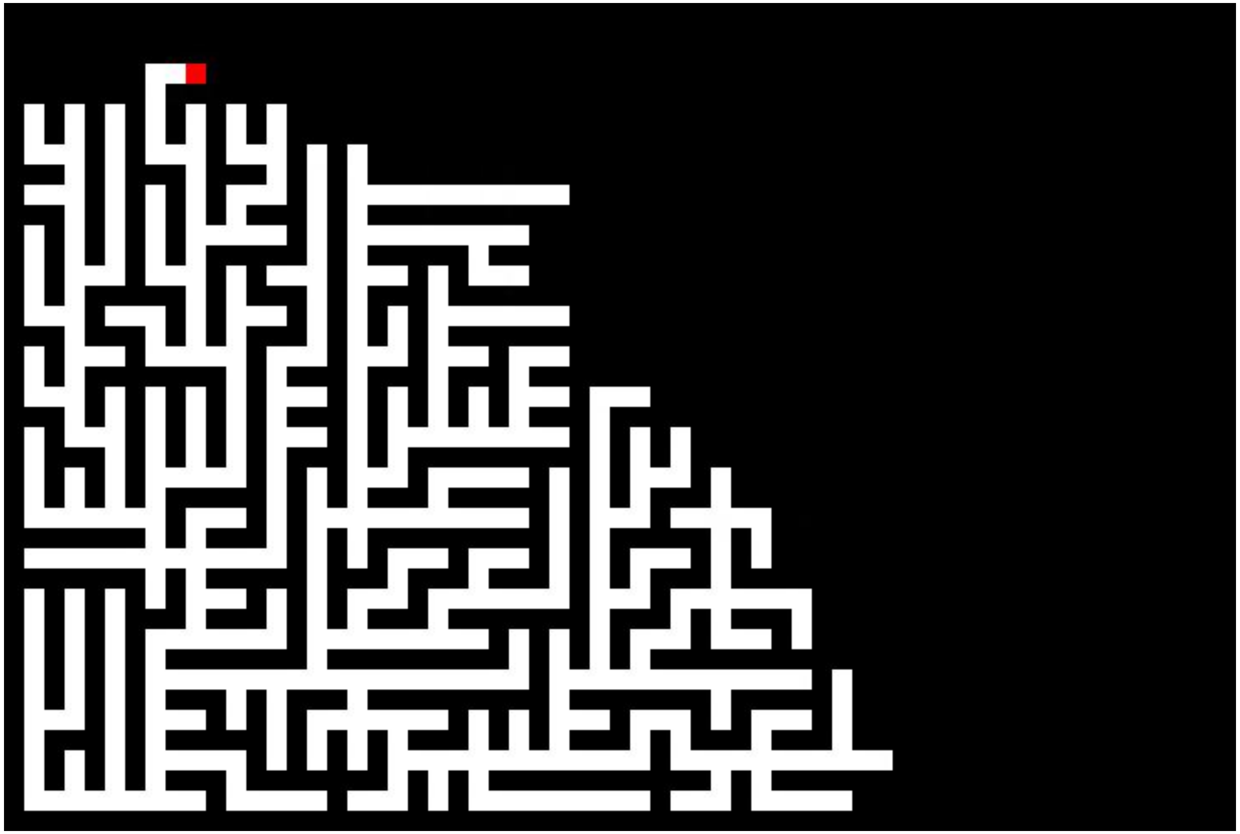


Spanning Trees

Lecture 22

CS2110 – Spring 2017

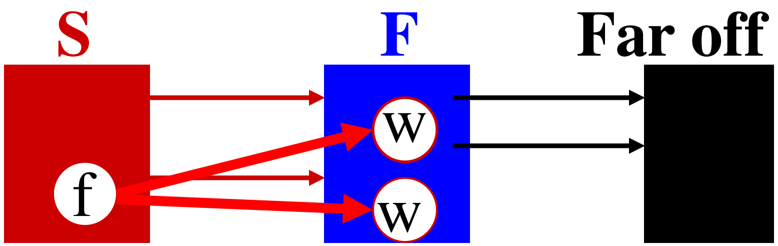
Prelim 2, assignments

- Prelim 2 is Tuesday. See the course webpage for details.
- Scope: up to but not including today's lecture. See the review guide for details.
  - Deadline for submitting conflicts has passed.

- A6 was due last night. Late penalty 3 points per day, up to 3 days. No exceptions – the solution is used in A7!

- A7 due next Thursday 4/27. It's short; 30-40 lines including comments. Do it before the prelim and it doubles as studying Dijkstra!

A Note on Dijkstra



1. **For s, d[s]** is length of shortest v → s path.

2. **For f, d[f]** is length of shortest v → f path of form

3. Edges leaving S go to F.

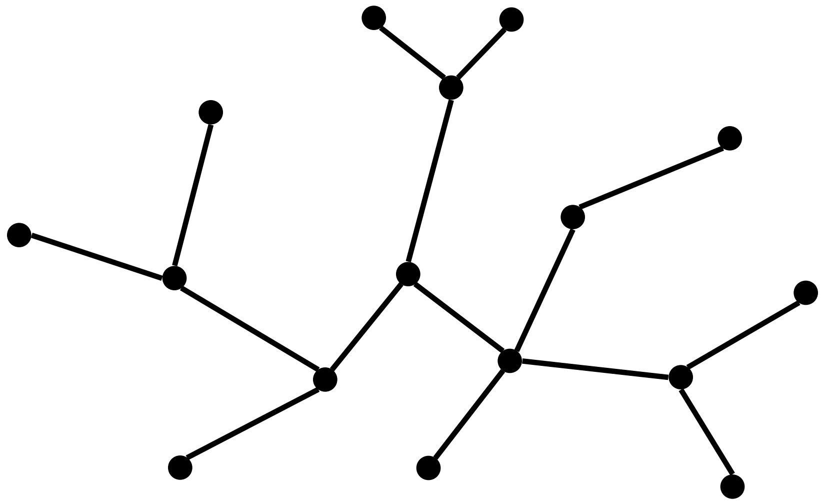
Theorem: For a node **f** in **F** with min d value, d[f] is its shortest path length

```
S= { }; F= { v }; d[v]= 0;
while  F ≠ {} {
    f= node in F with min d value;
    Remove f from F, add it to S;
    for each neighbor w of f {
        if (w not in S or F) {
            d[w]= d[f] + wgt(f, w);
            add w to F;
        } else
          if (d[f] + wgt (f,w) < d[w]) {
              d[w]= d[f] + wgt(f, w);
        }
    }
}
```

Undirected trees

An undirected graph is a *tree* if there is exactly one simple path between any pair of vertices

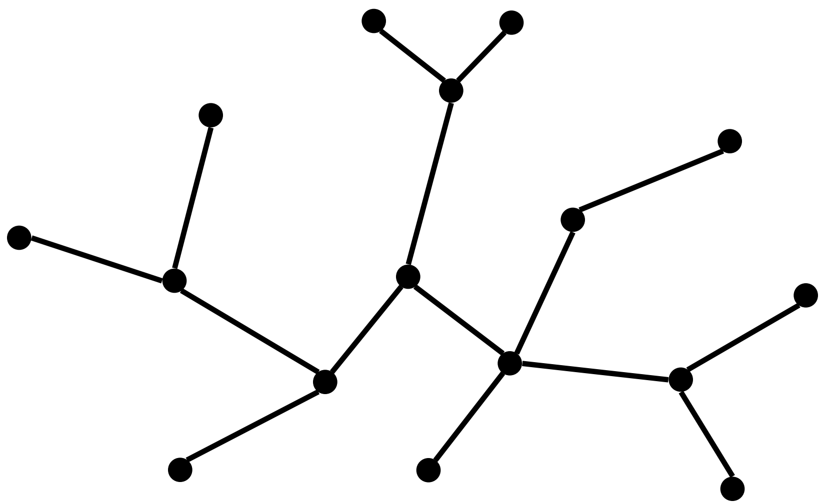
What's the root?
It doesn't matter!
Any vertex can be root.



Facts about trees

- $\#E = \#V - 1$
- connected
- no cycles

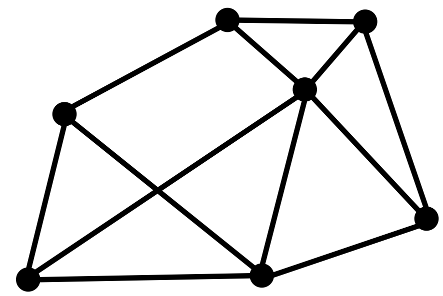
Any two of these properties imply the third and thus imply that the graph is a tree



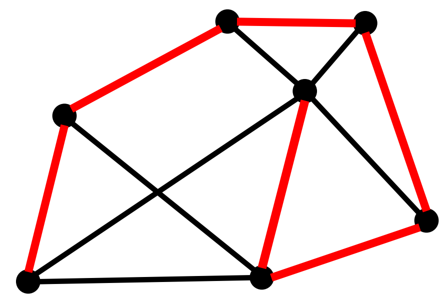
Spanning trees

A ***spanning tree*** of a **connected undirected** graph (V, E) is a subgraph (V, E') that is a tree

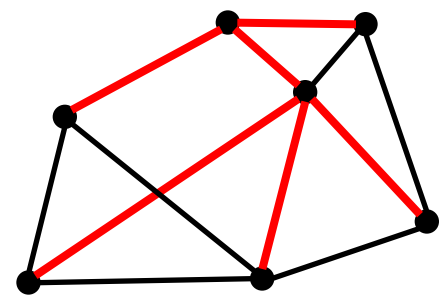
- Same set of vertices V
- E' $\subseteq$ E
- (V, E') is a tree



- Same set of vertices V
- Maximal set of edges that contains no cycle

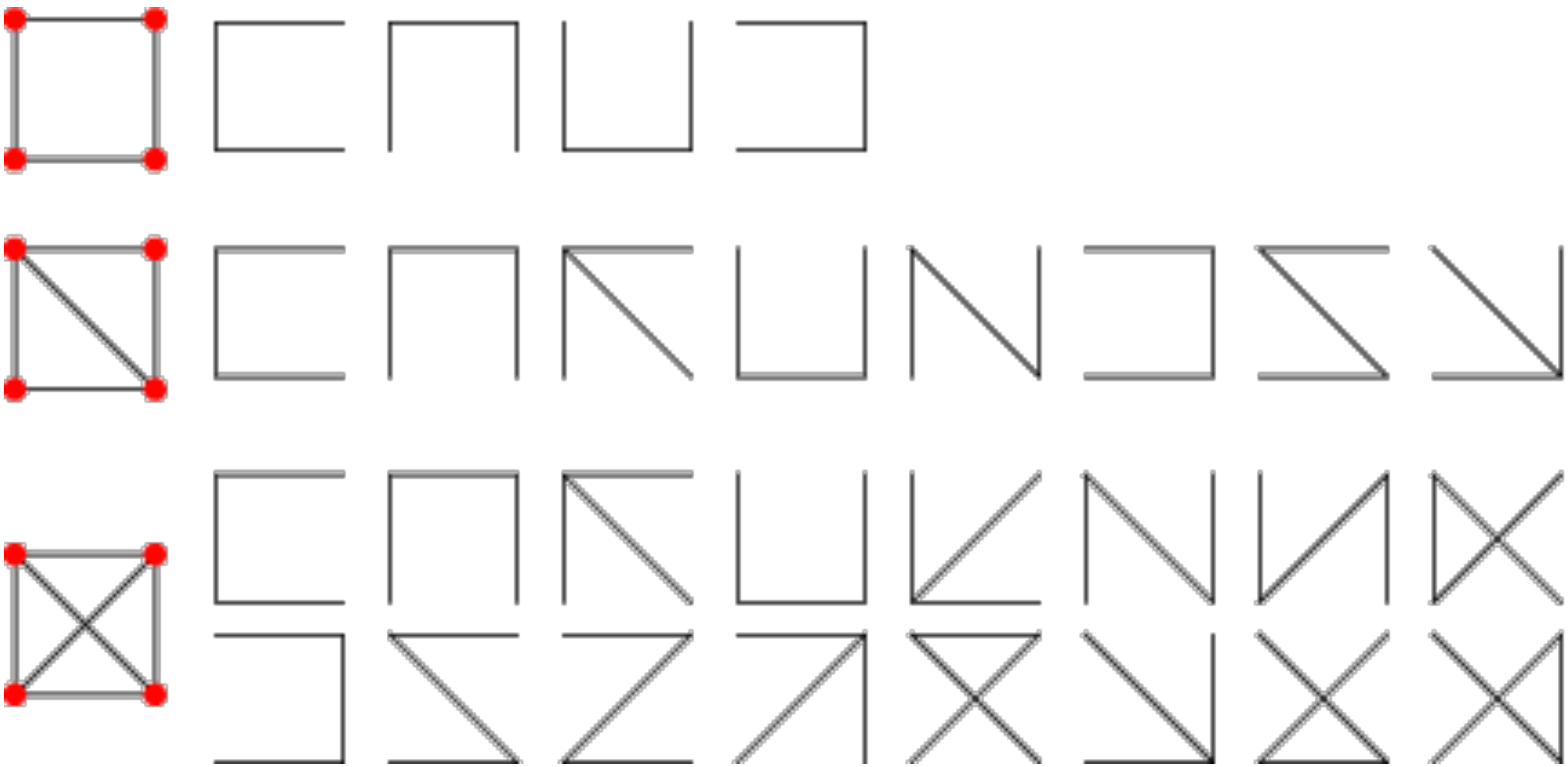


- Same set of vertices V
- Minimal set of edges that connect all vertices



Three equivalent definitions

Spanning trees: examples



http://mathworld.wolfram.com/SpanningTree.html

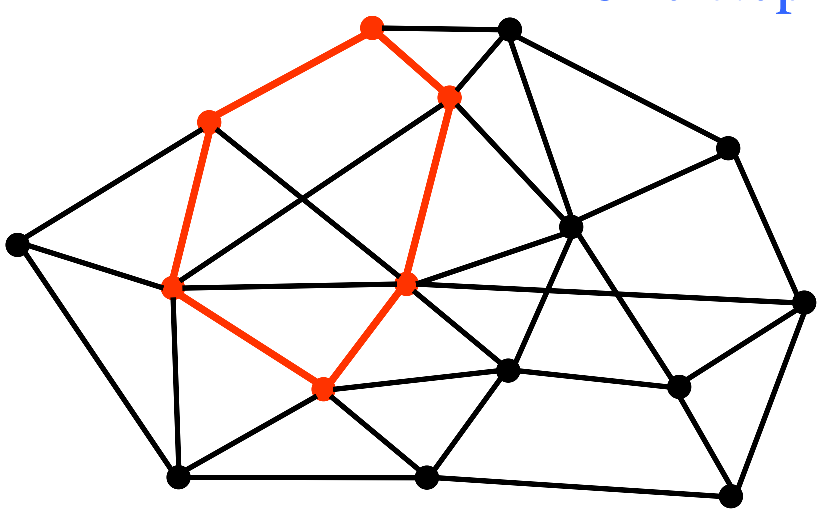
Finding a spanning tree: Subtractive method

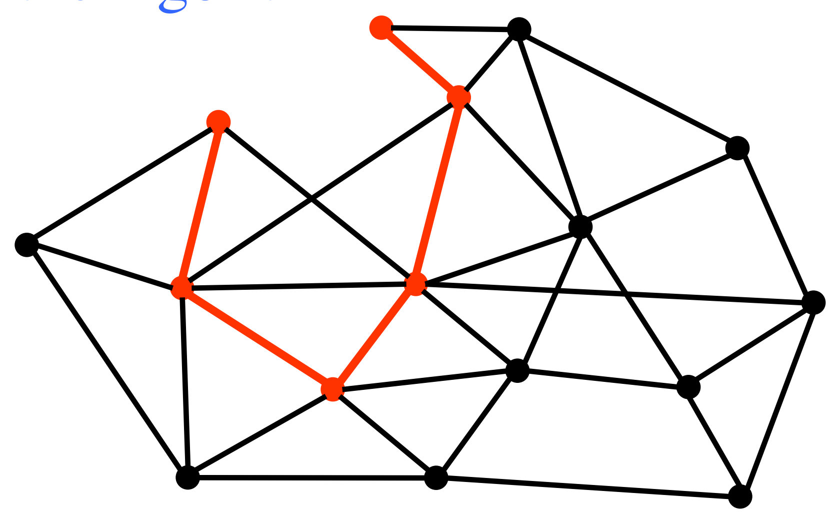
- Start with the whole graph – it is connected
- While there is a cycle:
  Pick an edge of a cycle and throw it out – the graph is still connected (why?)

Maximal set of edges that contains no cycle

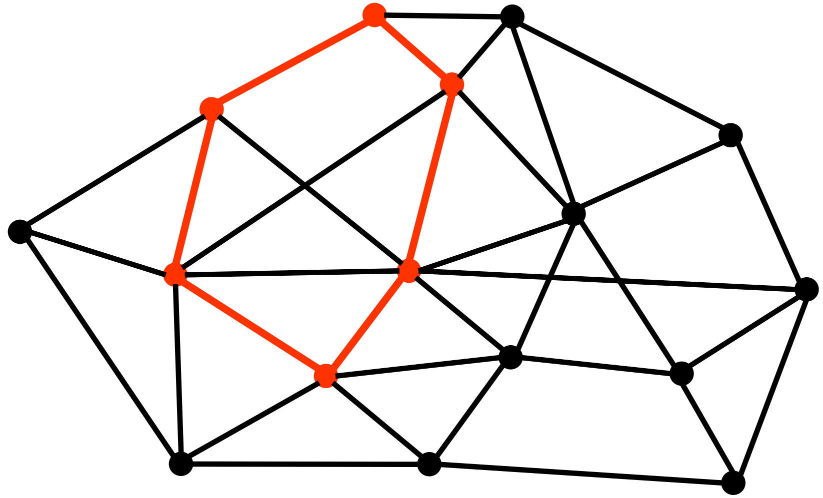
nondeterministic algorithm

One step of the algorithm



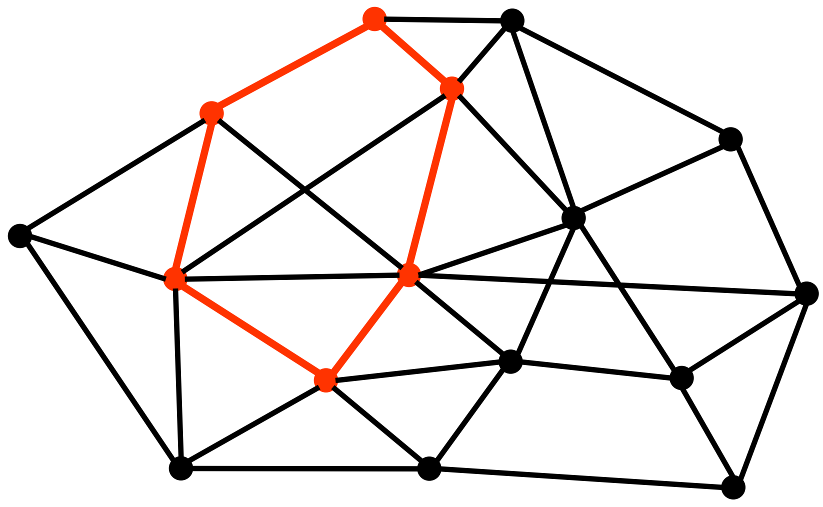


Aside: How can you find a cycle in an undirected graph?



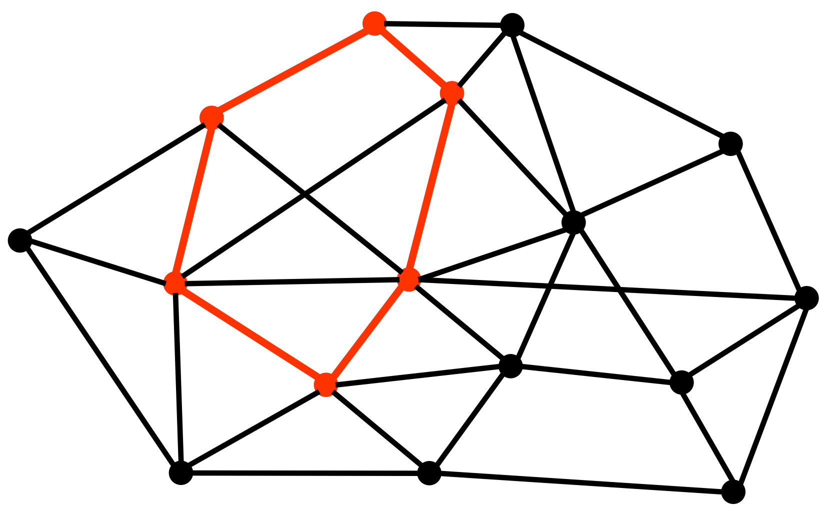
Aside: How can you find a cycle in an undirected graph?

```
/** Visit all nodes REACHABLE* from u. Pre: u is unvisited. . */
public static void dfs(int u) {
    Stack s= (u);
    while (s is not empty) {
        u= s.pop();
        if (u has not been visited) {
            visit u;
            for each edge (u, v) leaving u:
                s.push(v);
        }
    }
}
```



Aside: How can you find a cycle in an undirected graph?

```
/** true if the nodes reachable from u have a cycle. */
public static boolean hasCycle(int u) {
    Stack s= (u);
    while (s is not empty) {
        u= s.pop();
        if (u has been visited) {
            return true;
        } else {
            visit u;
            for each edge (u, v) leaving u:
                s.push(v);
        }
    }
    return false;
}
```



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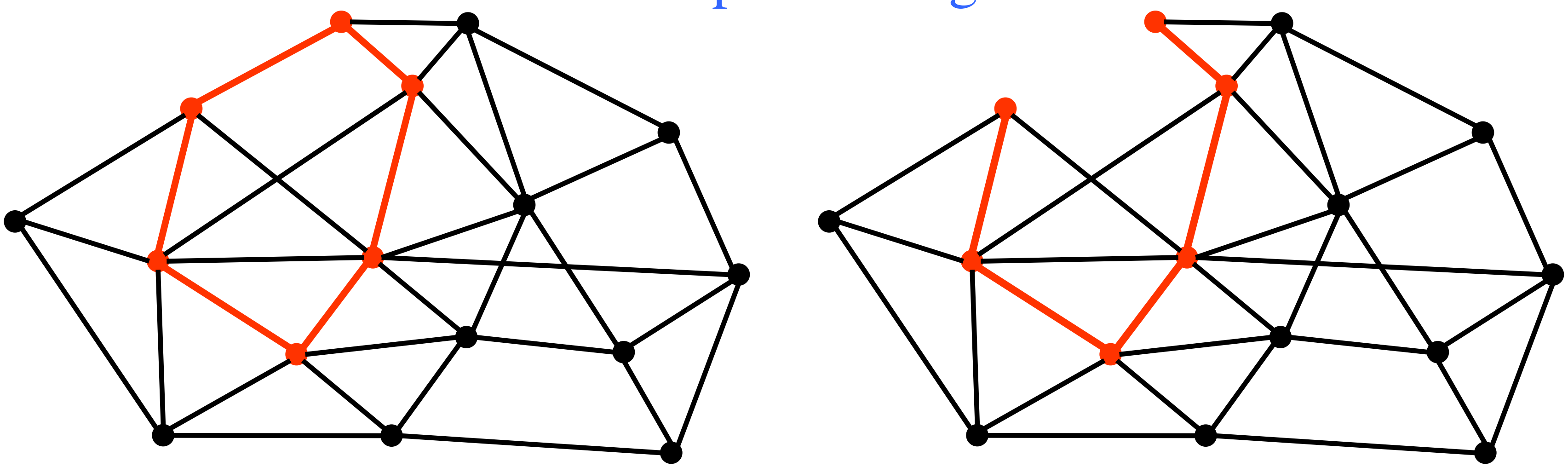
Maximal set of edges that contains no cycle

nondeterministic algorithm

One step of the algorithm



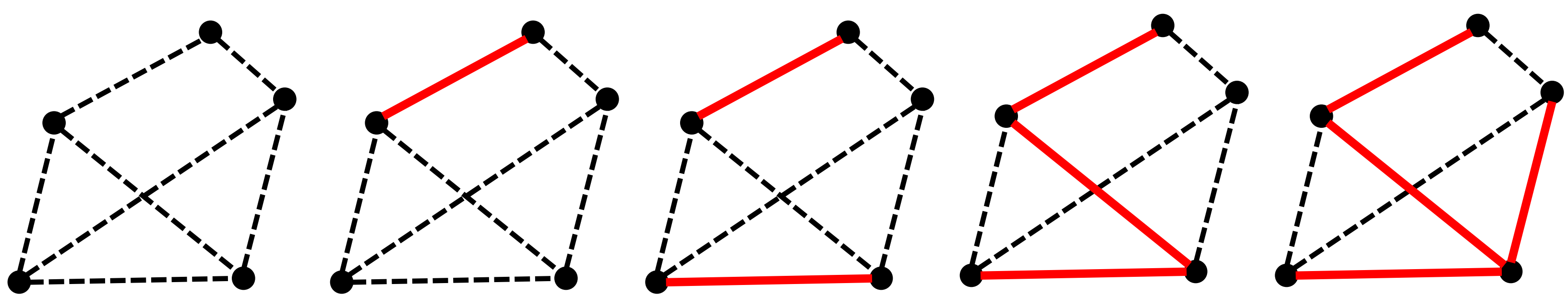
Finding a spanning tree: Additive method

- Start with no edges
- While the graph is not connected:
  Choose an edge that connects 2 **connected components** and add it – the graph still has no cycle (why?)

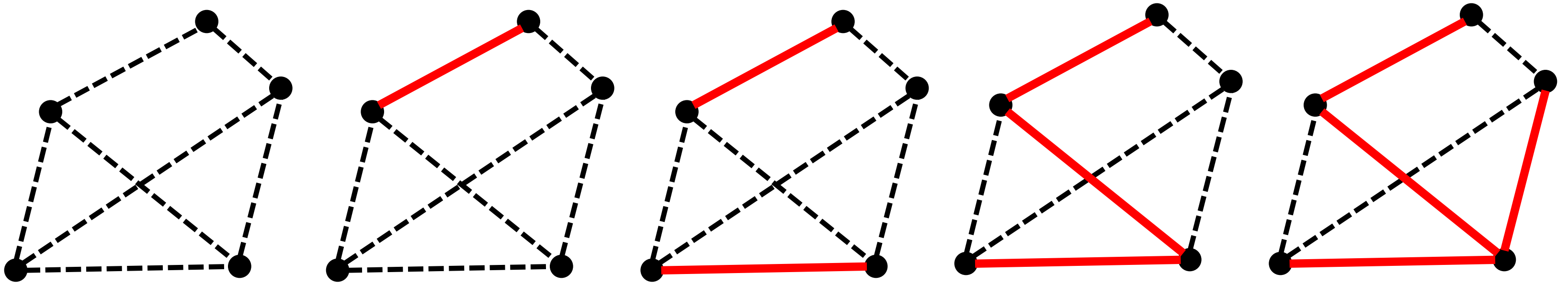
Minimal set of edges that connect all vertices

nondeterministic algorithm

Tree edges will be red.
Dashed lines show original edges.
Left tree consists of 5 connected components, each a node

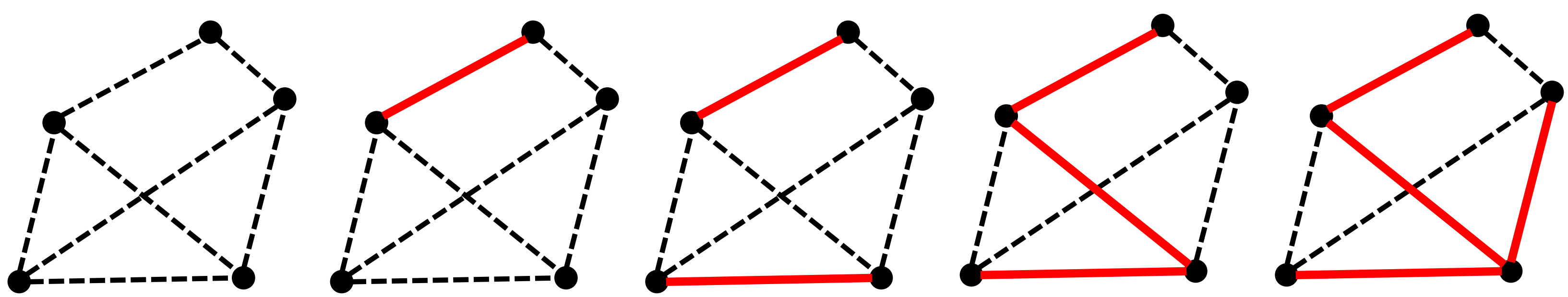


Aside: How do you find connected components?



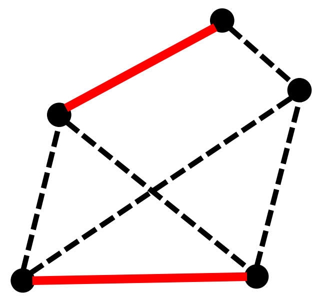
Aside: How do you find connected components?

```
/** Visit all nodes REACHABLE* from u. Pre: u is unvisited. . */
public static void dfs(int u) {
    Stack s= (u);
    while (s is not empty) {
        u= s.pop();
        if (u has not been visited) {
            visit u;
            for each edge (u, v) leaving u:
                s.push(v);
        }
    }
}
```



Aside: How do you find connected components?

```
/** Return the set of nodes in u's connected component. */
public static Set<int> getComponent(int u) {
    Stack s= (u);
    Set C = ();
    while (s is not empty) {
        u= s.pop();
        if (u has not been visited) {
            visit u;
            C.add(u);
            for each edge (u, v) leaving u:
                s.push(v);
        }
    }
    return C;
}
```



Finding a spanning tree: Additive method

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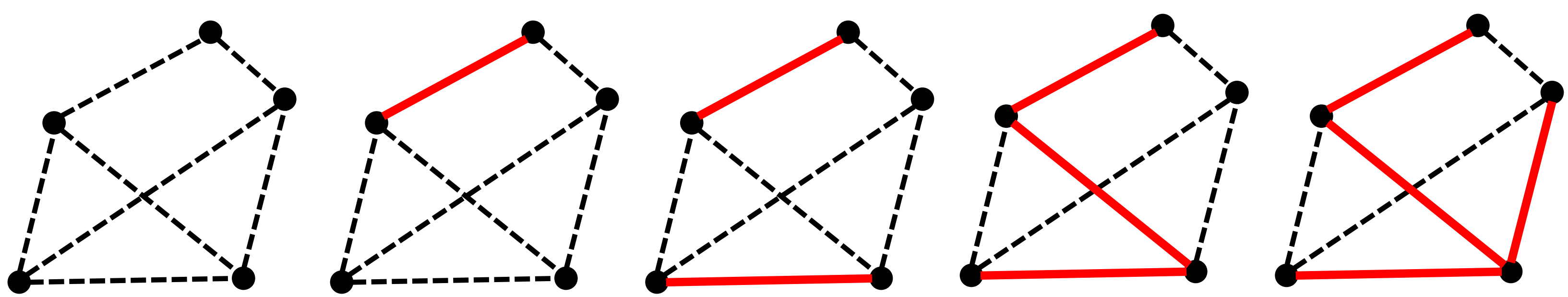


Spanning trees: examples

http://mathworld.wolfram.com/SpanningTree.html

Minimum spanning trees

- Suppose edges are weighted (> 0)
- We want a spanning tree of *minimum cost* (sum of edge weights)

- Some graphs have exactly one minimum spanning tree. Others have several trees with the same minimum cost, each of which is a minimum spanning tree

- Useful in network routing & other applications. For example, to stream a video

Greedy algorithm

A greedy algorithm follows the heuristic of making a locally optimal choice at each stage, with the hope of finding a global optimum.

Example. Make change using the fewest number of coins.
Make change for n cents, n < 100 (i.e. < $1)
Greedy: At each step, choose the largest possible coin

If n >= 50 choose a half dollar and reduce n by 50;
If n >= 25 choose a quarter and reduce n by 25;
As long as n >= 10, choose a dime and reduce n by 10;
If n >= 5, choose a nickel and reduce n by 5;
Choose n pennies.

Greedy algorithm —doesn't always work!

A greedy algorithm follows the heuristic of making a locally optimal choice at each stage, with the hope of finding a global optimum. Doesn't always work

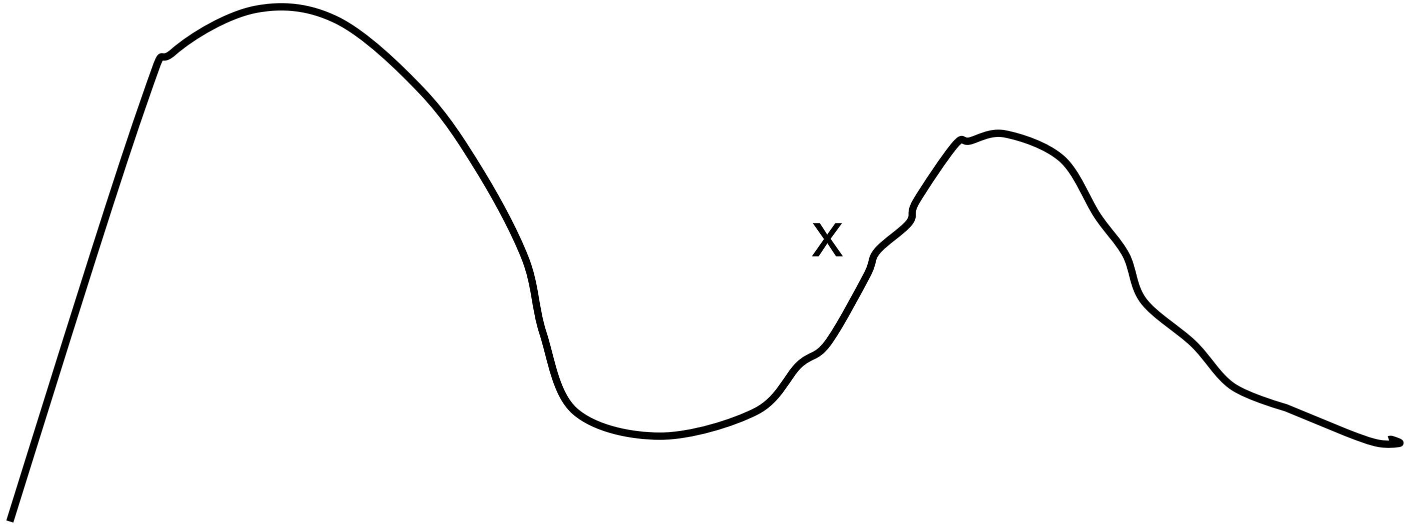
Example. Make change using the fewest number of coins.
Coins have these values: 7, 5, 1
Greedy: At each step, choose the largest possible coin

Consider making change for 10.
The greedy choice would choose: 7, 1, 1, 1.
But 5, 5 is only 2 coins.

Greediness doesn't work here

You're standing at point x, and your goal is to climb the highest mountain.

Two possible steps: down the hill or up the hill. The greedy step is to walk up hill. But that is a local optimum choice, not a global one. Greediness fails in this case.



Finding a minimal spanning tree

Suppose edges have > 0 weights
Minimal spanning tree: sum of weights is a minimum

We show two greedy algorithms for finding a minimal spanning tree.

They are versions of the basic additive method we have already seen: at each step add an edge that does not create a cycle.

Kruskal: add an edge with minimum weight. Can have a forest of trees.

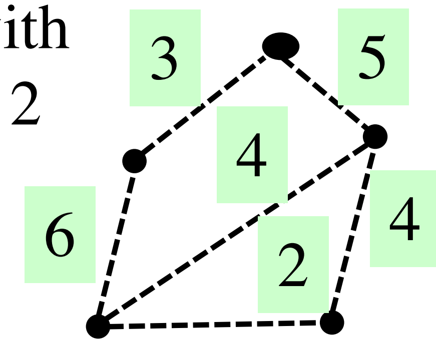
Prim (JPD): add an edge with minimum weight but so that the added edges (and the nodes at their ends) form *one* tree

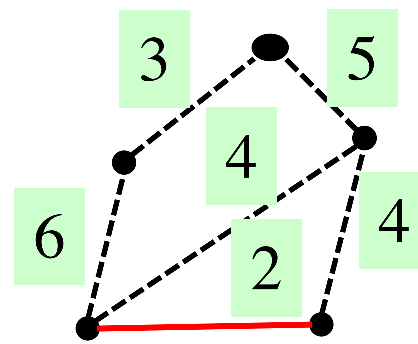
MST using Kruskal's algorithm

Minimal set of edges that connect all vertices

At each step, add an edge (that does not form a cycle) with minimum weight

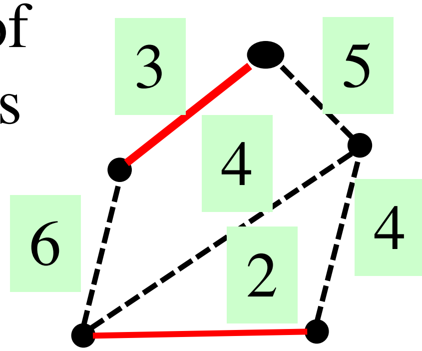
edge with weight 2



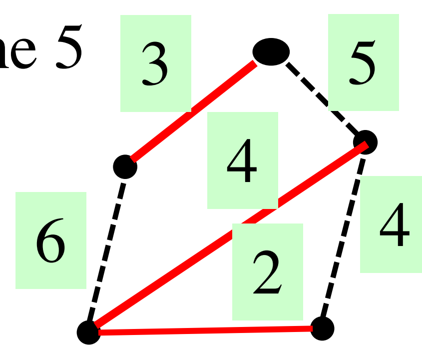


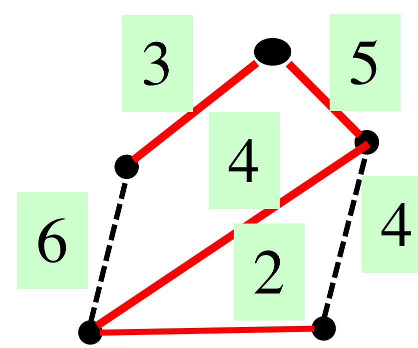
edge with weight 3

One of the 4's



The 5





Red edges need not form tree (until end)

Kruskal

Minimal set of edges that connect all vertices

Start with the all the nodes and no edges, so there is a forest of trees, each of which is a single node (a leaf).

At each step, add an edge (that does not form a cycle) with minimum weight

We do not look more closely at how best to implement Kruskal's algorithm —which data structures can be used to get a really efficient algorithm.

Leave that for later courses, or you can look them up online yourself.

We now investigate Prim's algorithm

MST using "Prim's algorithm"
(should be called "JPD algorithm")

Developed in 1930 by Czech mathematician **Vojtěch Jarník**. Práce Moravské Přírodovědecké Společnosti, 6, 1930, pp. 57–63. (in Czech)

Help:IPA for Czech

Developed in 1957 by computer scientist **Robert C. Prim**. *Bell System Technical Journal*, 36 (1957), pp. 1389–1401

From Wikipedia, the free encyclopedia

Vojtěch Jarník (Czech pronunciation: [ˈvojcɛx ˈjarɲiːk];

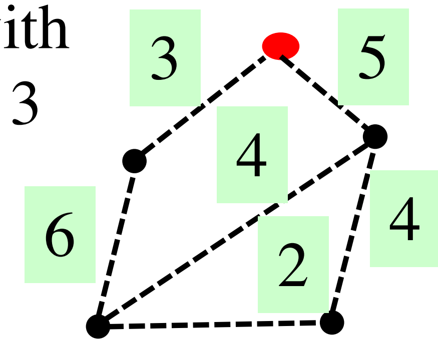
Developed about 1956 by **Edsger Dijkstra** and published in in 1959. *Numerische Mathematik* 1, 269–271 (1959)

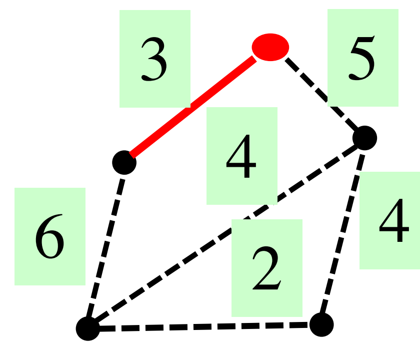
Prim’s algorithm

At each step, add an edge (that does not form a cycle) with minimum weight, but keep added edge connected to the start (red) node

Minimal set of edges that connect all vertices

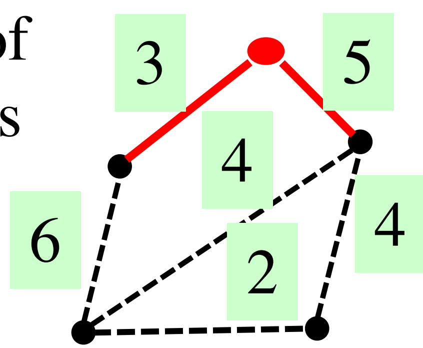
edge with weight 3



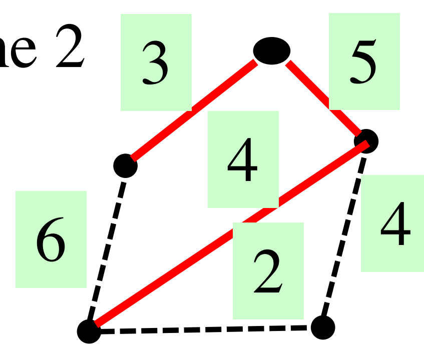


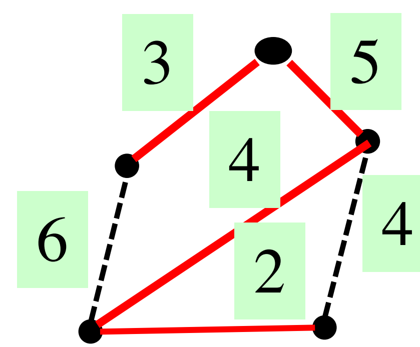
edge with weight 5

One of the 4’s



The 2





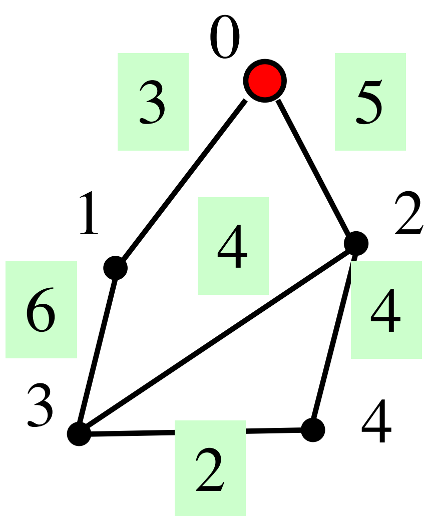
Difference between Prim and Kruskal

Minimal set of edges that connect all vertices

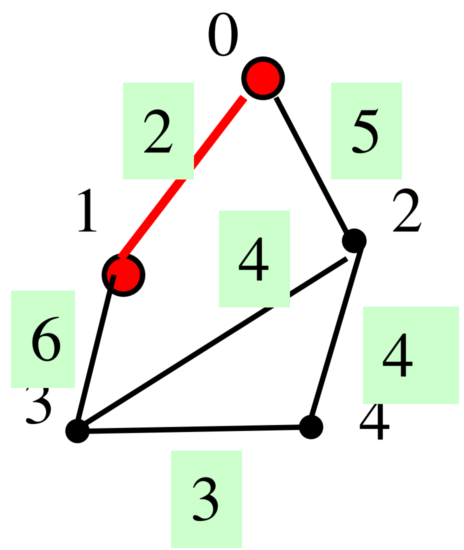
Prim requires that the constructed red tree always be connected.
Kruskal doesn't

But: Both algorithms find a minimal spanning tree

Here, Prim chooses (0, 1)
Kruskal chooses (3, 4)



Here, Prim chooses (0, 2)
Kruskal chooses (3, 4)



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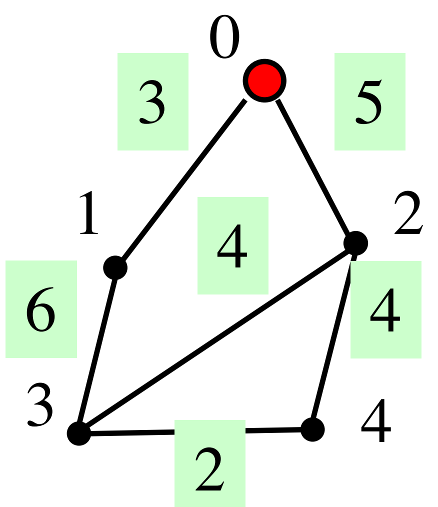

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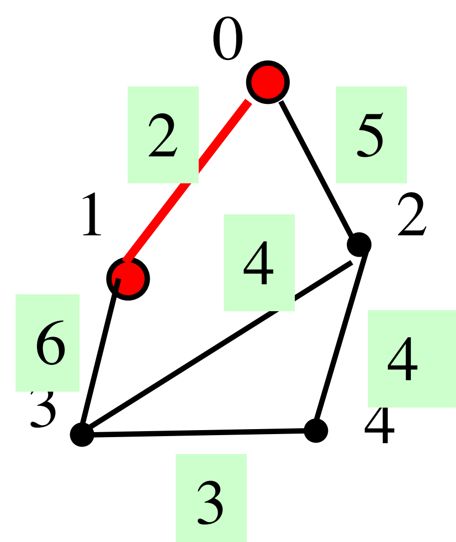


Difference between Prim and Kruskal

Minimal set of edges that connect all vertices

Prim requires that the constructed red tree always be connected.
Kruskal doesn't

But: Both algorithms find a minimal spanning tree

If the edge weights are all different, the Prim and Kruskal algorithms construct the same tree.

Prim's (JPD) spanning tree algorithm

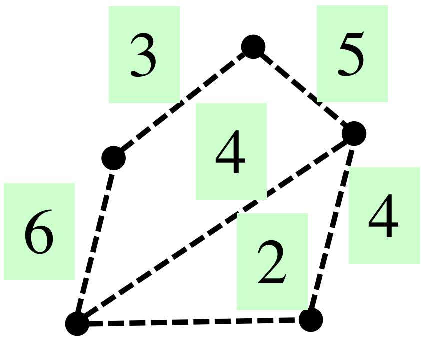
Given: graph (V, E)   (sets of vertices and edges)

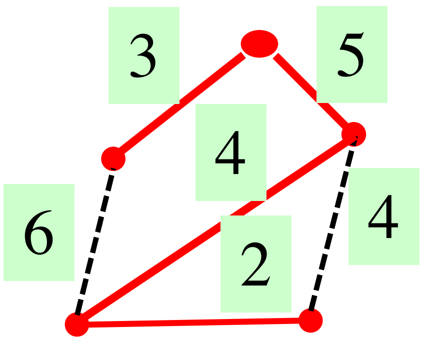
Output: tree (V1, E1), where

V1 = V

E1 is a subset of E

(V1, E1) is a minimal spanning tree –sum of edge weights is minimal



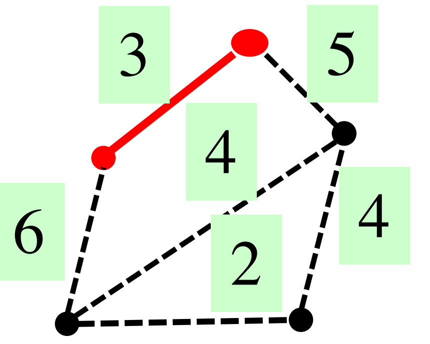


Prim's (JPD) spanning tree algorithm

V1= {an arbitrary node of V}; E1= {};

//inv: (V1, E1) is a tree, V1 ≤ V, E1 ≤ E

while (V1.size() < V.size()) {
    Pick an edge (u,v) with:
        min weight, u in V1,
        v not in V1;
    Add v to V1;
    Add edge (u, v) to E1
}



V1: 2 red nodes

E1: 1 red edge

S: 2 edges leaving red nodes

Consider having a set S of edges with the property:
If (u, v) an edge with u in V1 and v not in V1, then (u,v) is in S

Prim's (JPD) spanning tree algorithm

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//inv: (V1, E1) is a tree, V1 ≤ V, E1 ≤ E

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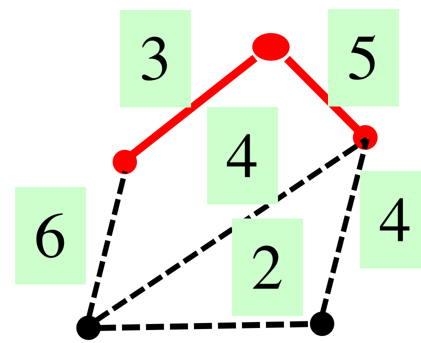


V1: 3 red nodes
E1: 2 red edges
S: 3 edges leaving red nodes

Consider having a set S of edges with the property:
If (u, v) an edge with u in V1 and v not in V1, then (u,v) is in S

Prim's (JPD) spanning tree algorithm

V1= {an arbitrary node of V}; E1= {};

//inv: (V1, E1) is a tree, V1 ≤ V, E1 ≤ E

while (V1.size() < V.size()) {

    Pick an edge (u,v) with:

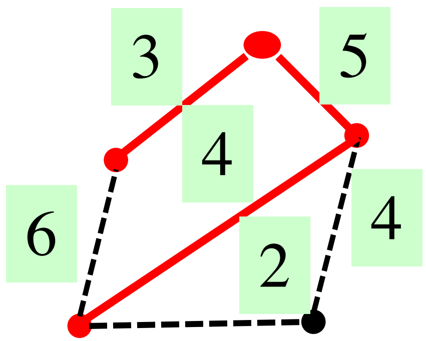
        min weight, u in V1,

        v not in V1;

    Add v to V1;

    Add edge (u, v) to E1

}



V1: 4 red nodes

E1: 3 red edges

S: 3 edges leaving red nodes

Note: the edge with weight 6 is not in in S – this avoids cycles

Consider having a set S of edges with the property:
If (u, v) an edge with u in V1 and v not in V1, then (u,v) is in S

Prim's (JPD) spanning tree algorithm

V1= {an arbitrary node of V}; E1= {};
//inv: (V1, E1) is a tree, V1 ≤ V, E1 ≤ E
S= set of edges leaving the single node in V1;
while (V1.size() < V.size()) {
~~Pick an edge (u,v) with:~~
~~min weight, u in V1,~~
~~v not in V1;~~
~~Add v to V1;~~
~~Add edge (u, v) to E1~~
}

Remove from S an edge
(u, v) with min weight

if v is not in V1:
add v to V1; add (u,v) to E1;
add edges leaving v to S

Consider having a set S of edges with the property:
If (u, v) an edge with u in V1 and v not in V1, then (u,v) is in S

Prim’s (JPD) spanning tree algorithm

```
V1= {start node};  E1=  {};
 S= set of edges leaving the single node in V1;
//inv: (V1, E1) is a tree, V1 ≤ V, E1 ≤ E,
//       All edges (u, v) in S have u in V1,
//        if edge (u, v) has u in V1 and v not in V1, (u, v) is in S
while (V1.size() < V.size()) {
    Remove from S an edge (u, v) with min weight;
    if (v not in V1) {
        add v to V1; add (u,v) to E1;
        add edges leaving v to S
    }
}
```

Question: How should we implement set S?

Prim's (JPD) spanning tree algorithm

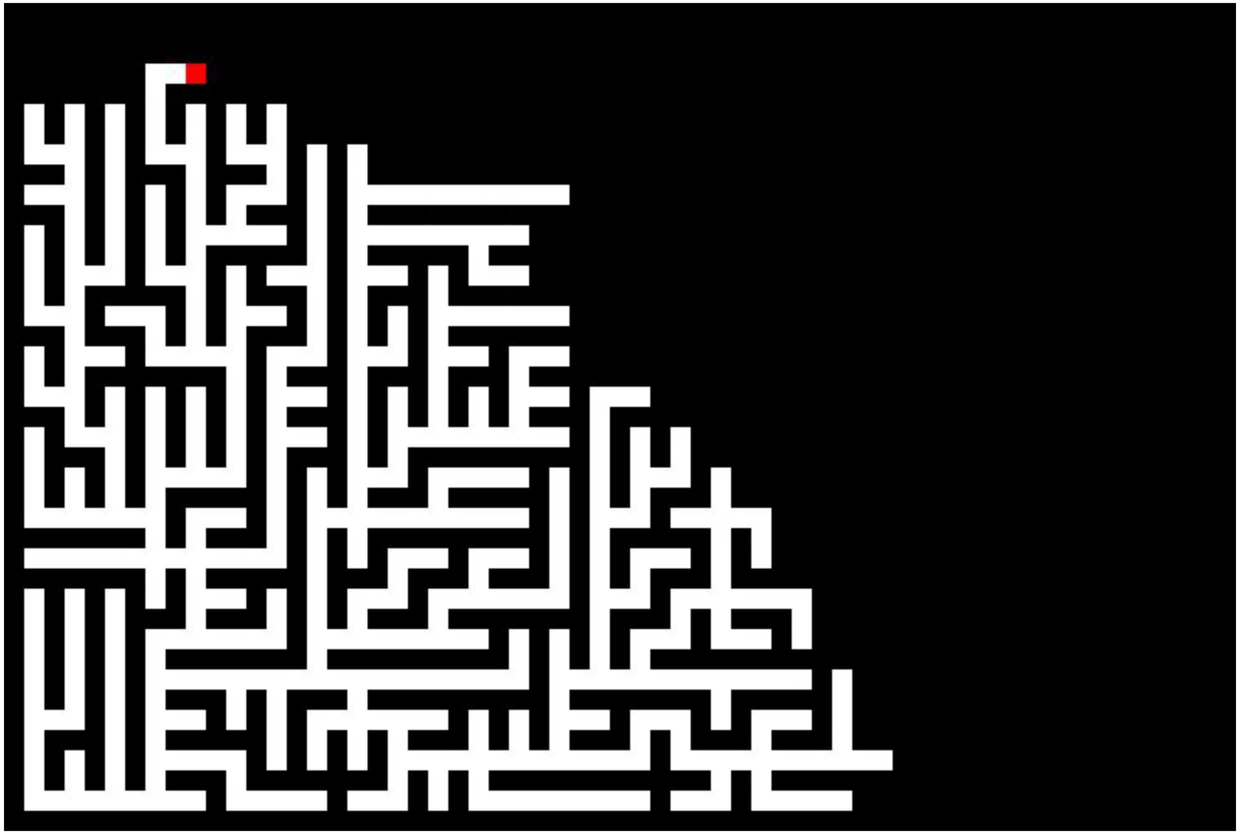
```
V1= {start node};  E1=  {};
 S= set of edges leaving the single node in V1;
//inv: (V1, E1) is a tree, V1 ≤ V, E1 ≤ E,
//       All edges (u, v) in S have u in V1,
//       if edge (u, v) has u in V1 and v not in V1, (u, v) is in S
while (V1.size() < V.size()) {
    Remove from S a min-weight edge (u, v);        #V   log #E
    if (v not in V1) {
        add v to V1; add (u,v) to E1;
        add edges leaving v to S                   #E   log #E
    }
}
```

Implement S as a heap.
Use adjacency lists for edges

Thought: Could we use for S a set of nodes instead of edges?
Yes. We don't go into that here

Application of minimum spanning tree

Maze generation using Prim's algorithm



The generation of a maze using Prim's algorithm on a randomly weighted grid graph that is 30x20 in size.

https://en.wikipedia.org/wiki/Maze_generation_algorithm#Randomized_Kruskal.27s_algorithm

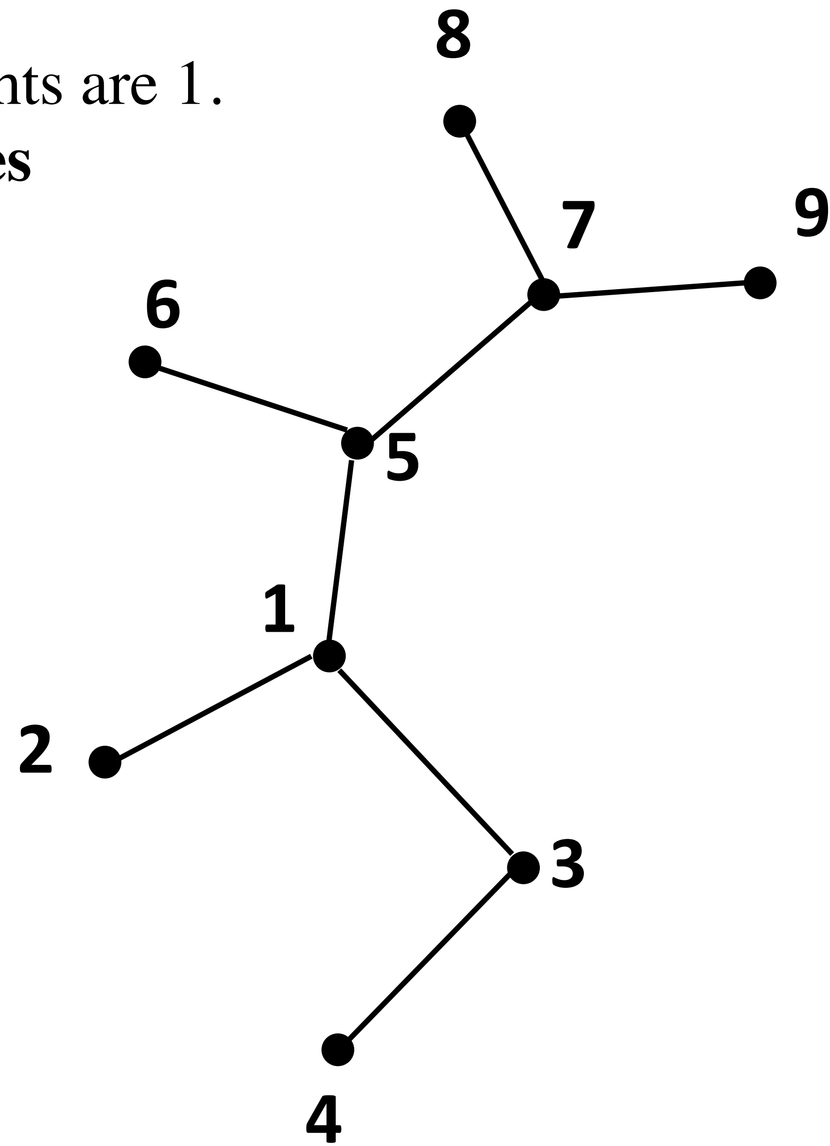
Graph algorithms MEGA-POLL!

In this undirected graph, all edge weights are 1.

Which of the following visit the nodes in the same order as Prim(1)?

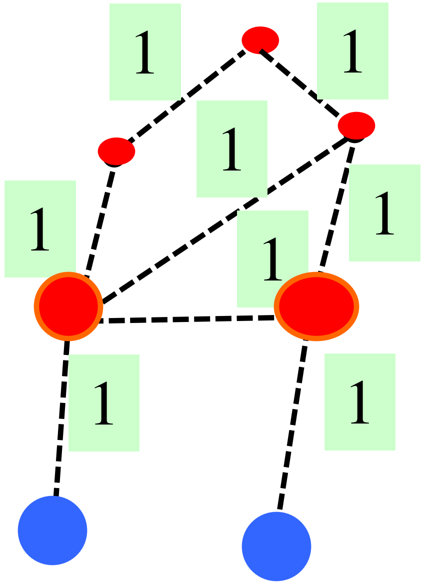
- Always break ties by choosing the lower-numbered node first.
- In tree traversals, use node 1 as the tree's root.

-Dijkstra(1)
-BFS(1)
-DFS(1)
-Preorder tree traversal
-Postorder tree traversal
-Level order tree traversal



Greedy algorithms

Suppose the weights are all 1.
Then Dijkstra's shortest-path
algorithm does a breath-first search!



Dijkstra's and Prim's algorithms look similar.
The steps taken are similar, but at each step

- Dijkstra's chooses an edge whose end node has a minimum path length from start node
- Prim's chooses an edge with minimum length

Breadth-first search, Shortest-path, Prim

Greedy algorithm: An algorithm that uses the heuristic of making the locally optimal choice at each stage with the hope of finding the global optimum.

Dijkstra's shortest-path algorithm makes a locally optimal choice: choosing the node in the Frontier with minimum L value and moving it to the Settled set. And, it is proven that it is not just a hope but a fact that it leads to the global optimum.

Similarly, Prim's and Kruskal's locally optimum choices of adding a minimum-weight edge have been proven to yield the global optimum: a minimum spanning tree.

BUT: Greediness does not always work!

Similar code structures

```
while (a vertex is unmarked) {
    v= best unmarked vertex
    mark v;
    for (each w adj to v)
        update D[w];
}
```

- Breadth-first-search (bfs)
  - best: next in queue
  - update: D[w] = D[v]+1
- Dijkstra's algorithm
  - best: next in priority queue
  - update: D[w] = min(D[w], D[v]+c(v,w))
- Prim's algorithm
  - best: next in priority queue
  - update: D[w] = min(D[w], c(v,w))

c(v,w) is the v→w edge weight

Traveling salesman problem

Given a list of cities and the distances between each pair, what is the shortest route that visits each city exactly once and returns to the origin city?

- The true TSP is very hard (called NP complete)… for this we want the *perfect* answer in all cases.
- Most TSP algorithms start with a spanning tree, then “evolve” it into a TSP solution. Wikipedia has a lot of information about packages you can download…

But really, how hard can it be?
How many paths can there be that visit all of 50 cities?
12,413,915,592,536,072,670,862,289,047,373,375,038,521,486,354,677,760,000,000,000

Graph Algorithms

- Search
  - Depth-first search
  - Breadth-first search
- Shortest paths
  - Dijkstra's algorithm
- Minimum spanning trees
  - Prim's algorithm
  - Kruskal's algorithm