RECURSION

Lecture 8

CS2110 – Fall 2016

Overview references to sections in text

- Note: We've covered everything in JavaSummary.pptx!
- What is recursion? 7.1-7.39   slide 1-7
- Base case   7.1-7.10 slide 13
- How Java stack frames work 7.8-7.10 slide 28-32

Next week's recitation

Study material on loop invariants here:

www.cs.cornell.edu/courses/CS2110/2016sp/online/index.html

Link: on links and on lecture notes pages of course website.

Do that BEFORE the MANDATORY recitation.

Then, do some problem solving as in this week's recitation

More work for us, not less
Doing this for first time in 2110. We will make mistakes.
Appreciate your tolerance and patience as we try something that studies show works better than conventional lectures

Why flip the class this way?

Usual way. 50-minute lecture, then study on your own. One hour? Total of, say, 2 hours.

Disadvantages:

- Hard to listen attentively for 50 minutes. Many people tune out, look at internet, videos, whatever
- Much time wasted here and there
- You don't always know just how to study. No problem sets, and if there are, no easy way to check answers.
- Study may consist of reading, not doing. Doesn't help.

Why flip the class this way?

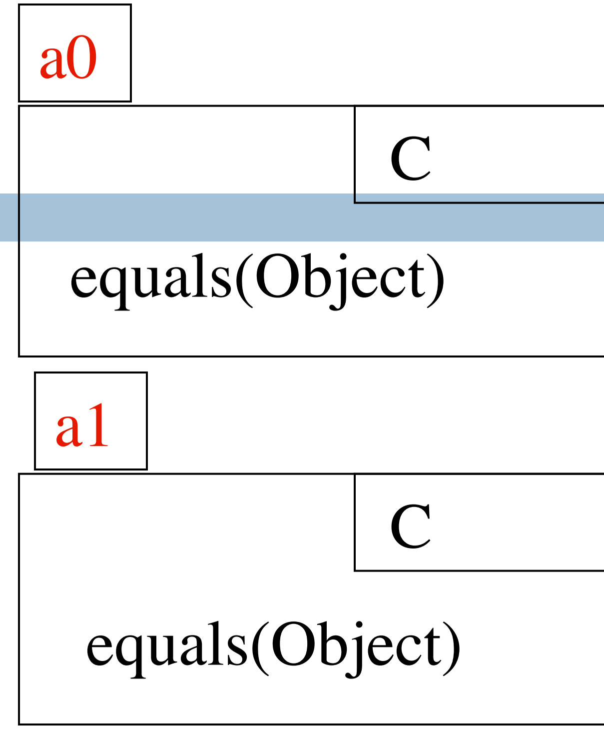
Flipped way. Watch short, usually 3-5 minute, videos on a topic. Then come to recitation and participate in solving problems.

Disadvantage: If you don't study the videos carefully, you are wasting your time.

Advantages

- Break up watching videos into shorter time periods.
- Watch parts of one several times.
- In recitation, you get to DO something, not just read, and you get to discuss with a partner and neighbors, ask TA questions, etc.

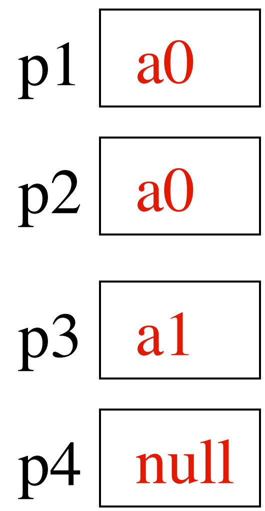
== versus equals



Use p1 == p2 or p1 != p2

to determine whether p1 and p2 point to the same object (or are both null).

Do NOT use p1.equals(p2) for this purpose, because it doesn't always tell whether they point to the same object! It depends on how equals is defined.



p4.equals(p1)

Null pointer exception!

p2 == p1 true

p3 == p1 false

p4 == p1 false

Sum the digits in a non-negative integer


```
/** = sum of digits in n.
  * Precondition:  n >= 0 */
 public static int sum(int n) {
     if (n < 10) return n;

    // { n has at least two digits }
    // return first digit + sum of rest
    return sum(n/10) +  n%10 ;
 }
```

sum calls itself!

E.g. sum(7) = 7

E.g. sum(8703) = sum(870) + 3;

Two issues with recursion


```
/** return sum of digits in n.
  * Precondition:  n >= 0 */
 public static int sum(int n) {
     if (n < 10) return n;

     // { n has at least two digits }
     // return first digit + sum of rest
     return sum(n/10)  +  n%10  + ;
 }
```

sum calls itself!

1. Why does it work? How does execution work?

2. How do we understand a given recursive method or how do we write/develop a recursive method?

Stacks and Queues

Stack: list with (at least) two basic ops:

* Push an element onto its top
* Pop (remove) top element

Last-In-First-Out (LIFO)

Like a stack of trays in a cafeteria

first second … last

Americans wait in a line. The Brits wait in a queue !

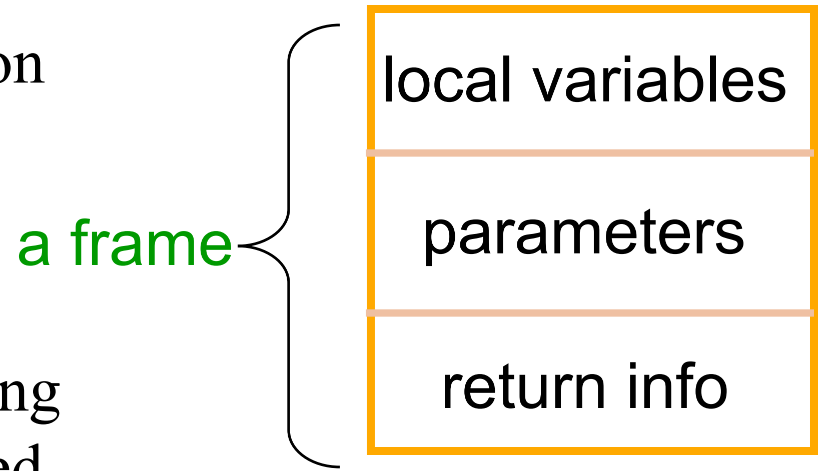
Queue: list with (at least) two basic ops:

* Append an element
* Remove first element

First-In-First-Out (FIFO)

Stack Frame

A "frame" contains information about a method call:



At runtime Java maintains a stack that contains frames for all method calls that are being executed but have not completed.

Method call: push a frame for call on stack assign argument values to parameters execute method body. Use the frame for the call to reference local variables parameters.

End of method call: pop its frame from the stack; if it is a function leave the return value on top of stack.

Frames for methods sum main method in the system


```
public static int sum(int n) {
    if (n < 10) return n;
    return sum(n/10)  +  n%10;
}

public static void main(
        String[] args) {
  int r= sum(824);
  System.out.println(r);
}
```

frame: n ___ return info

frame: r ___ args ___ return info

frame: ? return info

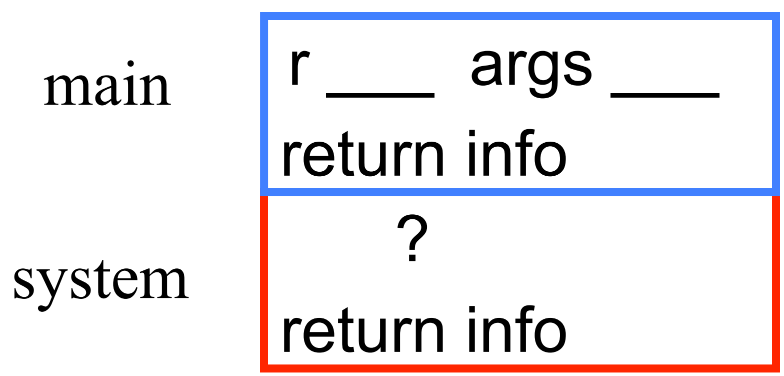
Frame for method in the system that calls method main

Example: Sum the digits in a non-negative integer


```
public static int sum(int n) {
    if (n < 10) return n;
    return sum(n/10)  +  n%10;
}

public static void main(
        String[] args) {
   int r= sum(824);
   System.out.println(r);
}
```

Frame for method in the system that calls method main: main is then called

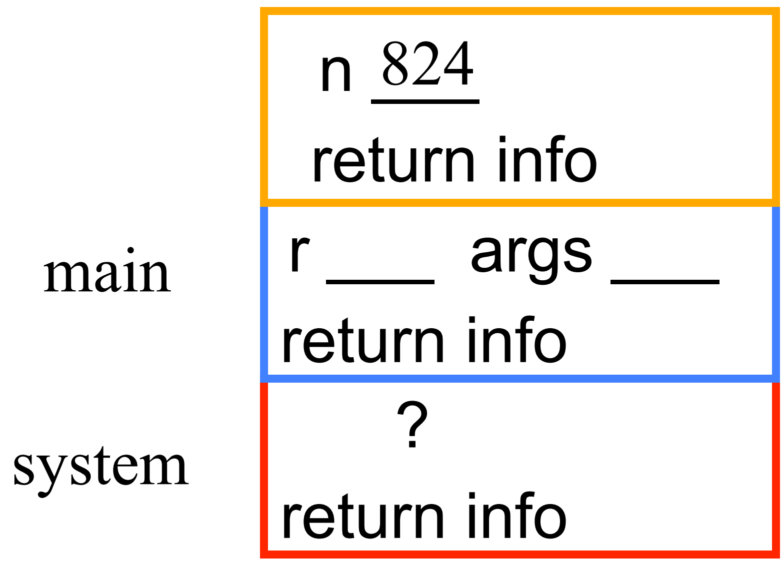


Example: Sum the digits in a non-negative integer


```
public static int sum(int n) {
    if (n < 10) return n;
    return sum(n/10)  +  n%10;
}

public static void main(
        String[] args) {
    int r= sum(824);
    System.out.println(r);
}
```

Method main calls sum:

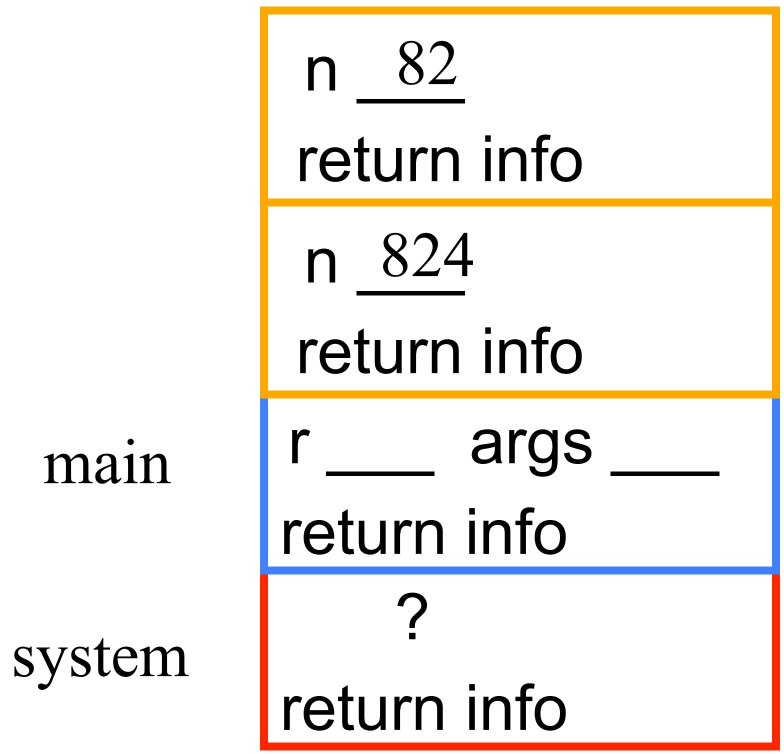


Example: Sum the digits in a non-negative integer


```
public static int sum(int n) {
    if (n < 10) return n;
    return sum(n/10)  +  n%10;
}

public static void main(
       String[] args) {
   int r= sum(824);
   System.out.println(r);
}
```

n >= 10 sum calls sum:

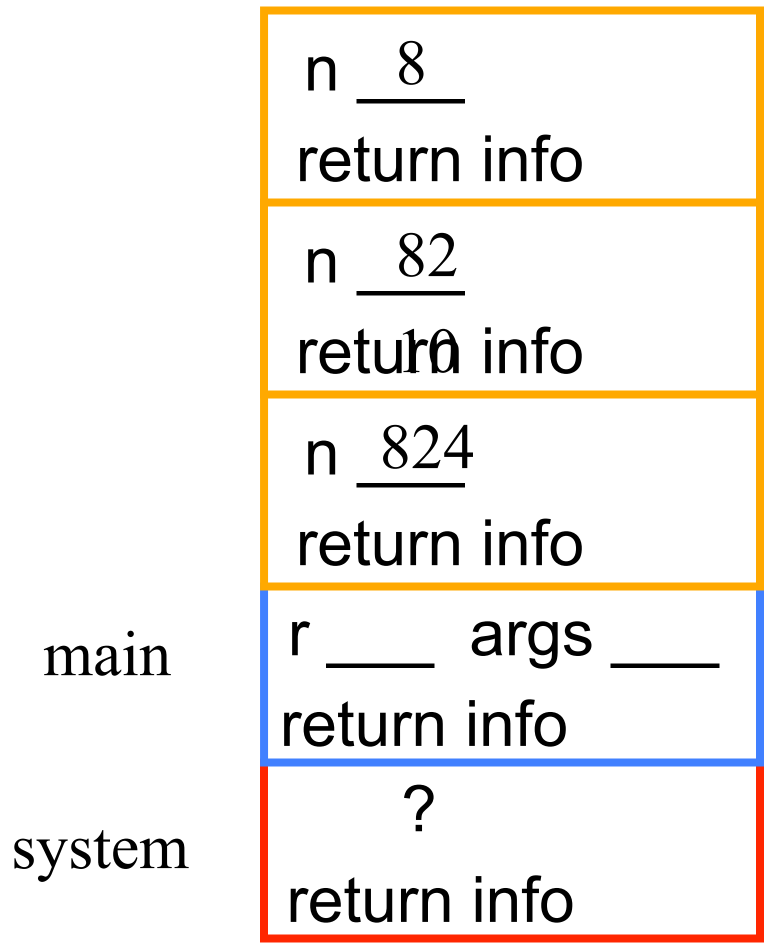


Example: Sum the digits in a non-negative integer


```
public static int sum(int n) {
    if (n < 10) return n;
    return sum(n/10)  +  n%10;
}

public static void main(
        String[] args) {
  int r= sum(824);
  System.out.println(r);
}
```

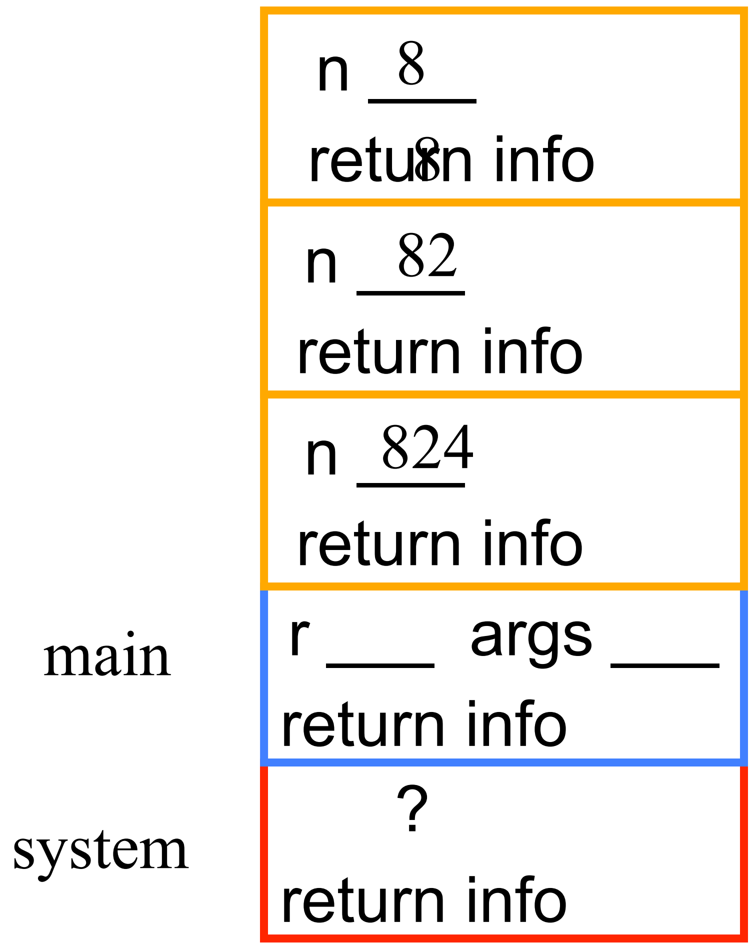
n >= 10. sum calls sum:



Example: Sum the digits in a non-negative integer


```
public static int sum(int n) {
    if (n < 10) return n;
    return sum(n/10)  +  n%10;
}

public static void main(
       String[] args) {
   int r= sum(824);
   System.out.println(r);
}
```

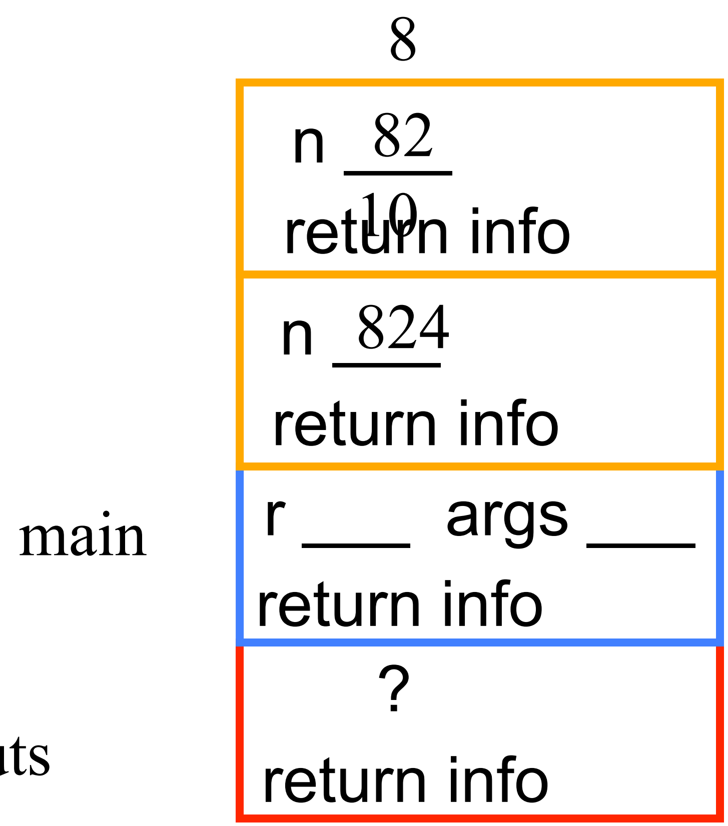


n < 10 sum stops: frame is popped and n is put on stack:

Example: Sum the digits in a non-negative integer


```
public static int sum(int n) {
     if (n < 10) return n;
     return sum(n/10)  +  n%10;
}

public static void main(
        String[] args) {
   int r= sum(824);
   System.out.println(r);
}
```

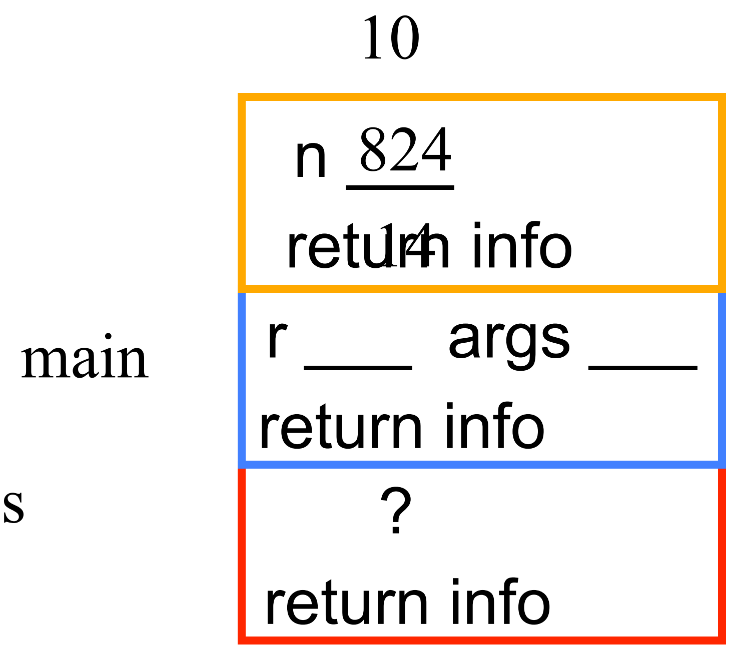


Using return value 8 stack computes
8 + 2 = 10 pops frame from stack puts
return value 10 on stack

Example: Sum the digits in a non-negative integer


```
public static int sum(int n) {
    if (n < 10) return n;
    return sum(n/10)  +  n%10;
}

public static void main(
       String[] args) {
   int r= sum(824);
   System.out.println(r);
}
```



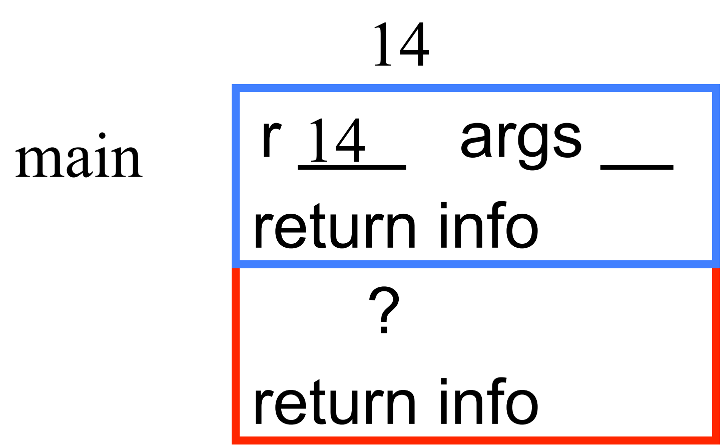
Using return value 10 stack computes
10 + 4 = 14 pops frame from stack
puts return value 14 on stack

Example: Sum the digits in a non-negative integer

```
public static int sum(int n) {
    if (n < 10) return n;
    return sum(n/10)  +  n%10;
}

public static void main(
      String[] args) {
  int r= sum(824);
  System.out.println(r);
}
```

Using return value 14 main stores
14 in r and removes 14 from stack



Memorize method call execution!

A frame for a call contains parameters, local variables, and other information needed to properly execute a method call.

To execute a method call:

1. push a frame for the call on the stack,
2. assign argument values to parameters,
3. execute method body,
4. pop frame for call from stack, and (for a function) push returned value on stack

When executing method body look in frame for call for parameters and local variables.

Questions about local variables


```
public static void m(…) {
    …
    while (…) {
        int d= 5;
        …
    }
}
```

```
public static void m(…) {
    int d;
    …
    while (…) {
        d= 5;
        …
    }
}
```

In a call m(…)
when is local variable d created and when is it destroyed?
Which version of procedure m do you like better? Why?

Recursion is used extensively in math

Math definition of n factorial

$0! = 1$

$n! = n * (n-1)!$ for $n > 0$

E.g. $3! = 3*2*1 = 6$

Easy to make math definition into a Java function!

```
public static int fact(int n) {
   if (n == 0) return 1;

   return n * fact(n-1);
}
```

Math definition of b^c for $c >= 0$

$b^0 = 1$

$b^c = b * b^{c-1}$ for $c > 0$

Lots of things defined recursively:
expression grammars trees ….
We will see such things later

Two views of recursive methods

- How are calls on recursive methods executed?
  We saw that. Use this only to gain understanding / assurance that recursion works
- How do we understand a recursive method — know that it satisfies its specification? How do we write a recursive method?
  This requires a totally different approach. Thinking about how the method gets executed will confuse you completely! We now introduce this approach.

How to understand what a call does

Make a copy of the method spec, replacing the parameters of the method by the arguments

spec says that the value of a call equals the sum of the digits of n

sum(654)

sum of digits of **n**

sum of digits of **654**

```
/** =  sum of the digits of n.
  * Precondition:  n >= 0 */
public static int sum(int n) {
    if (n < 10) return n;
    // n has at least two digits
    return sum(n/10)  +  n%10 ;
}
```

Understanding a recursive method

Step 1. Have a precise spec!

Step 2. Check that the method works in the base case(s): Cases where the parameter is small enough that the result can be computed simply and without recursive calls.

If n < 10 then n consists of a single digit. Looking at the spec we see that that digit is the required sum.

```
/** = sum of digits of n.
  * Precondition: n >= 0 */
public static int sum(int n) {
    if (n < 10) return n;

    // n has at least two digits
    return sum(n/10) + n%10 ;
}
```

Understanding a recursive method

Step 1. Have a precise spec!

Step 2. Check that the method works in the base case(s).

Step 3. Look at the recursive case(s). In your mind replace each recursive call by what it does according to the method spec and verify that the correct result is then obtained.

```
/** = sum of digits of n.
  * Precondition: n >= 0 */
public static int sum(int n) {
    if (n < 10) return n;

    // n has at least two digits
    return sum(n/10) + n%10 ;
}
```

```
return sum(n/10) + n%10;

return (sum of digits of n/10) + n%10;      // e.g. n = 843
```

Understanding a recursive method

Step 1. Have a precise spec!

Step 2. Check that the method works in the base case(s).

Step 3. Look at the recursive case(s). In your mind replace each recursive call by what it does acc. to the spec and verify correctness.

```
/** = sum of digits of n.
  * Precondition: n >= 0 */
public static int sum(int n) {
    if (n < 10) return n;

    // n has at least two digits
    return sum(n/10) + n%10 ;
}
```

Step 4. (No infinite recursion) Make sure that the args of recursive calls are in some sense smaller than the pars of the method.

$n/10 < n$

Understanding a recursive method

Step 1. Have a precise spec!

Important! Can't do step 3 without it

Step 2. Check that the method works in the base case(s).

Step 3. Look at the recursive case(s). In your mind replace each recursive call by what it does according to the spec and verify correctness.

Once you get the hang of it this is what makes recursion easy! This way of thinking is based on math induction which we don't cover in this course.

Step 4. (No infinite recursion) Make sure that the args of recursive calls are in some sense smaller than the pars of the method

Writing a recursive method

Step 1. Have a precise spec!

Step 2. Write the base case(s): Cases in which no recursive calls are needed Generally for “small” values of the parameters.

Step 3. Look at all other cases. See how to define these cases in terms of smaller problems of the same kind. Then implement those definitions using recursive calls for those smaller problems of the same kind. Done suitably point 4 is automatically satisfied.

Step 4. (No infinite recursion) Make sure that the args of recursive calls are in some sense smaller than the pars of the method

Examples of writing recursive functions

For the rest of the class we demo writing recursive functions using the approach outlined below. The java file we develop will be placed on the course webpage some time after the lecture.

Step 1. Have a precise spec!

Step 2. Write the base case(s).

Step 3. Look at all other cases. See how to define these cases in terms of smaller problems of the same kind. Then implement those definitions using recursive calls for those smaller problems of the same kind.

The Fibonacci Function

Mathematical definition:

fib(0) = 0

fib(1) = 1

two base cases!

fib(n) = fib(n − 1) + fib(n − 2) $n \geq 2$

Fibonacci sequence: 0 1 1 2 3 5 8 13 …

```
/** = fibonacci(n). Pre: n >= 0 */
static int fib(int n) {
  if (n <= 1) return n;
  // { 1 < n }
  return fib(n-2) + fib(n-1);
}
```

Fibonacci (Leonardo Pisano) 1170-1240?

Statue in Pisa Italy

Giovanni Paganucci

1863

Check palindrome-hood

A String palindrome is a String that reads the same backward and forward.

A String with at least two characters is a palindrome if

- (0) its first and last characters are equal and
- (1) chars between first & last form a palindrome:

have to be the same

e.g. AMANAPLANACANALPANAMA

have to be a palindrome

A recursive definition!

Example: Is a string a palindrome?


```
/** = "s is a palindrome" */
public static boolean isPal(String s) {
    if (s.length() <= 1)
        return true;

    // { s has at least 2 chars }
    int n= s.length()-1;
    return s.charAt(0) == s.charAt(n)  &&  isPal(s.substring(1,n));
}
```

Substring from s[1] to s[n-1]

isPal("racecar") returns true

isPal("pumpkin") returns false

- A man a plan a caret a ban a myriad a sum a lac a liar a hoop a pint a catalpa a gas an oil a bird a yell a vat a caw a pax a wag a tax a nay a ram a cap a yam a gay a tsar a wall a car a luger a ward a bin a woman a vassal a wolf a tuna a nit a pall a fret a watt a bay a daub a tan a cab a datum a gall a hat a fag a zap a say a jaw a lay a wet a gallop a tug a trot a trap a tram a torr a caper a top a tonk a toll a ball a fair a sax a minim a tenor a bass a passer a capital a rut an amen a ted a cabal a tang a sun an ass a maw a sag a jam a dam a sub a salt an axon a sail an ad a wadi a radian a room a rood a rip a tad a pariah a revel a reel a reed a pool a plug a pin a peek a parabola a dog a pat a cud a nu a fan a pal a rum a nod an eta a lag an eel a batik a mug a mot a nap a maxim a mood a leek a grub a gob a gel a drab a citadel a total a cedar a tap a gag a rat a manor a bar a gal a cola a pap a yaw a tab a raj a gab a nag a pagan a bag a jar a bat a way a papa a local a gar a baron a mat a rag a gap a tar a decal a tot a led a tic a bard a leg a bog a burg a keel a doom a mix a map an atom a gum a kit a baleen a gala a ten a don a mural a pan a faun a ducat a pagoda a lob a rap a keep a nip a gulp a loop a deer a leer a lever a hair a pad a tapir a door a moor an aid a raid a wad an alias an ox an atlas a bus a madam a jag a saw a mass an anus a gnat a lab a cadet an em a natural a tip a caress a pass a baronet a minimax a sari a fall a ballot a knot a pot a rep a carrot a mart a part a tort a gut a poll a gateway a law a jay a sap a zag a fat a hall a gamut a dab a can a tabu a day a batt a waterfall a patina a nut a flow a lass a van a mow a nib a draw a regular a call a war a stay a gam a yap a cam a ray an ax a tag a wax a paw a cat a valley a drib a lion a saga a plat a catnip a pooh a rail a calamus a dairyman a bater a canal Panama

Example: Count the e's in a string


```
/** =  number of times c occurs in s */
public static int countEm(char c String s) {
   if (s.length() == 0) return 0;

   // { s has at least 1 character }
   if (s.charAt(0) != c)
        return countEm(c s.substring(1));

   // { first character of s is c}
   return 1 + countEm (c s.substring(1));
}
```

substring s[1..]
i.e. s[1] …
s(s.length()-1)

- countEm('e' "it is easy to see that this has many e's") = 4
- countEm('e' "Mississippi") = 0