SORTING

Insertion sort
Selection sort
Quicksort
Mergesort
And their asymptotic time complexity

See lecture notes page, row in table for this lecture, for file searchSortAlgorithms.zip

Lecture 11
CS2110 – Spring 2017

A3 and Prelim

- 379/607 (62%) people got 65/65 for correctness on A3
- 558/607 (92%) got at least 60/65 for correctness on A3
- Prelim: Next Tuesday evening, March 14
  Read the Exams page on course website to determine when you take the prelim (5:30 or 7:30) and what to do if you have a conflict.
- If necessary, complete CMS assignment P1Conflict by the end of Wednesday (tomorrow).
- So far, only 15 people filled it out!

InsertionSort

pre: b [0 ? b.length]   post: b [0 sorted b.length]

inv: b [0 sorted i ? b.length]

or: b[0..i-1] is sorted

inv: b [0 processed i ? b.length]

A loop that processes elements of an array in increasing order has this invariant

for (int i= 0; i < b.length; i= i+1) { maintain invariant }

Each iteration, i= i+1; How to keep inv true?

inv: b [0 sorted i ? b.length]

e.g. b [0 2 5 5 5 7 i 3 ? b.length]

b [0 2 3 5 5 5 i 7 ? b.length]

Push b[i] down to its shortest position in b[0..i]

Will take time proportional to the number of swaps needed

What to do in each iteration?

inv: b [0 sorted i ? b.length]

e.g. b [0 2 5 5 5 7 i 3 ? b.length]

Loop body (inv true before and after)

2 5 5 5 3 | 7 ?
2 5 5 3 5 | 7 ?
2 5 3 5 5 | 7 ?
2 3 5 5 5 | 7 ?

Push b[i] to its sorted position in b[0..i], then increase i

b [0 2 3 5 5 5 7 i ? b.length]

InsertionSort

```
// sort b[], an array of int
// inv: b[0..i-1] is sorted
for (int i= 0; i < b.length; i= i+1) {
    Push b[i] down to its sorted
    position in b[0..i]
}
```

Note English statement in body. **Abstraction**. Says **what** to do, not **how.**

This is the best way to present it. We expect you to present it this way when asked.

Later, can show how to implement that with an inner loop

Many people sort cards this way
Works well when input is *nearly sorted*

Push b[i] down …

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```
// Q: b[0..i-1] is sorted
// Push b[i] down to its sorted position in b[0..i]
int k= i;
while (k > 0  &&  b[k] < b[k-1]) {
      Swap b[k] and b[k-1]
      k= k–1;
}
// R: b[0..i] is sorted
```

start?
stop?
progress?
maintain invariant?

invariant P: b[0..i] is sorted
except that b[k] may be < b[k-1]

k i
| 2 5 3 5 5 | 7 ? |

example

How to write nested loops

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```
// sort b[], an array of int
// inv: b[0..i-1] is sorted
for (int i= 0; i < b.length; i= i+1) {
   Push b[i] down to its sorted
      position in b[0..i]
}
```

Present algorithm like this

If you are going to show implementation, *put in "WHAT IT DOES" as a comment*

```
// sort b[], an array of int
// inv: b[0..i-1] is sorted
for (int i= 0; i < b.length; i= i+1) {
   //Push b[i] down to its sorted
   //position in b[0..i]
   int k= i;
   while (k > 0  &&  b[k] < b[k-1]) {
      swap b[k] and b[k-1];
      k= k-1;
   }
}
```

InsertionSort

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```
// sort b[], an array of int
// inv: b[0..i-1] is sorted
for (int i= 0; i < b.length; i= i+1) {
   Push b[i] down to its sorted position
   in b[0..i]
}
```

Let n = b.length

- Worst-case: O(n^2) (reverse-sorted input)
- Best-case: O(n) (sorted input)
- Expected case: O(n^2)

Pushing b[i] down can take i swaps.
Worst case takes

1 + 2 + 3 + … n-1 = (n-1)*n/2

Swaps.

SelectionSort

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pre: b [0 … b.length: ?] post: b [0 … b.length: sorted]

inv: b [0 … i: sorted , <= b[i..] | i … b.length: >= b[0..i-1]]

Additional term in invariant

Keep invariant true while making progress?

e.g.: b [0 … i: 1 2 3 4 5 6 | i … b.length: 9 9 9 7 8 6 9]

Increasing i by 1 keeps inv true only if b[i] is min of b[i..]

SelectionSort

```
//sort b[], an array of int
// inv: b[0..i-1] sorted  AND
//       b[0..i-1]  <=  b[i..]
for (int i= 0; i < b.length; i= i+1) {
   int m= index of minimum of b[i..];
   Swap b[i] and b[m];
}
```

Another common way for people to sort cards

Runtime with n = b.length
- Worst-case O(n^2)
- Best-case O(n^2)
- Expected-case O(n^2)

b [0 … i: sorted, smaller values | i … length: larger values]

Each iteration, swap min value of this section into b[i]

Swapping b[i] and b[m]

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```
// Swap b[i] and b[m]
int t= b[i];
b[i]= b[m];
b[m]= t;
```

Partition algorithm of quicksort

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pre: h h+1 k

x	?

x is called the pivot

Swap array values around until b[h..k] looks like this:

post: h j k

<= x	x	>= x

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20	31	24	19	45	56	4	20	5	72	14	99

pivot

partition

j

19	4	5	14	20	31	24	45	56	20	72	99

Not yet sorted

Not yet sorted

these can be in any order

these can be in any order

The 20 could be in the other partition

Partition algorithm

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pre: h h+1 k

b

x	?

post: h j k

b

<= x	x	>= x

Combine pre and post to get an invariant

invariant needs at least 4 sections

h j t k

b

<= x	x	?	>= x

Partition algorithm

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h j t k

b

<= x	x	?	>= x

Initially, with j = h and t = k, this diagram looks like the start diagram

```
j= h; t= k;
while (j < t) {
   if (b[j+1] <= b[j]) {
      Swap b[j+1] and b[j];   j= j+1;
   } else {
      Swap b[j+1] and b[t];   t= t-1;
   }
}
```

Takes linear time: O(k+1-h)

Terminate when j = t, so the “?” segment is empty, so diagram looks like result diagram

---

QuickSort procedure

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```
/** Sort b[h..k]. */
public static void QS(int[] b, int h, int k) {
   if (b[h..k] has < 2 elements) return;
   int j= partition(b, h, k);
      // We know b[h..j–1] <= b[j] <= b[j+1..k]
      // Sort b[h..j-1] and b[j+1..k]
      QS(b, h, j-1);
      QS(b, j+1, k);
}
```

Base case

Function does the partition algorithm and returns position j of pivot

h j k

<= x	x	>= x

QuickSort

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Quicksort developed by Sir Tony Hoare (he was knighted by the Queen of England for his contributions to education and CS).

83 years old.

Developed Quicksort in 1958. But he could not explain it to his colleague, so he gave up on it.

Later, he saw a draft of the new language Algol 58 (which became Algol 60). It had recursive procedures. First time in a procedural programming language. “Ah!,” he said. “I know how to write it better now.” 15 minutes later, his colleague also understood it.

Tony Hoare

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Speaking in Olin 155
in 2004

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Elaine Gries, Edsger and Ria Dijkstra, Tony and Jill Hoare
1980s.

Worst case quicksort: pivot always smallest value

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j		n	
x0	>= x0		partioning at depth 0

	j		
x0	x1	>= x1	partioning at depth 1

		j		
x0	x1	x2	>= x2	partioning at depth 2

```
/** Sort b[h..k]. */
public static void QS(int[] b, int h, int k) {
    if (b[h..k] has < 2 elements) return;
    int j= partition(b, h, k);
    QS(b, h, j-1);     QS(b, j+1, k);
```

Depth of recursion: O(n)

Processing at depth i: O(n-i)

O(n*n)

Best case quicksort: pivot always middle value

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0	j	n
<= x0	x0	>= x0

depth 0. 1 segment of size ~n to partition.

<=x1	x1	>= x1	x0	<=x2	x2	>=x2

Depth 2. 2 segments of size ~n/2 to partition.

Depth 3. 4 segments of size ~n/4 to partition.

Max depth: O(log n). Time to partition on each level: O(n)
Total time: O(n log n).

Average time for Quicksort: n log n. Difficult calculation

QuickSort complexity to sort array of length n

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Time complexity
Worst-case: O(n*n)
Average-case: O(n log n)

```
/** Sort b[h..k]. */
public static void QS(int[] b, int h, int k) {
   if (b[h..k] has < 2 elements) return;
   int j= partition(b, h, k);
   // We know b[h..j–1] <= b[j] <= b[j+1..k]
   // Sort b[h..j-1] and b[j+1..k]
  QS(b, h, j-1);
  QS(b, j+1, k);
}
```

Worst-case space: ?
What's depth of recursion?

Worst-case space: O(n)!
  --depth of recursion can be n
Can rewrite it to have space O(log n)
Show this at end of lecture if we have time

Partition. Key issue. How to choose pivot

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	h		k
pre: b	x	?	

	h	j	k
post: b	<= x	x	>= x

Choosing pivot
Ideal pivot: the median, since it splits array in half
But computing is O(n), quite complicated

Popular heuristics: Use
- first array value (not so good)
- middle array value (not so good)
- Choose a random element (not so good)
- median of first, middle, last, values (often used)!

Merge two adjacent sorted segments

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/* Sort b[h..k]. Precondition: b[h..t] and b[t+1..k] are sorted. */
public static merge(int[] b, int h, int t, int k) {
}

	h				t				k
b	4	7	7	8	9	3	4	7	8

h t k
sorted | sorted

b	3	4	4	7	7	7	8	8	9

h k
merged, sorted

Merge two adjacent sorted segments

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/* Sort b[h..k]. Precondition: b[h..t] and b[t+1..k] are sorted. */
public static merge(int[] b, int h, int t, int k) {
Copy b[h..t] into a new array c;
Merge c and b[t+1..k] into b[h..k];
}

Runs in time linear in size of b[h..k].
Look at this method in file searchSortAlgorithms.zip found in row for lecture on Lecture notes page of course website

h t k
sorted | sorted

h k
merged, sorted

Merge two adjacent sorted segments

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// Merge sorted c and b[t+1..k] into b[h..k]

pre: c [0 x t-h]   b [h ? t y k]   x, y are sorted

post: b [h x and y, sorted k]

invariant: c [0 head of x i tail of x c.length]

b [h u ? v tail of y k]

head of x and head of y, sorted

b[h..u-1] ≤ c[i..]

b[h..u-1] ≤ b[v..k]

Mergesort

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```
/** Sort b[h..k] */
public static void mergesort(int[] b, int h, int k]) {
    if (size b[h..k] < 2)
        return;
    int t= (h+k)/2;
    mergesort(b, h, t);
    mergesort(b, t+1, k);
    merge(b, h, t, k);
}
```

h t k
sorted | sorted

h k
merged, sorted

Mergesort

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```
/** Sort b[h..k] */
public static void mergesort(
        int[] b, int h, int k]) {
    if (size b[h..k] < 2)
        return;
    int t= (h+k)/2;
    mergesort(b, h, t);
    mergesort(b, t+1, k);
    merge(b, h, t, k);
}
```

Let n = size of b[h..k]

Merge: time proportional to n

Depth of recursion: log n

Can therefore show (later) that time taken is proportional to n log n

But space is also proportional to n

QuickSort versus MergeSort

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```
/** Sort b[h..k] */
public static void QS
        (int[] b, int h, int k) {
    if (k – h < 1) return;
    int j= partition(b, h, k);
    QS(b, h, j-1);
    QS(b, j+1, k);
}
```

```
/** Sort b[h..k] */
public static void MS
        (int[] b, int h, int k) {
    if (k – h < 1) return;
    MS(b, h, (h+k)/2);
    MS(b, (h+k)/2 + 1, k);
    merge(b, h, (h+k)/2, k);
}
```

One processes the array then recurses.
One recurses then processes the array.

Analysis of Matrix Multiplication

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Multiply n-by-n matrices A and B:

Convention, matrix problems measured in terms of n, the number of rows, columns

- Input size is really $2n^2$, not n
- Worst-case time: $O(n^3)$
- Expected-case time: $O(n^3)$

```
for (i = 0; i < n; i++)
  for (j = 0; j < n; j++) {
    c[i][j] = 0;
    for (k = 0; k < n; k++)
      c[i][j] += a[i][k]*b[k][j];
  }
```

```
for (i = 0; i < n; i++)
  for (j = 0; j < n; j++) {
    throw new Exception();
}}
```

An aside. Will not be tested. Lower Bound for Comparison Sorting

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Goal: Determine minimum time *required* to sort *n* items

Note: we want *worst-case*, not *best-case* time

- Best-case doesn't tell us much. E.g. Insertion Sort takes O(n) time on already-sorted input
- Want to know *worst-case time* for *best possible* algorithm

- How can we prove anything about the *best possible* algorithm?
- Want to find characteristics that are common to *all* sorting algorithms
- Limit attention to *comparison-based algorithms* and try to count number of comparisons

An aside. Will not be tested. Lower Bound for Comparison Sorting

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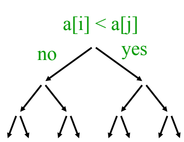
- Comparison-based algorithms make decisions based on comparison of data elements
- Gives a *comparison tree*
- If algorithm fails to terminate for some input, comparison tree is infinite
- Height of comparison tree represents *worst-case number of comparisons* for that algorithm
- Can show: *Any* correct comparison-based algorithm must make at least n log n comparisons in the worst case



An aside. Will not be tested. Lower Bound for Comparison Sorting

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- Say we have a correct comparison-based algorithm
- Suppose we want to sort the elements in an array b[]
- Assume the elements of b[] are distinct
- Any permutation of the elements is initially possible
- When done, b[] is sorted
- But the algorithm could not have taken the same path in the comparison tree on different input permutations

An aside. Will not be tested. Lower Bound for Comparison Sorting

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How many input permutations are possible? $n! \sim 2^{n \log n}$

For a comparison-based sorting algorithm to be correct, it must have at least that many leaves in its comparison tree

To have at least $n! \sim 2^{n \log n}$ leaves, it must have height at least n log n (since it is only binary branching, the number of nodes at most doubles at every depth)

Therefore its longest path must be of length at least n log n, and that is its worst-case running time

Quicksort with logarithmic space

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Problem is that if the pivot value is always the smallest (or always the largest), the depth of recursion is the size of the array to sort.

Eliminate this problem by doing some of it iteratively and some recursively

Quicksort with logarithmic space

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Problem is that if the pivot value is always the smallest (or always the largest), the depth of recursion is the size of the array to sort.

Eliminate this problem by doing some of it iteratively and some recursively. We may show you this later. Not today!

It's on the next two slides. You do not have to study this for the prelim!

QuickSort with logarithmic space

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```
/** Sort b[h..k]. */
public static void QS(int[] b, int h, int k) {
   int h1= h; int k1= k;
   // invariant b[h..k] is sorted if b[h1..k1] is sorted
   while (b[h1..k1] has more than 1 element) {
        Reduce the size of b[h1..k1], keeping inv true
   }
}
```

QuickSort with logarithmic space

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```
/** Sort b[h..k]. */
public static void QS(int[] b, int h, int k) {
   int h1= h; int k1= k;
   // invariant b[h..k] is sorted if b[h1..k1] is sorted
   while (b[h1..k1] has more than 1 element) {
        int j= partition(b, h1, k1);
        // b[h1..j-1] <= b[j] <= b[j+1..k1]
        if (b[h1..j-1] smaller than b[j+1..k1])
              {  QS(b, h, j-1);  h1= j+1; }
        else
              {QS(b, j+1, k1);  k1= j-1; }
   }
}
```

Only the smaller segment is sorted recursively. If b[h1..k1] has size n, the smaller segment has size < n/2. Therefore, depth of recursion is at most log n