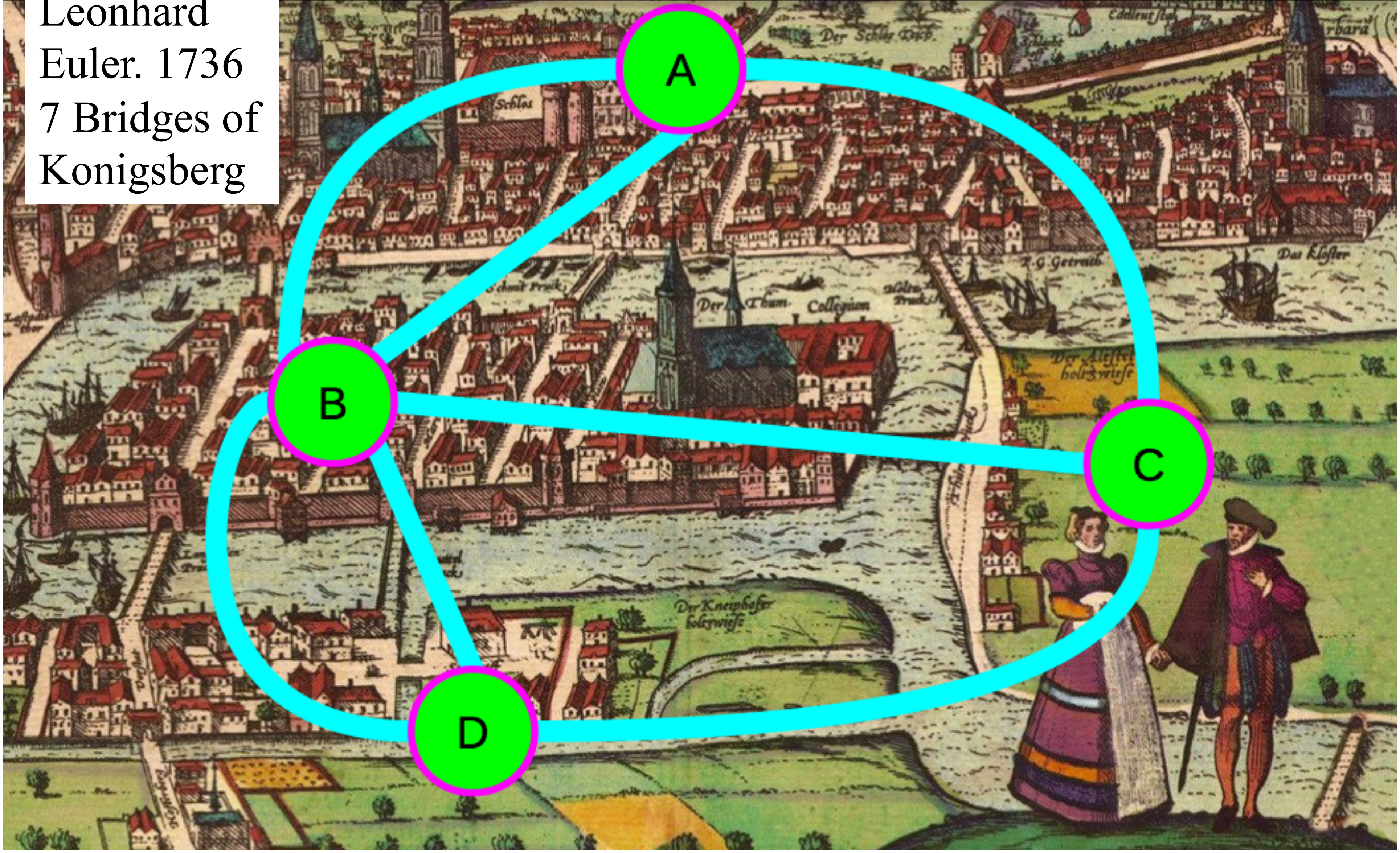
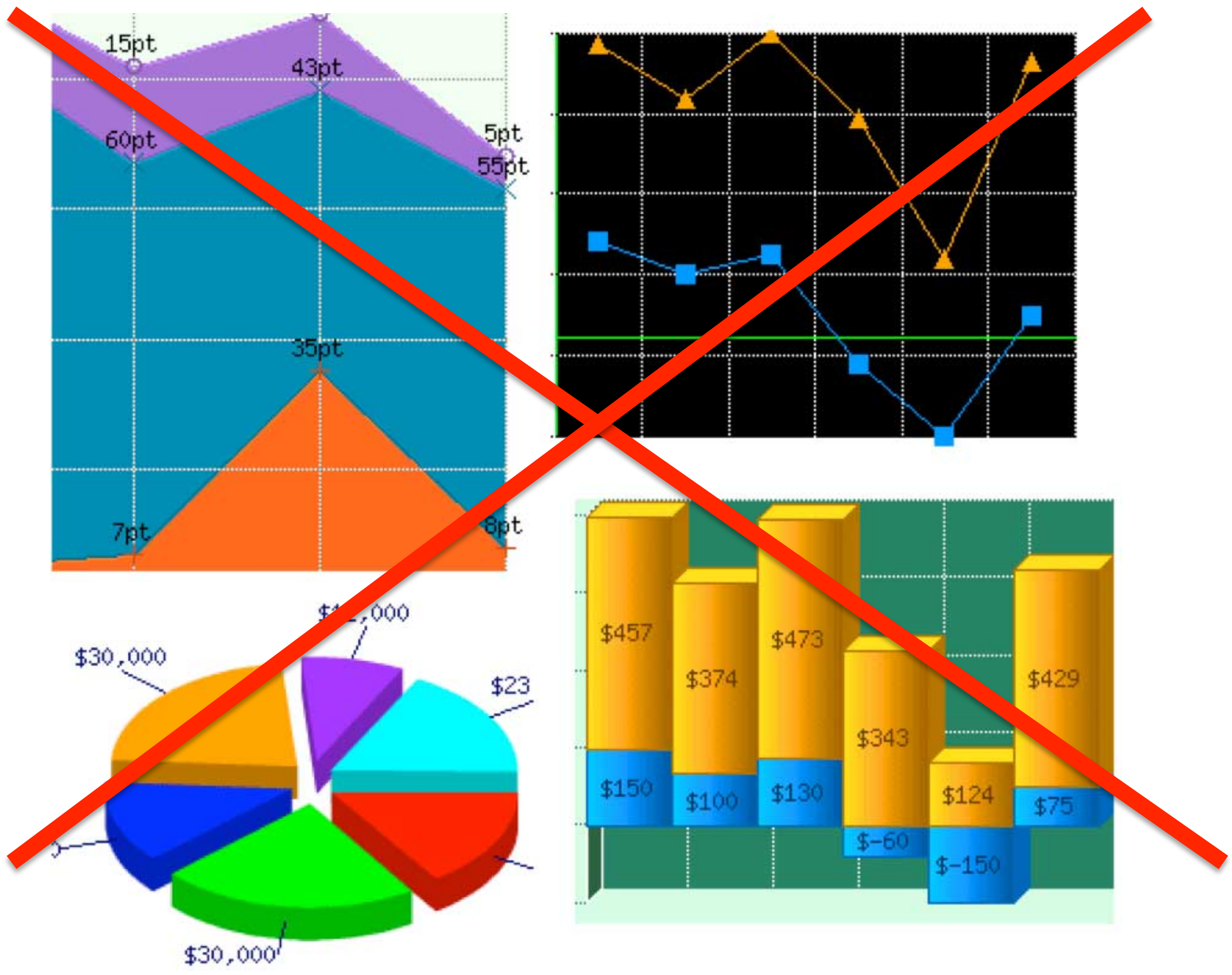
Graphs - I

CS 2110, Spring 2017

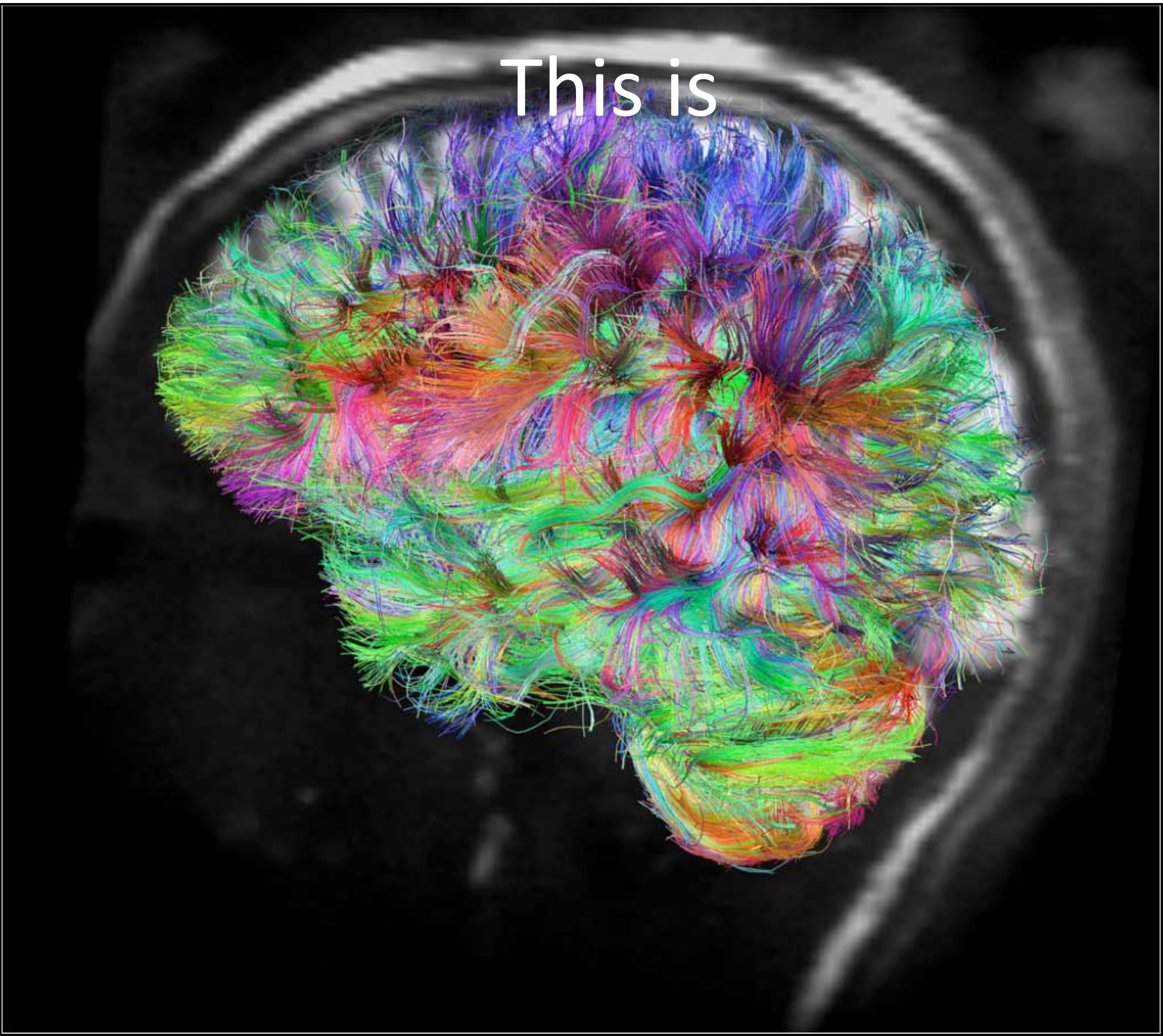
Leonhard Euler. 1736
7 Bridges of Konigsberg



These *aren't* the graphs we're interested in

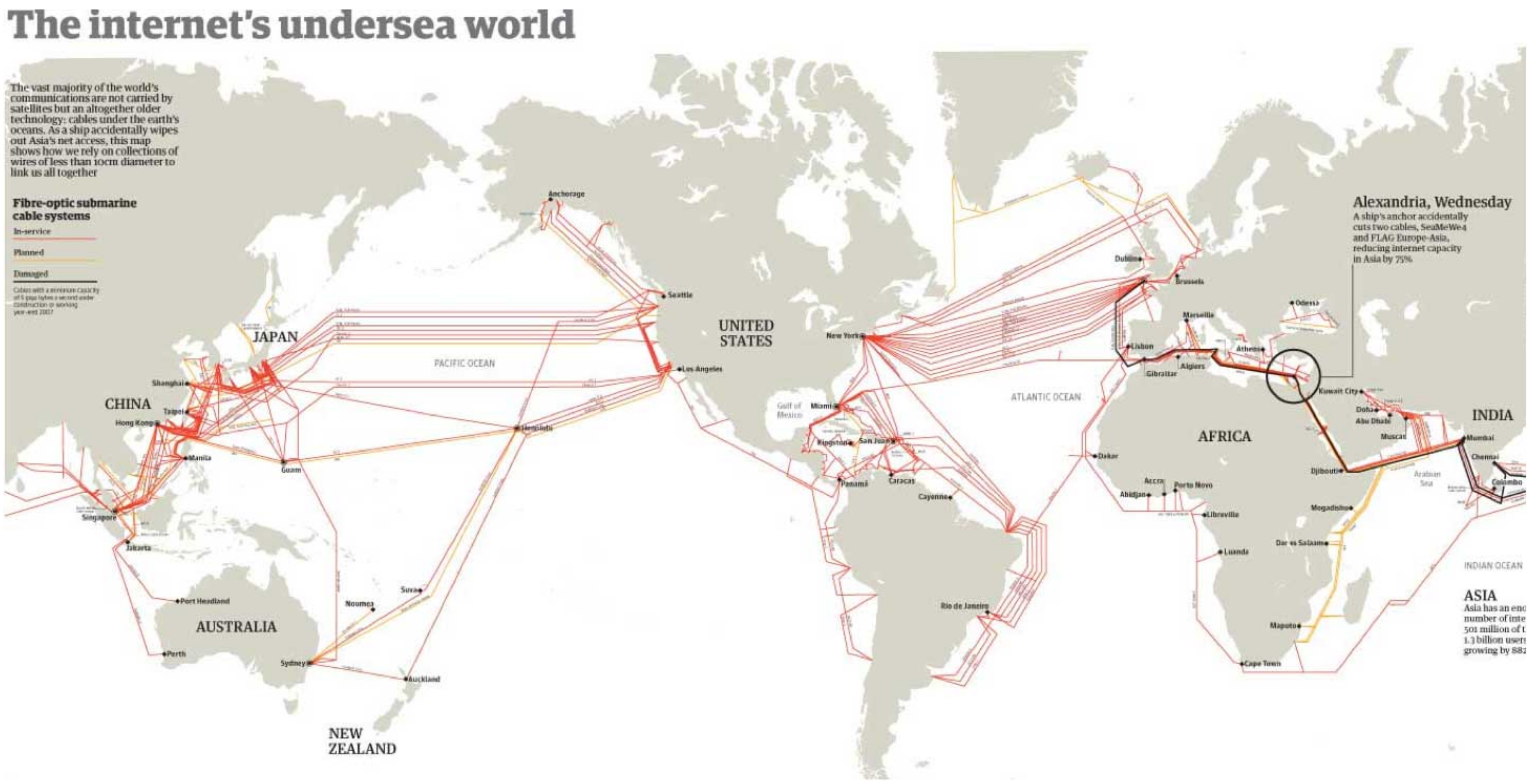


This is



V.J. Wedeen and L.L. Wald, Martinos Center for Biomedical Imaging at MGH

And so is this



The internet's undersea world

The vast majority of the world's communications are not carried by satellites but an altogether older technology: cables under the earth's oceans. As a ship accidentally wipes out Asia's net access, this map shows how we rely on collections of wires of less than 10cm diameter to link us all together

Fibre-optic submarine cable systems

In-service

Planned

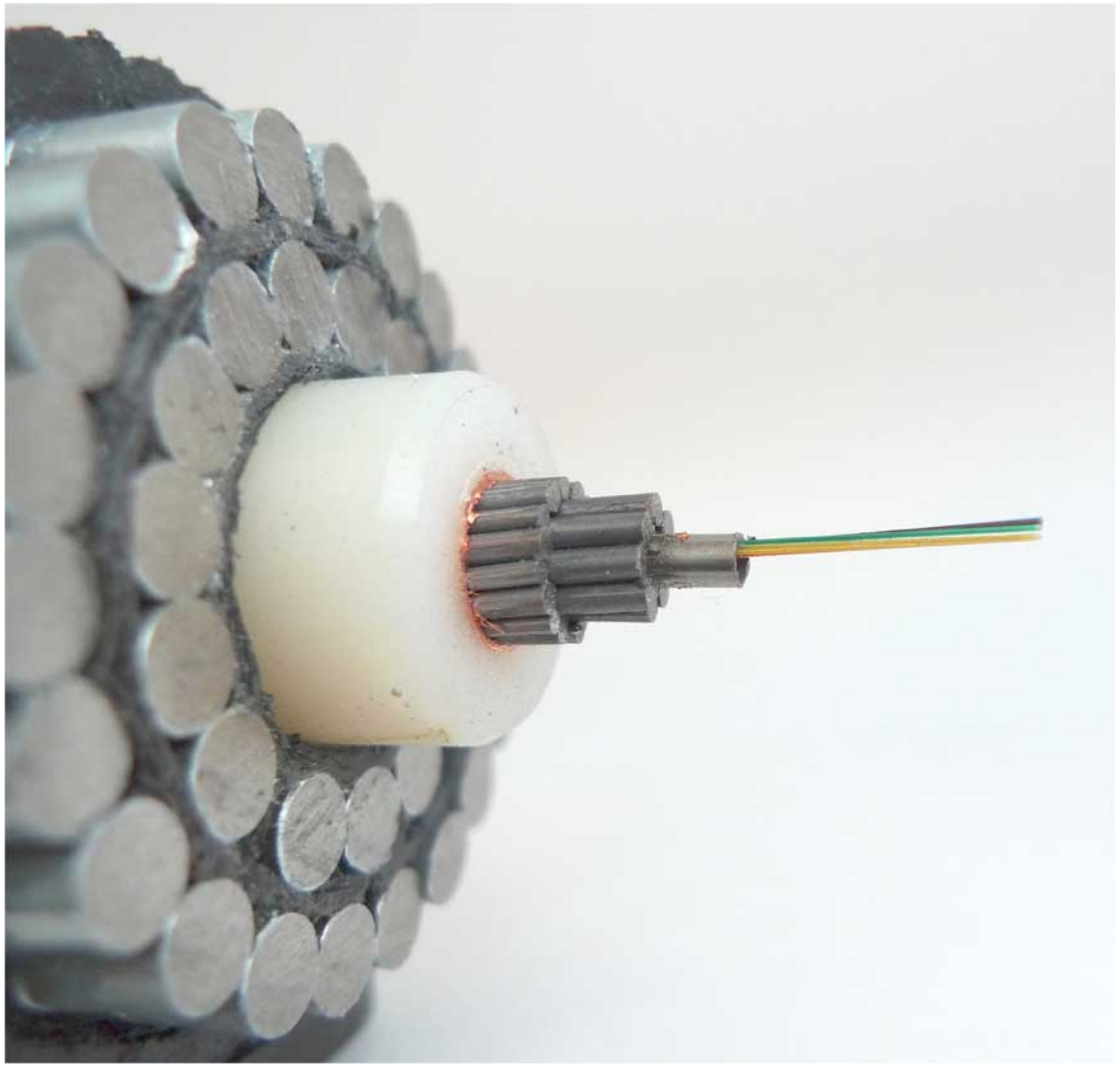
Damaged

Cables with a minimum capacity of 5 giga bytes a second under construction or working year-end 2007

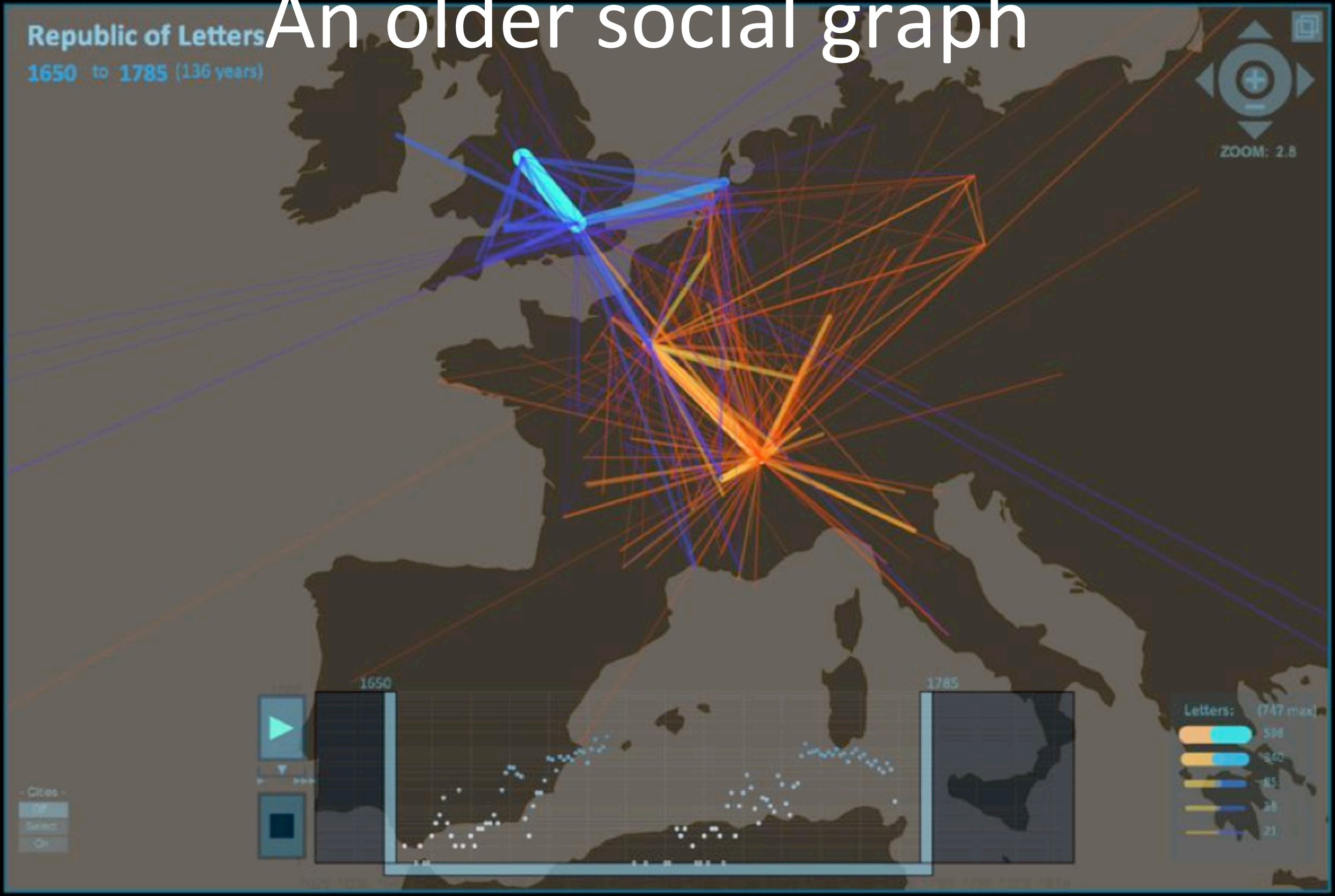
Alexandria, Wednesday
A ship's anchor accidentally cuts two cables, SeaMeWe4 and FLAG Europe-Asia, reducing internet capacity in Asia by 75%

ASIA
Asia has an enc
number of inte
501 million of t
1.3 billion user
growing by 882

This carries Internet traffic across the oceans

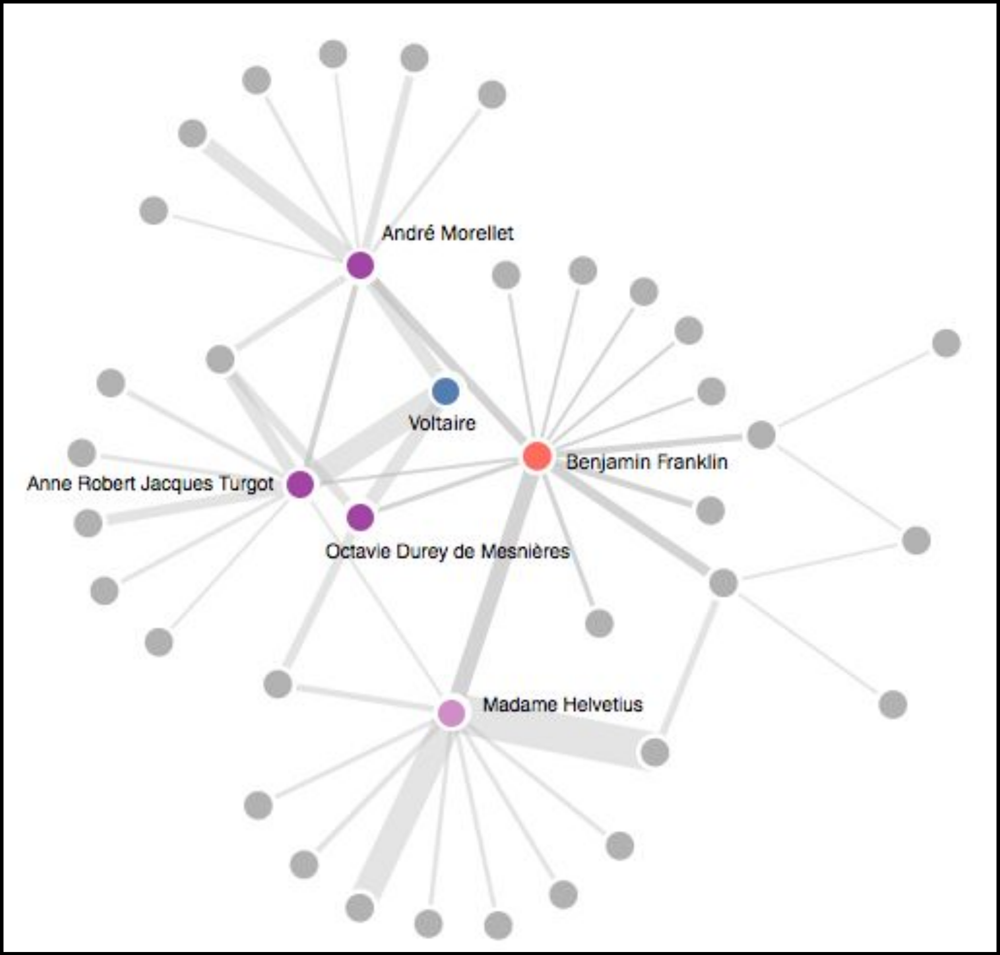


An older social graph



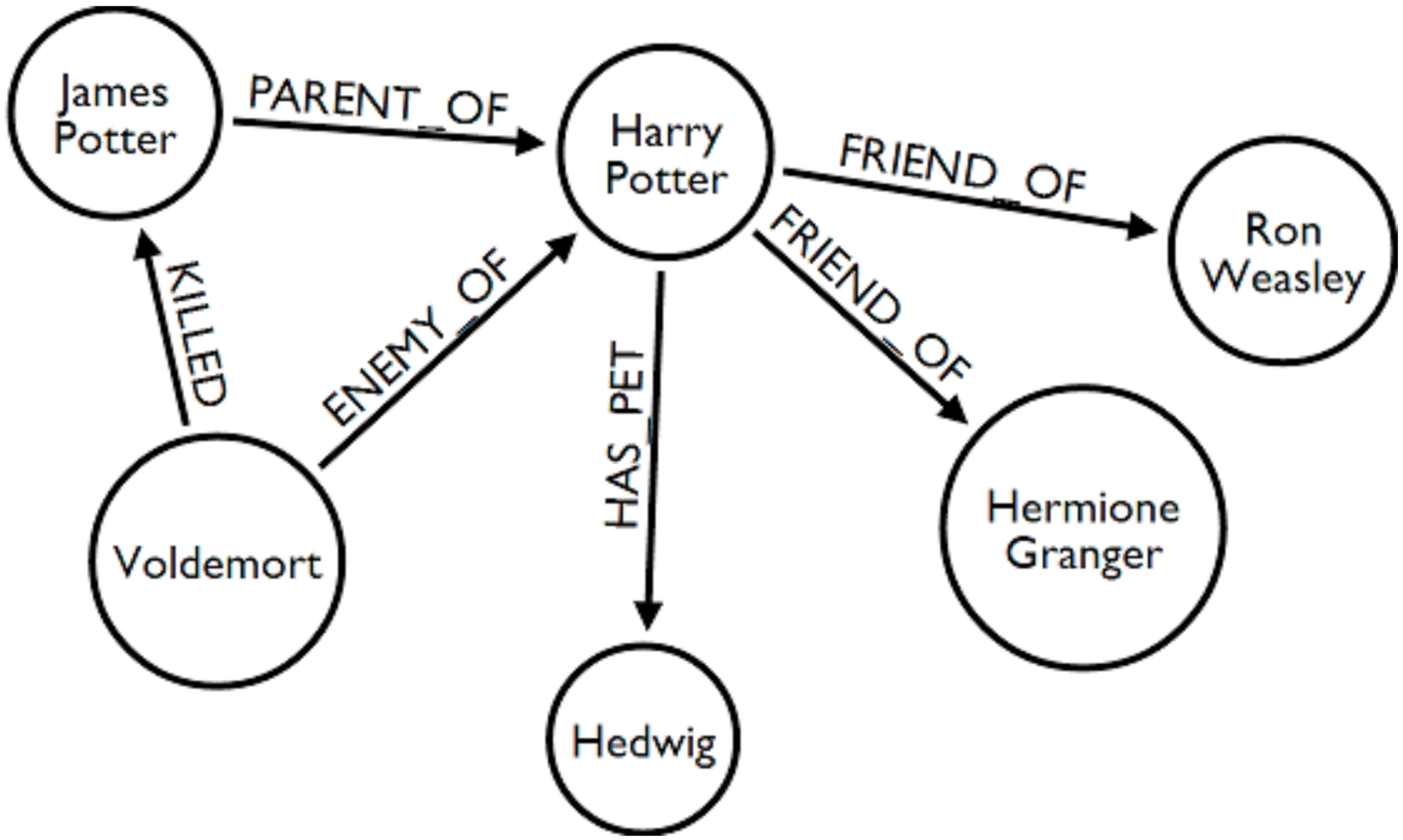
Locke's (blue) and Voltaire's (yellow) correspondence.
Only letters for which complete location information is available are shown.
Data courtesy the Electronic Enlightenment Project, University of Oxford.

An older social graph

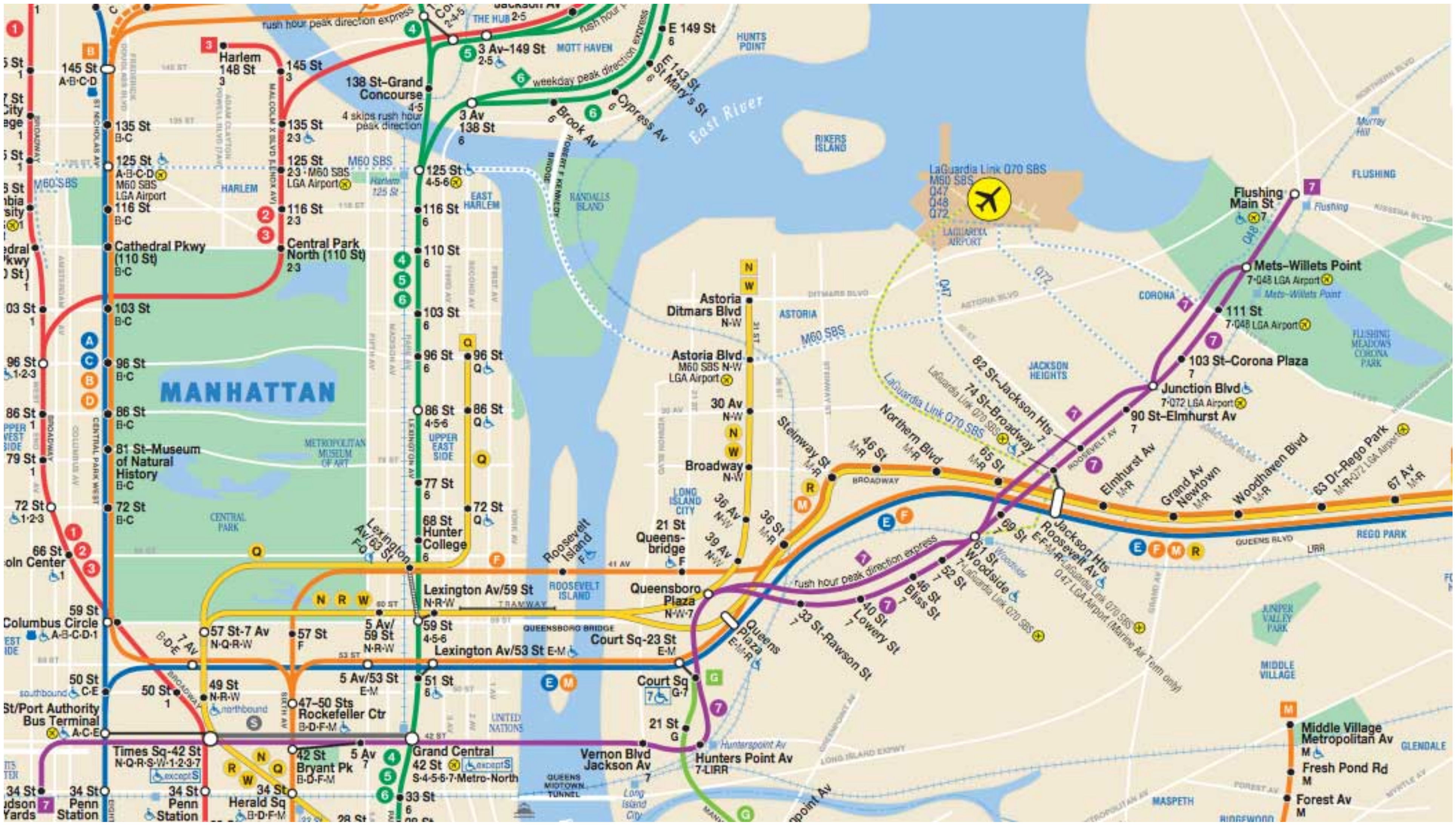


Voltaire and Benjamin Franklin

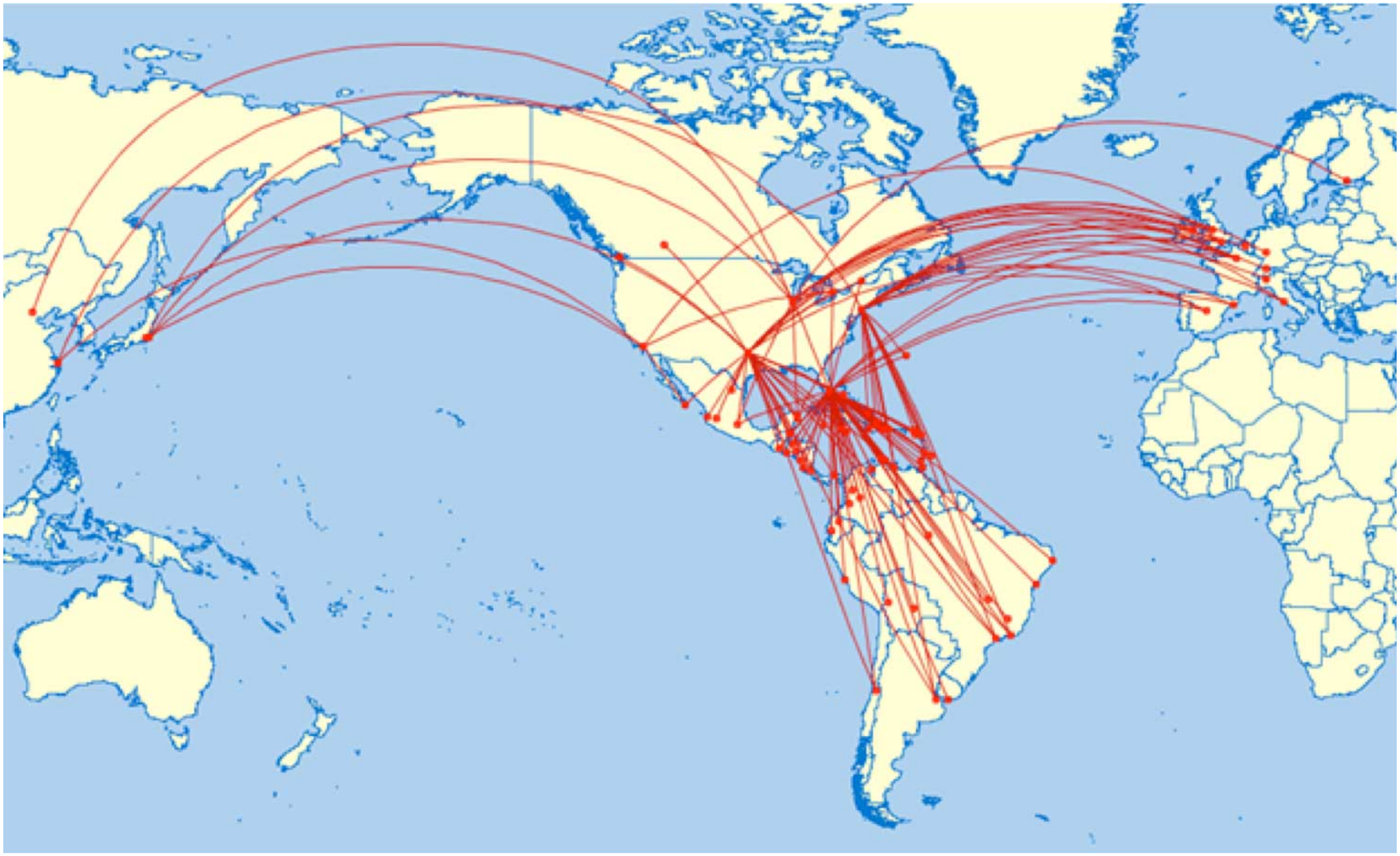
A fictional social graph



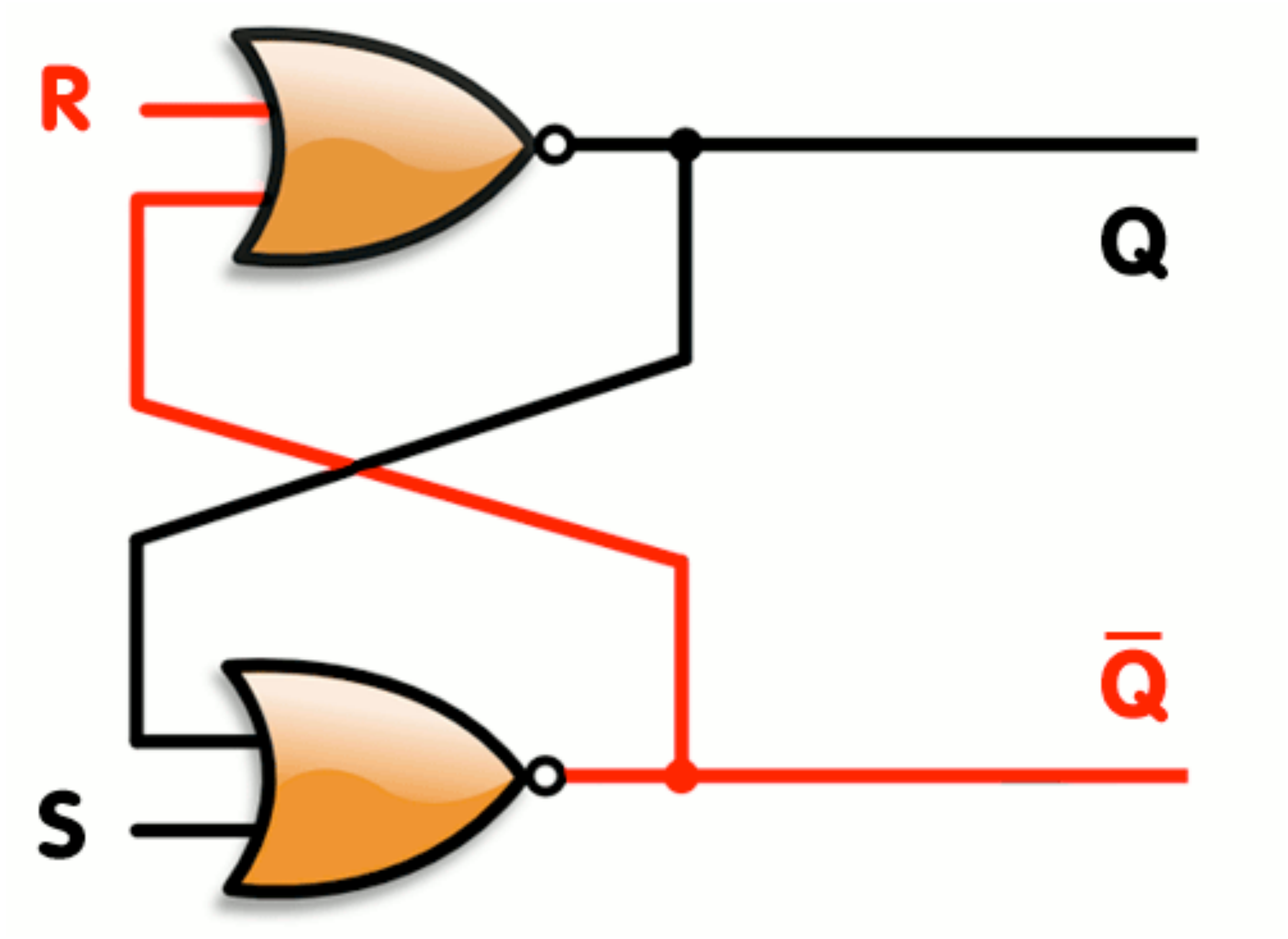
A transport graph: NY subway system



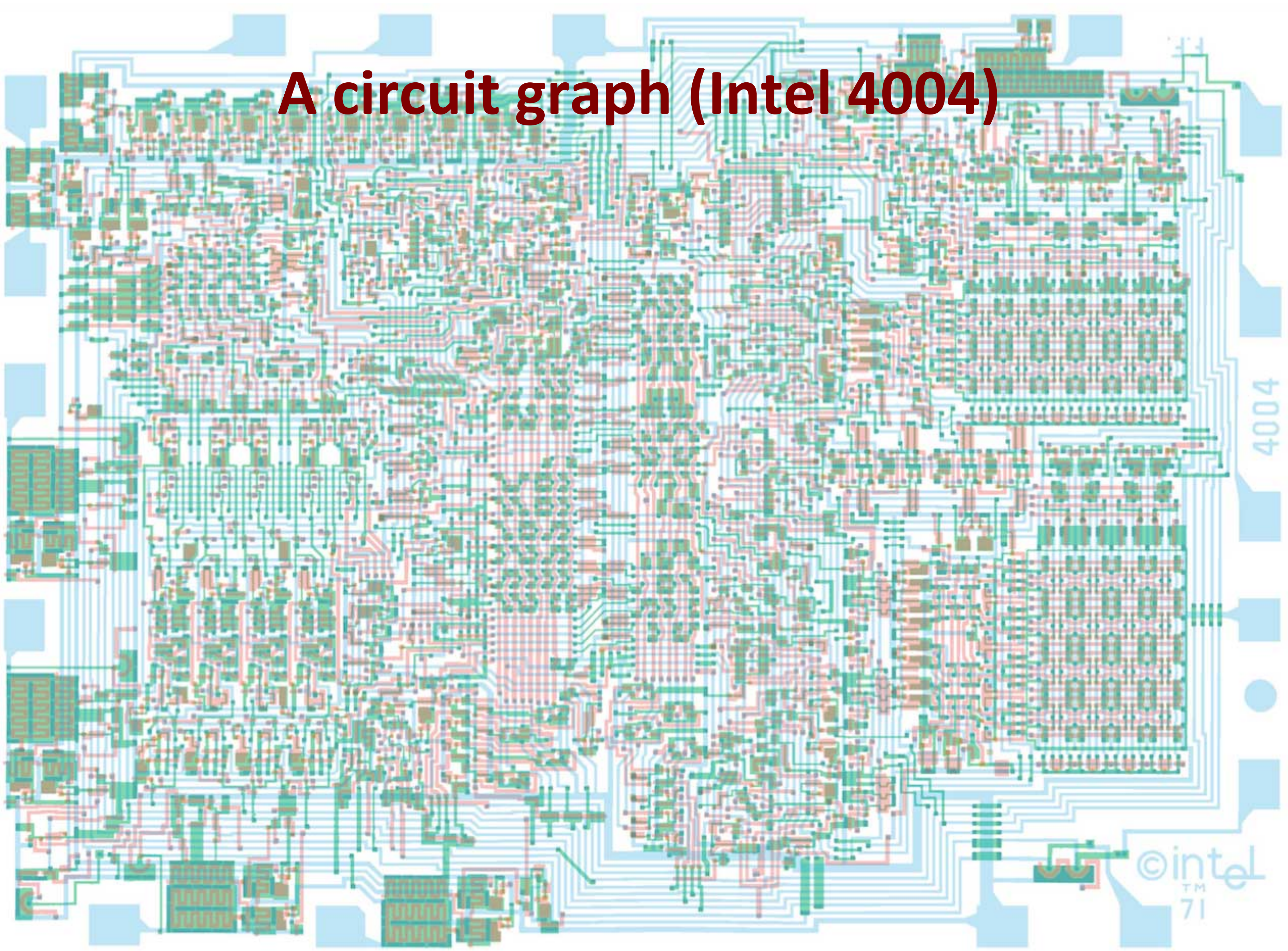
Another transport graph



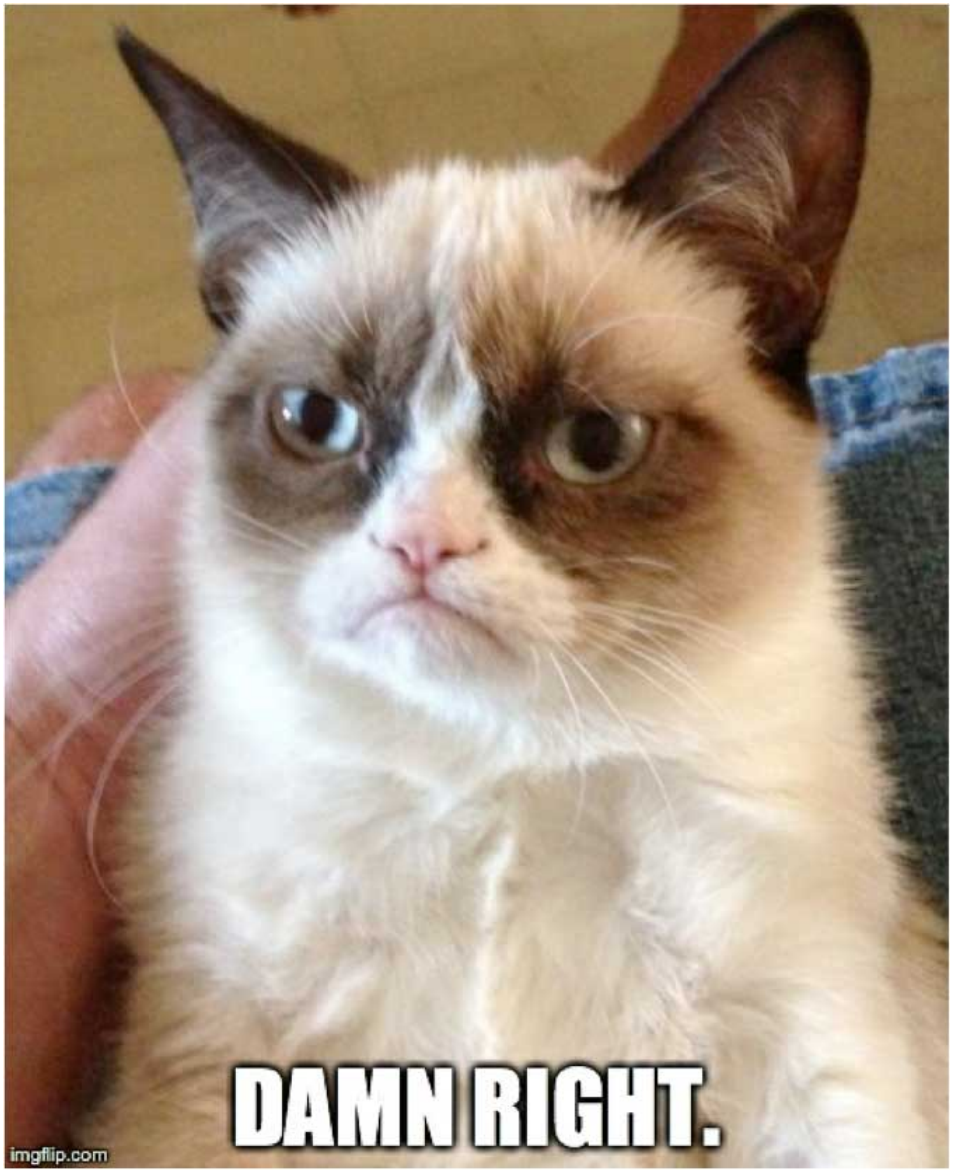
A circuit graph (flip-flop)



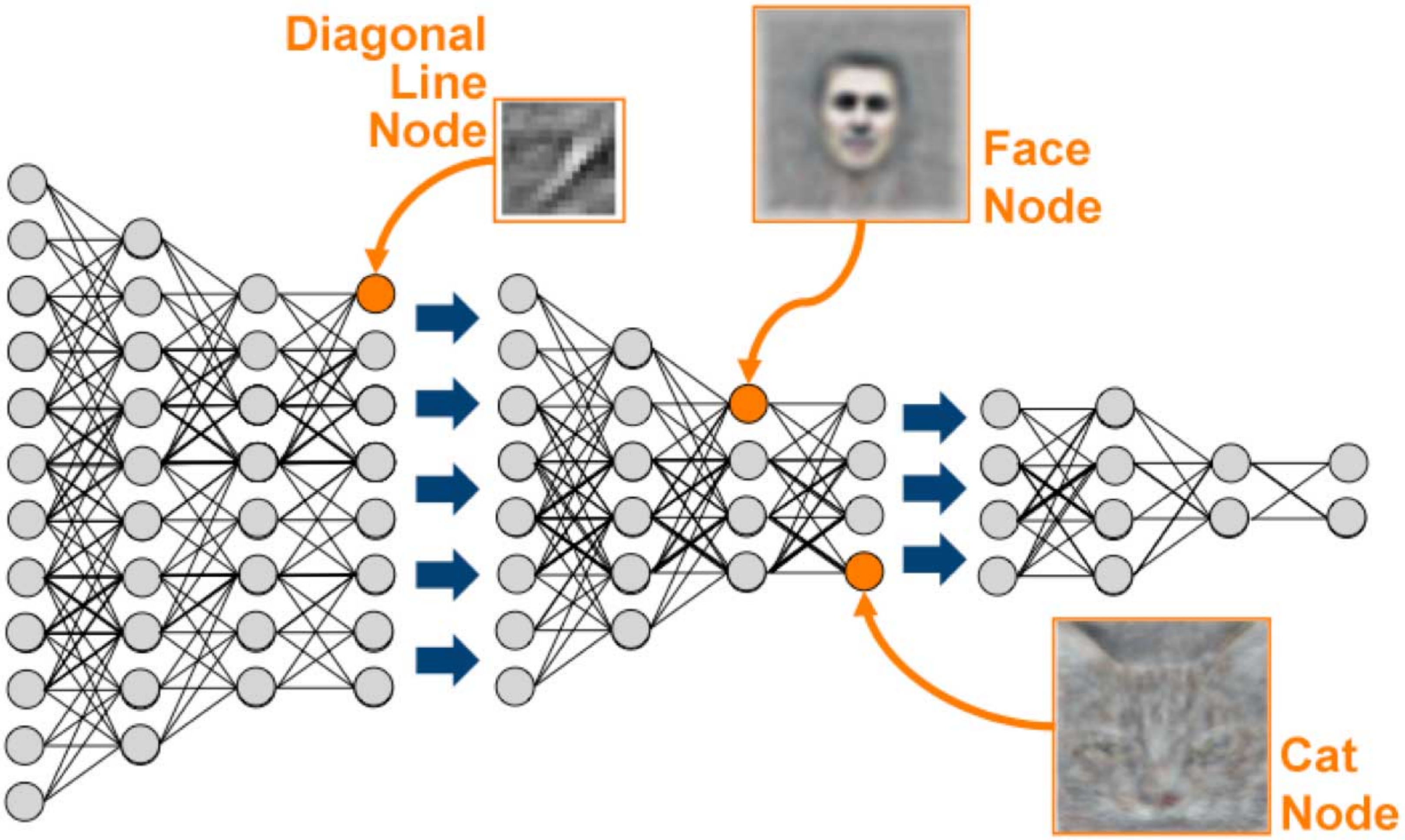
A circuit graph (Intel 4004)



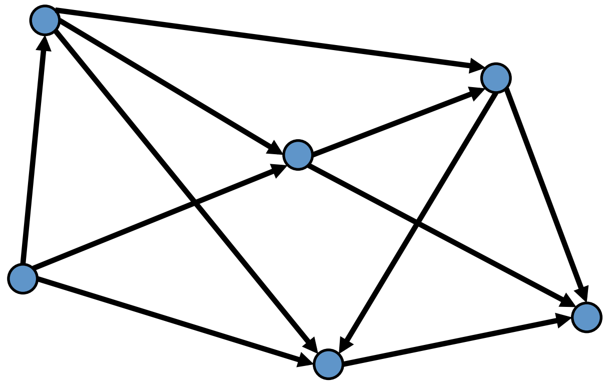
This is not a graph, this is a cat

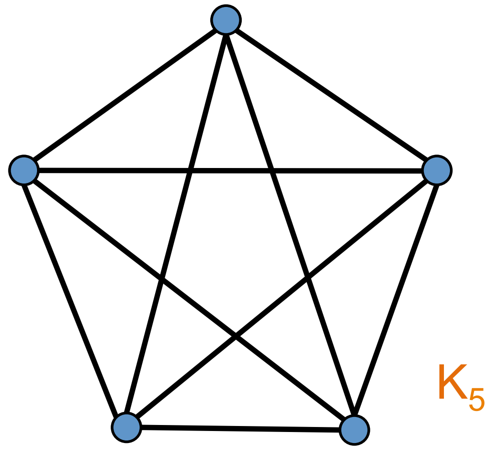


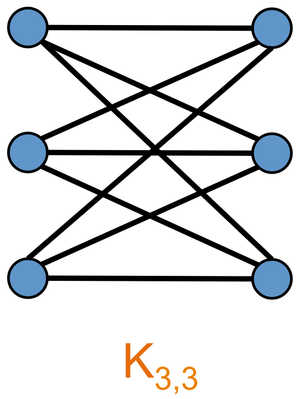
This is a graph(ical model) that has learned to recognize cats

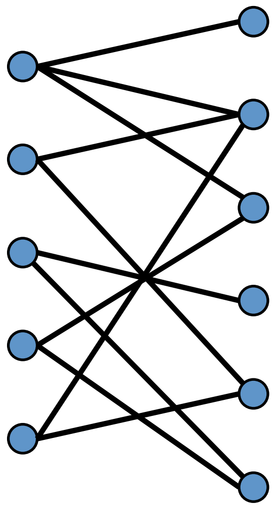


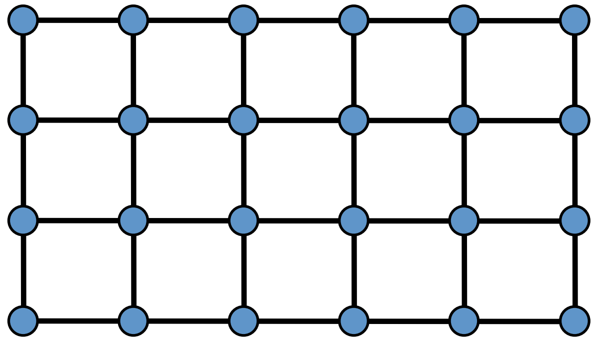
Some abstract graphs

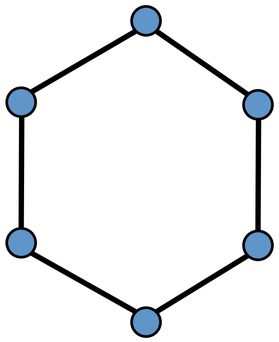












Applications of Graphs

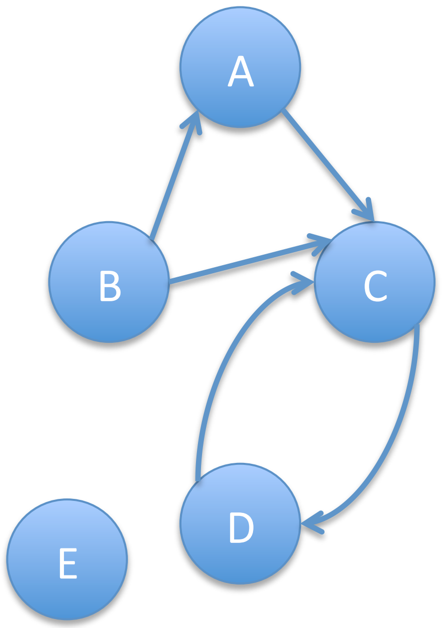
- Communication networks
- Social networks
- Routing and shortest path problems
- Commodity distribution (network flow)
- Traffic control
- Resource allocation
- Numerical linear algebra (sparse matrices)
- Geometric modeling (meshes, topology, …)
- Image processing (e.g. graph cuts)
- Computer animation (e.g. motion graphs)
- Systems biology
- Digital humanities (e.g. Republic of Letters)
- …

Directed graphs

- A directed graph (digraph) is a pair (V, E) where
  - V is a (finite) set
  - E is a set of ***ordered*** pairs (u, v) where $u,v \in V$
    - Often require $u \neq v$ (i.e. no self-loops)

- An element of V is called a vertex or node
- An element of E is called an edge or arc

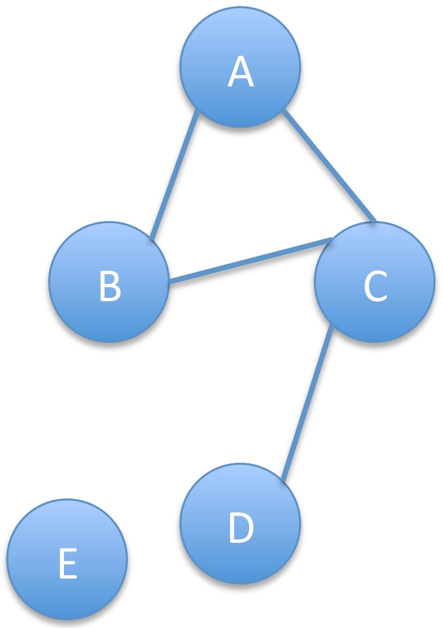
- $|V|$ = size of V, often denoted by n
- $|E|$ = size of E, often denoted by m



$\boldsymbol{V} = \{A, B, C, D, E\}$
$\boldsymbol{E} = \{(A,C), (B,A), (B,C), (C,D), (D,C)\}$
$|\boldsymbol{V}| = 5$
$|\boldsymbol{E}| = 5$

Undirected Graphs

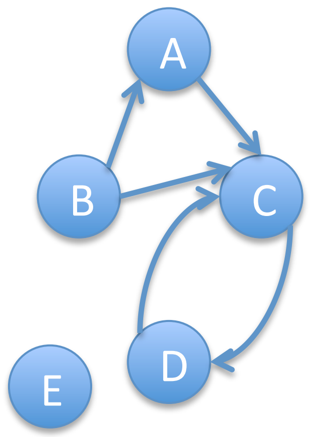
- An undirected graph is just like a directed graph!
  - … except that E is now a set of ***unordered*** pairs $\{u, v\}$ where $u,v \in V$
- Every undirected graph can be easily converted to an equivalent directed graph via a simple transformation:
  - Replace every undirected edge with two directed edges in opposite directions
- … but not vice versa

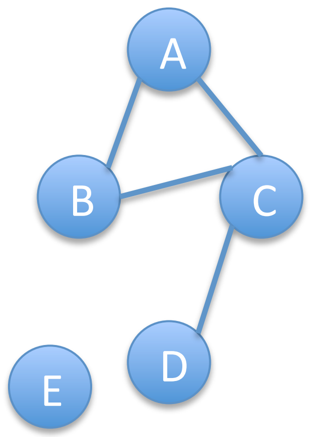


$\boldsymbol{V} = \{A, B, C, D, E\}$
$\boldsymbol{E} = \{\{A,C\}, \{B,A\}, \{B,C\}, \{C,D\}\}$
$|\boldsymbol{V}| = 5$
$|\boldsymbol{E}| = 4$

Graph terminology

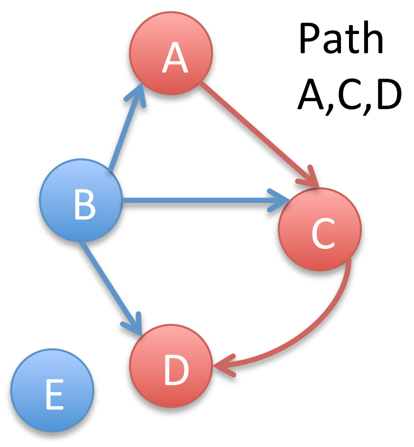
- Vertices u and v are called
  - the source and sink of the directed edge (u, v), respectively
  - the endpoints of (u, v) or $\{u, v\}$
- Two vertices are adjacent if they are connected by an edge
- The outdegree of a vertex u in a directed graph is the number of edges for which u is the source
- The indegree of a vertex v in a directed graph is the number of edges for which v is the sink
- The degree of a vertex u in an undirected graph is the number of edges of which u is an endpoint



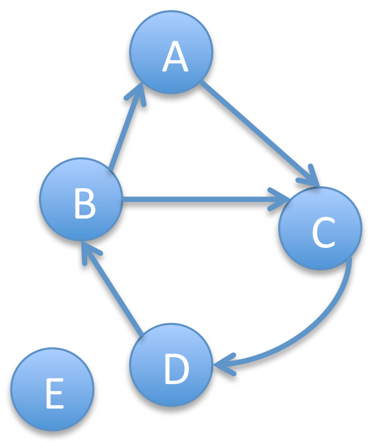


More graph terminology

- A path is a sequence $v_0,v_1,v_2,\ldots,v_p$ of vertices such that for $0 \leq i < p$,
  - $(v_i, v_{i+1}) \in E$ if the graph is directed
  - $\{v_i, v_{i+1}\} \in E$ if the graph is undirected
- The length of a path is its number of edges
- A path is simple if it doesn't repeat any vertices
- A cycle is a path $v_0, v_1, v_2, \ldots, v_p$ such that $v_0 = v_p$
- A cycle is simple if it does not repeat any vertices except the first and last
- A graph is acyclic if it has no cycles
- A ***d***irected ***a***cyclic ***g***raph is called a DAG

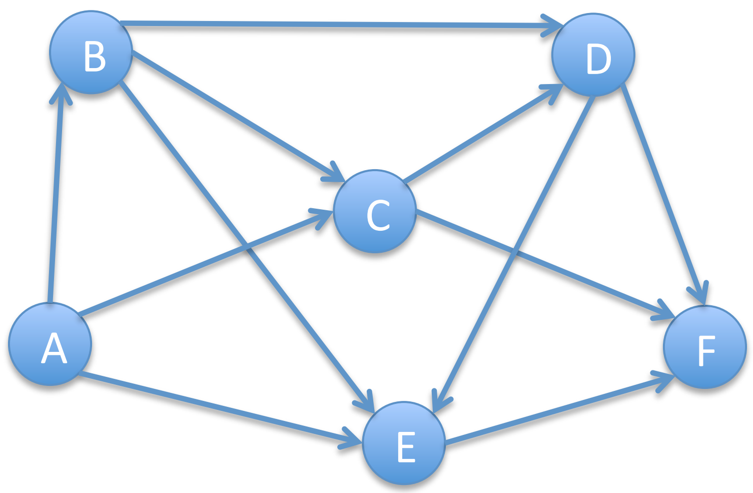


DAG



Not a DAG

Is this a DAG?



- Intuition:
  - If it's a DAG, there must be a vertex with indegree zero
- This idea leads to an *algorithm*
  - A digraph is a DAG if and only if we can iteratively delete indegree-0 vertices until the graph disappears

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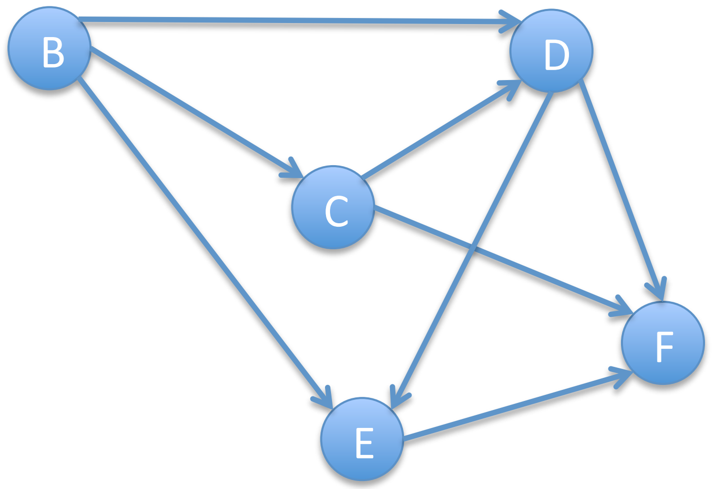

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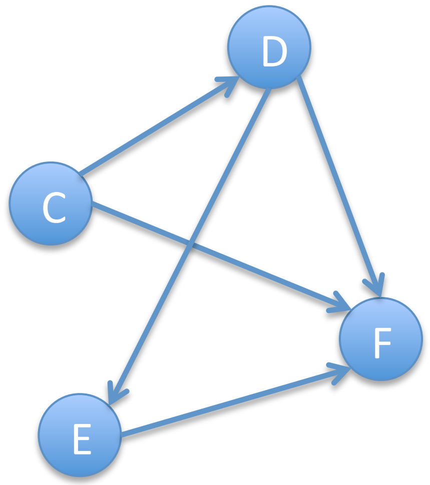

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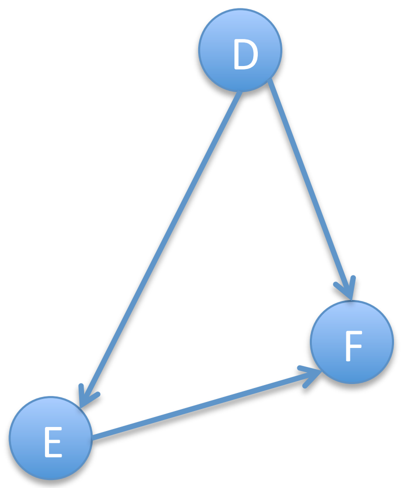

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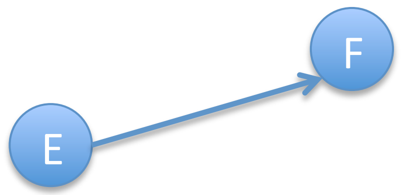

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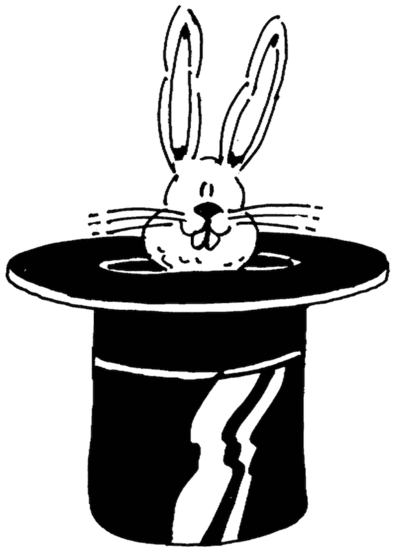
Is this a DAG?

YES!



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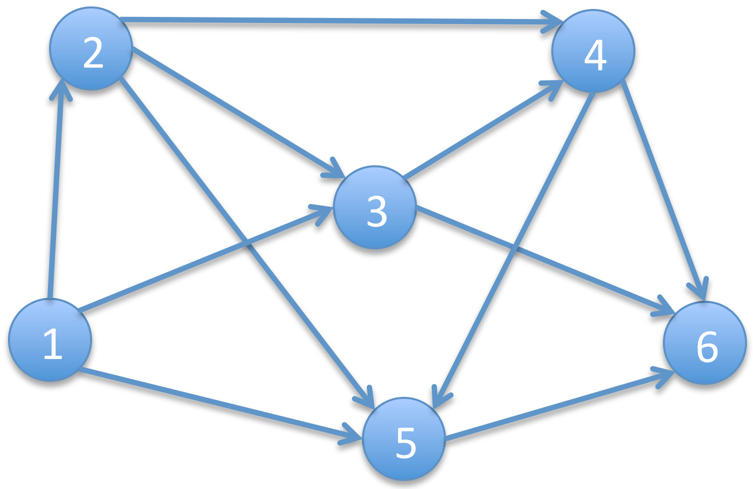
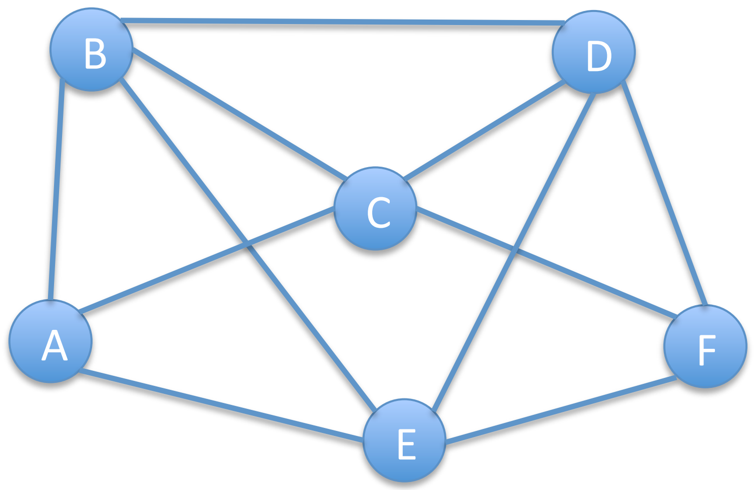
Topological sort



- We just computed a topological sort of the DAG
  - This is a numbering of the vertices such that all edges go from lower- to higher-numbered vertices
  - Useful in job scheduling with precedence constraints

Graph coloring

- A coloring of an undirected graph is an assignment of a color to each node such that no two adjacent vertices get the same color



- How many colors are needed to color this graph?

Graph coloring

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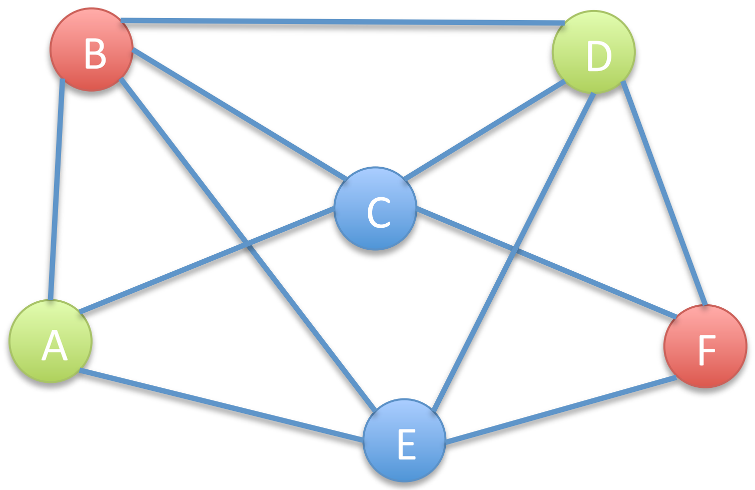
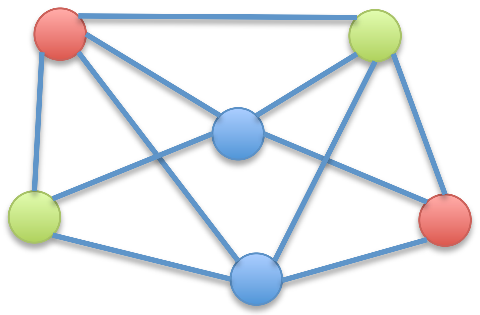


- How many colors are needed to color this graph?

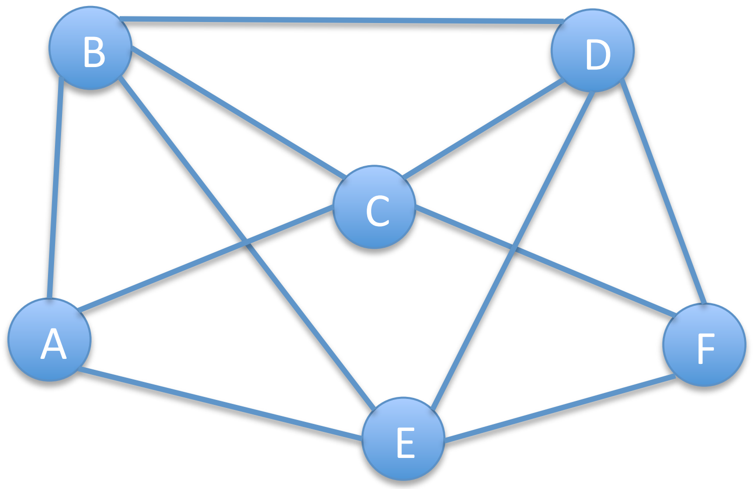
An application of coloring

- Vertices are **tasks**
- Edge (u, v) is present if tasks u and v each require access to the **same shared resource**, and thus cannot execute simultaneously
- Colors are **time slots** to schedule the tasks
- Minimum number of colors needed to color the graph = minimum number of time slots required



Planarity

- A graph is planar if it can be drawn in the plane without any edges crossing



- Is this graph planar?

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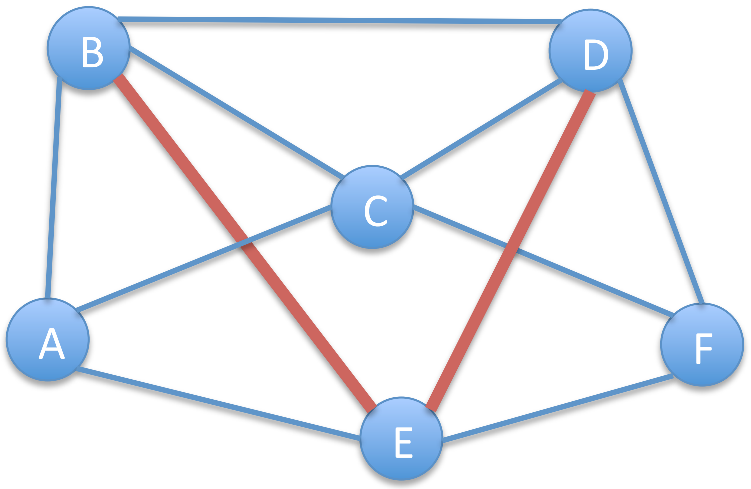

- Is this graph planar?
  - Yes!

Planarity

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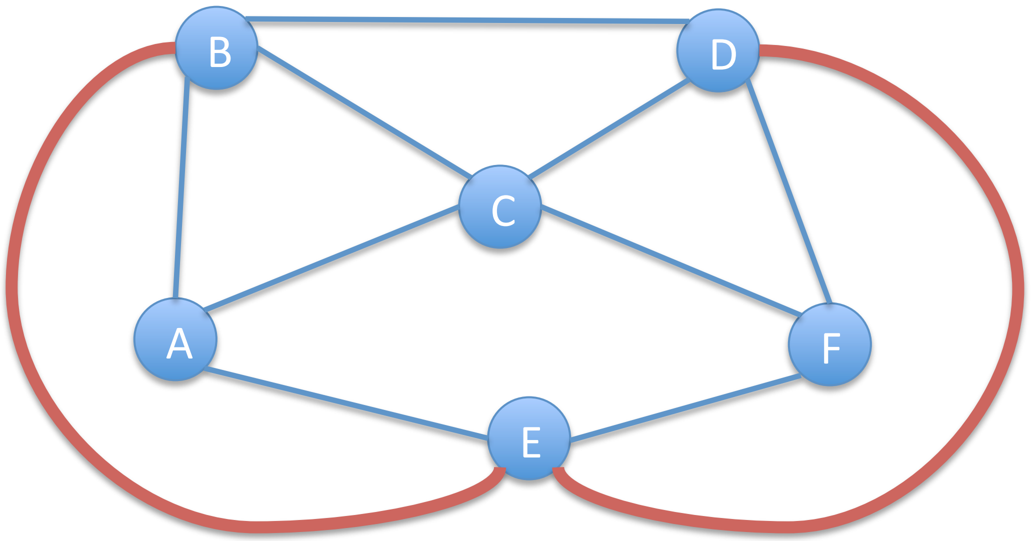
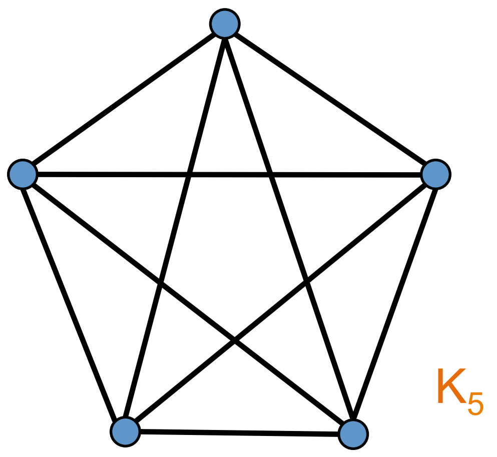
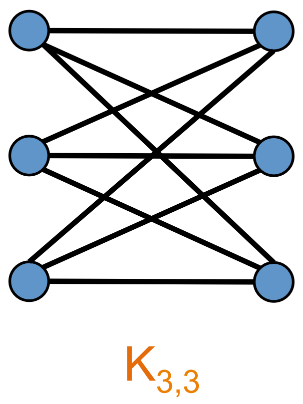


- Is this graph planar?
  - Yes!

Detecting Planarity

Kuratowski's Theorem:



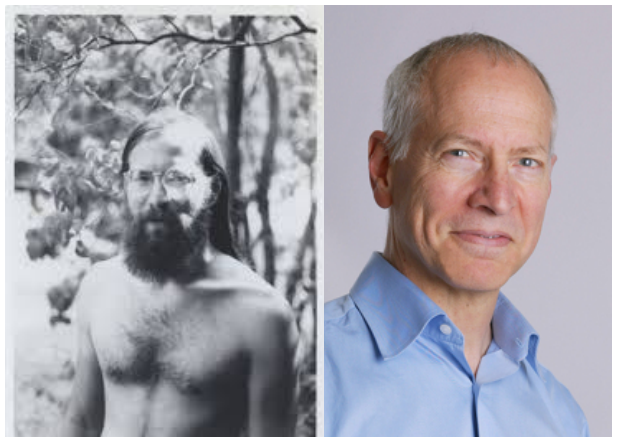


- A graph is planar if and only if it does not contain a copy of K_5 or $K_{3,3}$ (possibly with other nodes along the edges shown)

Detecting Planarity

In the early 1970's, Cornell Prof John Hopcroft spent a sabbatical at Stanford and worked with PhD student Bob Tarjan. They developed the first linear-time algorithm for testing whether a graph was planar. They later received the ACM Turing Award for their work on algorithms.

Tarjan was hired at one point in the 1970's into our department, but the Ithaca weather was too depressing for him and he left for Princeton.



Coloring a graph:

How many colors are needed to color the countries so that no two adjacent countries have the same color?

Question asked as early as 1852.

1879. Kemp publishes a theorem that only 4 colors are needed!

1880. Julius Peterson finds a flaw in the Kemp's proof!

Central Balkan Region
SLOVENIA
HUNGARY
CROATIA
ROMANIA
Vojvodina
BOSNIA AND HERZEGOVINA
Republika Srpska
Federation of Bosnia and Herzegovina
Serbia
Montenegro
Kosovo
BULG.
CROATIA
ALBANIA
THE FORMER YUGOSLAV REPUBLIC OF MACEDONIA
Adriatic Sea
ITALY
GREECE
Strait of Otranto
Scale 1:3,550,000
Lambert Conformal Conic Projection, standard parallels 40 N and 56 N

Four-Color Theorem:

Every planar graph is 4-colorable [Appel & Haken, 1976]

The proof rested on checking that 1,936 special graphs had a certain property.

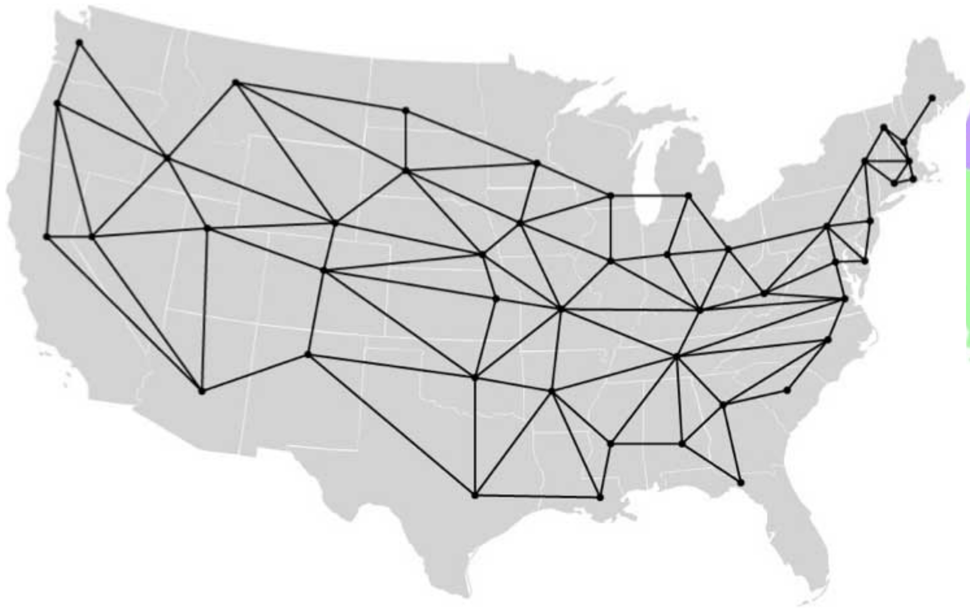
They used a computer to check that those 1, 936 graphs had that property!

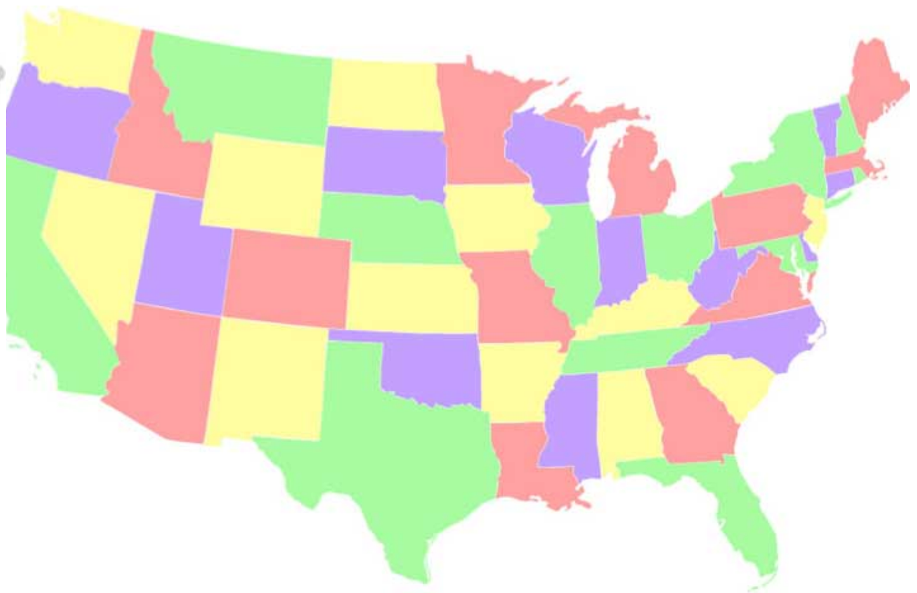
Basically the first time a computer was needed to check something. Caused a lot of controversy.

Gries looked at their computer program, a recursive program written in the assembly language of the IBM 7090 computer, and found an error, which was safe (it said something didn't have the property when it did) and could be fixed. Others did the same.

Since then, there have been improvements. And a formal proof has even been done in the Coq proof system

Another 4-colored planar graph



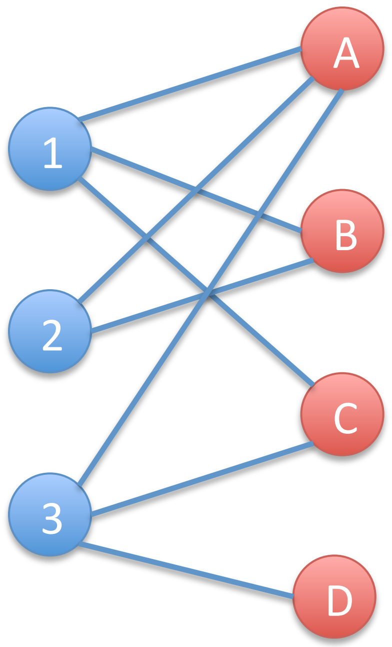


http://www.cs.cmu.edu/~bryant/boolean/maps.html

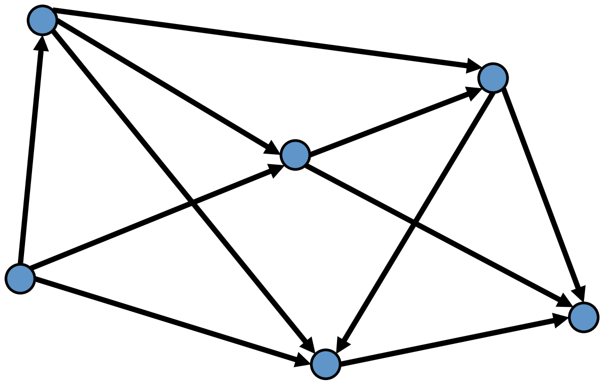
Bipartite graphs

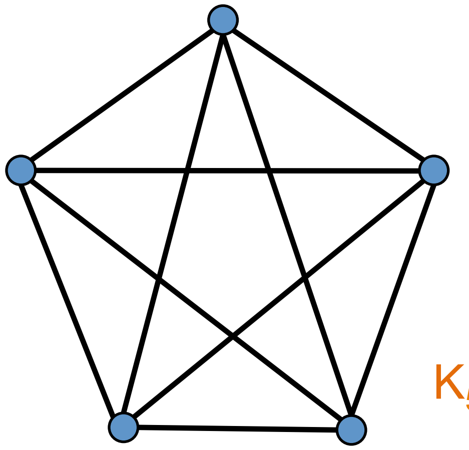
- A directed or undirected graph is bipartite if the vertices can be partitioned into two sets such that no edge connects two vertices in the same set

- The following are equivalent
  - G is bipartite
  - G is 2-colorable
  - G has no cycles of odd length

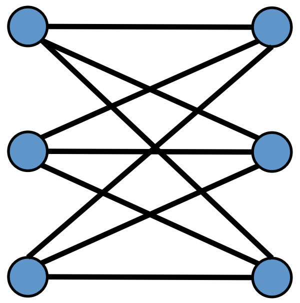


Some abstract graphs

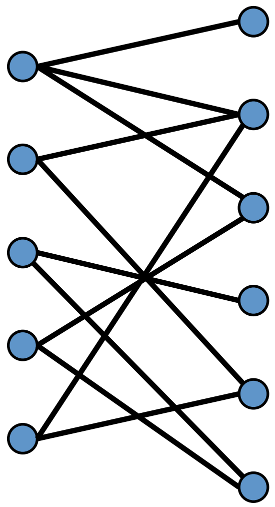


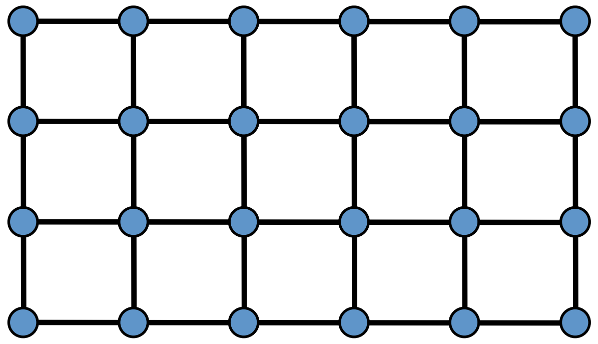


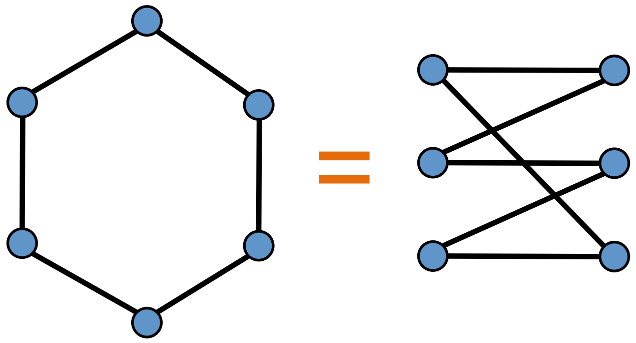
K_5



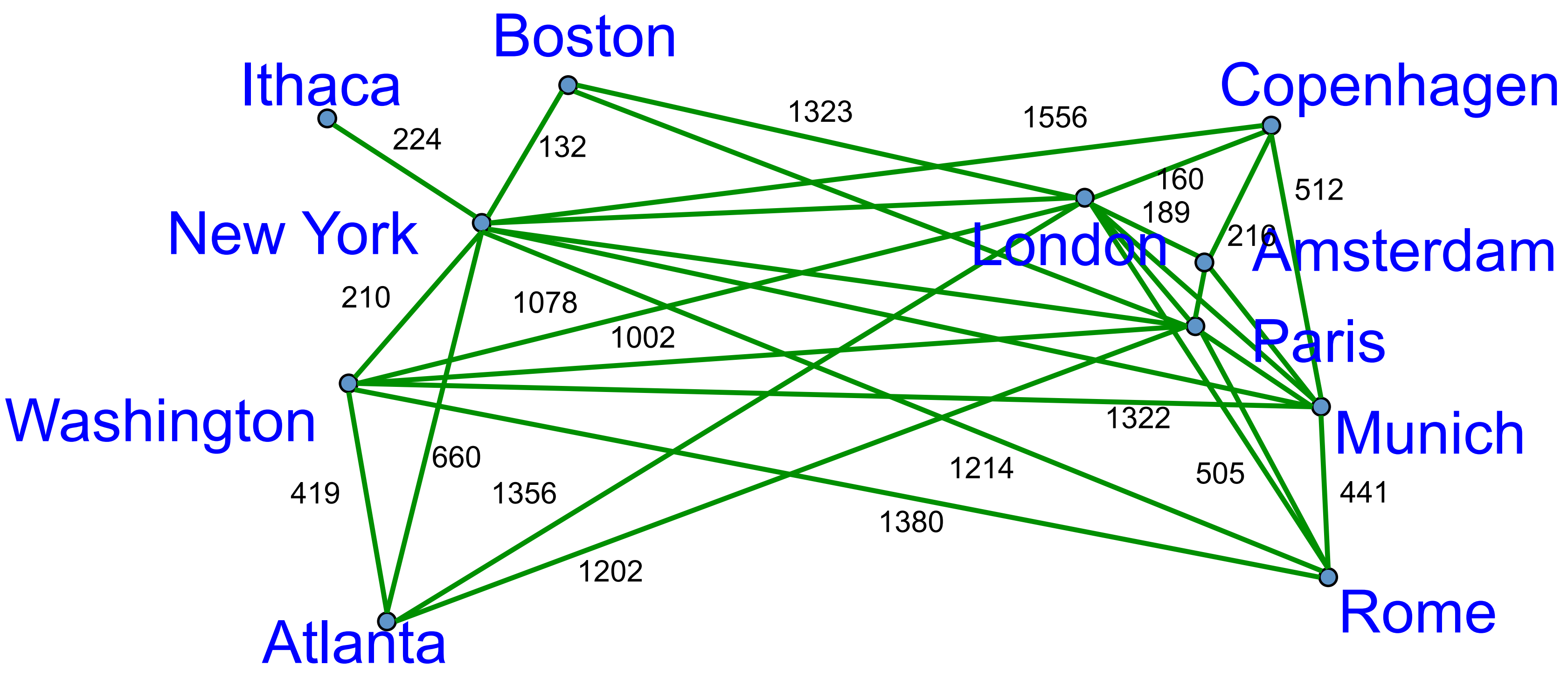
$K_{3,3}$





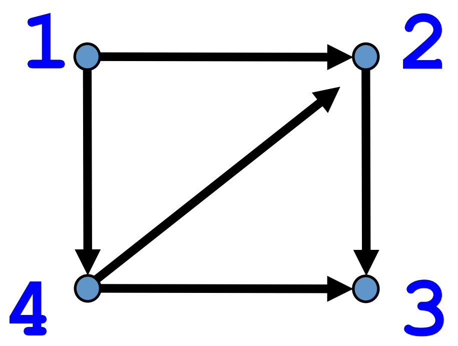


Traveling salesperson

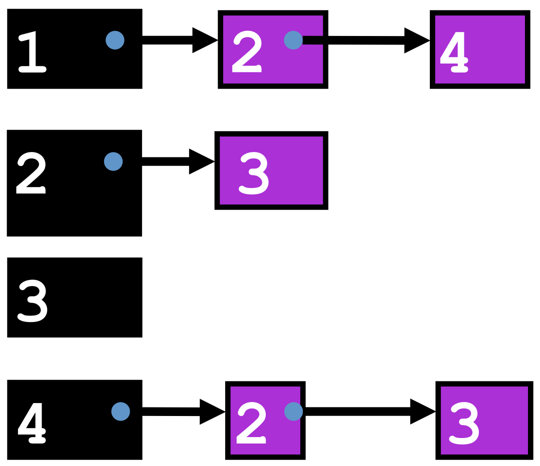


Find a path of minimum distance that visits every city

Representations of graphs



Adjacency List



Adjacency Matrix

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 1 & 0 & 1 \\ 2 & 0 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 & 0 \\ 4 & 0 & 1 & 1 & 0 \end{array}$$

Adjacency matrix or adjacency List?

- n = number of vertices
- m = number of edges
- $d(u)$ = degree of u = no. of edges leaving u

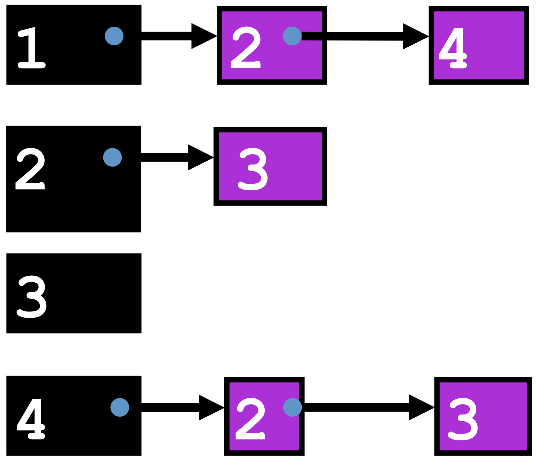
$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 1 & 0 & 1 \\ 2 & 0 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 & 0 \\ 4 & 0 & 1 & 1 & 0 \end{array}$$

- Adjacency Matrix
  - Uses space $O(n^2)$
  - Enumerate all edges in time $O(n^2)$
  - Answer "Is there an edge from u to v?" in $O(1)$ time
  - Better for dense graphs (lots of edges)

Adjacency matrix or adjacency list?

- n = number of vertices
- e = number of edges
- $d(u)$ = degree of u = no. edges leaving u

- Adjacency List
  - Uses space $O(e + n)$
  - Enumerate all edges in time $O(e + n)$
  - Answer “Is there an edge from u to v?” in $O(d(u))$ time
  - Better for sparse graphs (fewer edges)



Graph algorithms

- Search
  - Depth-first search
  - Breadth-first search
- Shortest paths
  - Dijkstra's algorithm
- Minimum spanning trees
  - Jarnik/Prim/Dijkstra algorithm
  - Kruskal's algorithm