot g(n)$

f(n) = O(g(n)) is simply WRONG. Mathematically, it is a disaster. You see it sometimes, even in textbooks. Don't read such things.

Here's an example to show what happens when we use = this way.

We know that n+2 is O(n) and n+3 is O(n). Suppose we use =

n+2 = O(n)
n+3 = O(n)

But then, by transitivity of equality, we have n+2 = n+3.
We have proved something that is false. Not good.

Problem-size examples

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- Suppose a computer can execute 1000 operations per second; how large a problem can we solve?

operations	1 second	1 minute	1 hour
n	1000	60,000	3,600,000
n log n	140	4893	200,000
n^2	31	244	1897
$3n^2$	18	144	1096
n^3	10	39	153
2^n	9	15	21

Commonly Seen Time Bounds

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O(1)	constant	excellent
O(log n)	logarithmic	excellent
O(n)	linear	good
O(n log n)	n log n	pretty good
$O(n^2)$	quadratic	maybe OK
$O(n^3)$	cubic	maybe OK
$O(2^n)$	exponential	too slow

Java Lists

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- java.util defines an interface List<E>
- implemented by multiple classes:
  - ArrayList
  - LinkedList

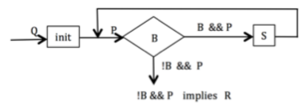
Linear search for v in b[0..]

20

// Store value in i to truthify b[0..i-1] < v <= b[i..]
// Precondition: b is sorted

If v in b, set i to index of first occurrence of v

If v not in b, set i so that all elements of b that are < v are to the left of index i.



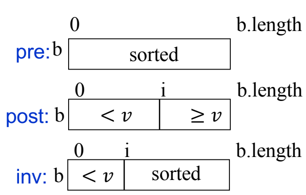
Methodology:
1. Define pre and post conditions.
2. Draw the invariant as a combination of pre and post.
3. Develop loop using 4 loopy questions.

Practice doing this!

Linear search for v in b[0..]

21

// Store value in i to truthify b[0..i-1] < v <= b[i..]
// Precondition: b is sorted

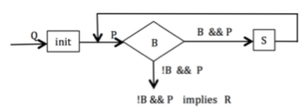


Methodology:
1. Define pre and post conditions.
2. Draw the invariant as a combination of pre and post.
3. Develop loop using 4 loopy questions.

Practice doing this!

The Four Loopy Questions

22

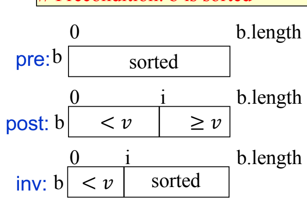


- Does it start right?
  Is {Q} init {P} true?
- Does it continue right?
  Is {P && B} S {P} true?
- Does it end right?
  Is P && !B => R true?
- Will it get to the end?
  Does it make progress toward termination?

Linear search for v in b[0..]

23

// Store value in i to truthify b[0..i-1] < v <= b[i..]
// Precondition: b is sorted




```
i= 0;
while ( i < b.length &&
        b[i] < v )       {
    i= i+1;
}
```

Each iteration takes constant time.

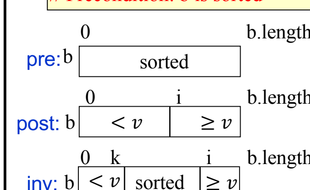
Worst case: b.length iterations

Linear algorithm: O(b.length)

Binary search for v in b[0..]

24

// Store value in i to truthify b[0..i-1] < v <= b[i..]
// Precondition: b is sorted



Methodology:
1. Define pre and post conditions.
2. Draw the invariant as a combination of pre and post.
3. Develop loop using 4 loopy questions.

Practice doing this!

Binary search for v in b[0..]

25

// Store value in i to truthify b[0..i-1] < v <= b[i..]
// Precondition: b is sorted

pre: b [sorted] (0 … b.length)

inv: b [< v | sorted | ≥ v] (0, k, i, b.length)

```
k= -1;
i= b.length;
```

Make invariant true initially

Binary search for v in b[0..]

26

// Store value in i to truthify b[0..i-1] < v <= b[i..]
// Precondition: b is sorted

post: b [< v | ≥ v] (0, i, b.length)

inv: b [< v | sorted | ≥ v] (0, k, i, b.length)

```
k= -1;
i= b.length;
while (k < i-1) {


}
```

Determine loop condition B:
!B && inv imply post

Binary search for v in b[0..]

27

// Store value in i to truthify b[0..i-1] < v <= b[i..]
// Precondition: b is sorted

inv: b [< v | sorted | ≥ v] (0, k, **j**, i, b.length)

```
k= -1;
i= b.length;
while (k < i-1) {
  int j= (k+i)/2;
  // k < j < i
  Set one of k, i to j

}
```

Figure out how to make progress toward termination.
Envision cutting size of b[k+1..i-1] in half

Binary search for v in b[0..]

28

// Store value in i to truthify b[0..i-1] < v <= b[i..]
// Precondition: b is sorted

inv: b [< v | sorted | ≥ v] (0, k, **j**, i, b.length)

inv: b [< v | < v | < v | | ≥ v] (0, k, j, i)

```
k= -1;
i= b.length;
while (k < i-1) {
  int j= (k+i)/2;
  // k < j < i
  if (b[j] < v) k= j
  else i= j;
}
```

Figure out how to make progress toward termination.
Cut size of b[k+1..i-1] in half

Binary search for v in b[0..]

29

// Store value in i to truthify b[0..i-1] < v <= b[i..]
// Precondition: b is sorted

This algorithm is better than binary searches that stop when v is found.

1. Gives good info when v not in b.
2. Works when b is empty.
3. Finds first occurrence of v, not arbitrary one.
4. Correctness, including making progress, easily seen using invariant

```
k= -1;
i= b.length;
while (k < i-1) {
  int j= (k+i)/2;
  if (b[j]<v) k= j;
  else i= j;
}
```

Each iteration takes constant time.

Worst case: log(b.length) iterations

Logarithmic: O(log(b.length))

Dutch National Flag Algorithm

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Dutch National Flag Algorithm

Dutch national flag. Swap b[0..n-1] to put the reds first, then the whites, then the blues. That is, given precondition Q, swap values of b[0.n] to truthify postcondition R:

	0			n
Q: b	?			
R: b	reds	whites	blues	
P1: b	reds	whites	blues	?
P2: b	reds	whites	?	blues

Dutch National Flag Algorithm: invariant P1

	0	h	k	p	n
Q: b	?				
R: b	reds	whites	blues		
P1: b	reds	whites	blues	?	

```
h= 0; k= h; p= k;
while ( p != n ) {
  if (b[p] blue)  p= p+1;
  else if (b[p] white) {
      swap b[p], b[k];
      p= p+1; k= k+1;
  }
  else { // b[p] red
      swap b[p], b[h];
      swap b[p], b[k];
      p= p+1; h=h+1; k= k+1;
  }
}
```

Dutch National Flag Algorithm: invariant P2

	0	h	k	p	n
Q: b	?				
R: b	reds	whites	blues		
P2: b	reds	whites	?	blues	

```
h= 0; k= h; p= n;
while ( k != p ) {
  if (b[k] white)  k= k+1;
  else if (b[k] blue) {
      p= p-1;
      swap b[k], b[p];
  }
  else { // b[k] is red
      swap b[k], b[h];
      h= h+1; k= k+1;
  }
}
```

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Asymptotically, which algorithm is faster?

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Invariant 1

0	h	k	p	n
reds	whites	blues	?	

```
h= 0; k= h; p= k;
while ( p != n ) {
  if (b[p] blue)       p= p+1;
  else if (b[p] white) {
      swap b[p], b[k];
      p= p+1; k= k+1;
  }
  else { // b[p] red
      swap b[p], b[h];
      swap b[p], b[k];
      p= p+1; h=h+1; k= k+1;
  }
}
```

Invariant 2

0	h	k	p	n
reds	whites	?	blues	

```
h= 0; k= h; p= n;
while ( k != p ) {
  if (b[k] white)
  else if (b[k] blue) {   k= k+1;
      p= p-1;
      swap b[k], b[p];
  }
  else { // b[k] is red
      swap b[k], b[h];
      h= h+1; k= k+1;
  }
}
```