CORRECTNESS ISSUES AND LOOP INVARIANTS

Lecture 8
CS2110 – Spring 2015

The next several lectures

Study algorithms for searching and sorting arrays.
Investigate their complexity –how much time and space they take
"Formalize" the notions of average-case and worst-case complexity

We want you to *know* these algorithms

- *Not* by memorizing code but by
- Being able to *develop* the algorithms from their specifications and, when necessary, a small idea

We give you some guidelines and instructions on how to develop an algorithm from its specification.
Deal mainly with developing loops.

Many (most) of you could use instruction on developing algorithms, keeping things simple

```
String[] dummy = s.split(""); // turns s into string array
int len = s.length()-1; // length of string s
String a = “”; // will be reverse of s

for (int b = len; b > -1; b--){
   a=  a.dummy[b];
 }

if (~~s.equals(a)) return true;~~
else  ~~return false;~~

return s.equals(b)
```

This submitted code for body of isPalindrome didn't work because split wasn't used properly – and it wasn't debugged

Why calculate the reverse of s?

Some principles and strategies for development

- Don't introduce a variable without a good reason.
- Put local variables as close to their first use as possible.
- Structure expressions to make them readable.
- Make the structure of the program reflect the structure of the data.
- Never have lots of syntax errors.
- Intersperse coding and testing: code a little, test a little.
- Write the class invariant while putting in field declarations.
- Write a method spec *before* writing the method body.
- Use assert statements to check method preconditions –as along as it doesn't complicate program too much and doesn't change the time-complexity of the method.

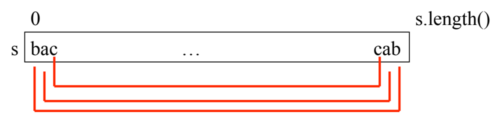
Show *development* of isPalindrome

```
/** Return true iff s is a palindrome */
public static boolean isPalindrome(String s)
```

Our instructions said to visit each char of s only once!

isPalindrome: Set ispal to "s is a palindrome" (forget about returns for now. Store value in ispal.

Think of checking equality of outer chars, then chars inside them, then chars inside them, etc.



Key idea:
Generalize this to a picture that is true before/after each iteration

isPalindrome: Set ispal to "s is a palindrome" (forget about returns for now. Store value in ispal.

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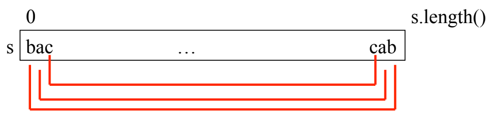
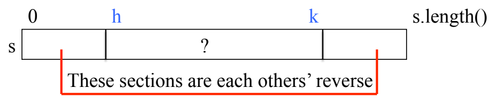
Generalize to a picture that is true before/after each iteration





isPalindrome: Set ispal to "s is a palindrome"

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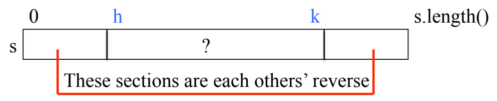
```
int h= 0;
int k= s.length() – 1;
// s[0..h-1] is the reverse of s[k+1..]
while (  h < k   &&  s.charAt(h) == s.charAt(k)  ) {
      h= h+1;  k= k-1;
}
ispal=   h >= k;
```

Initialization to make picture true

Stop when result is known
Continue when it's not

Make progress toward termination
AND keep picture true



isPalindrome

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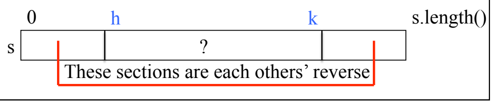
```
/** Return true iff s is a palindrome */
public static boolean isPal(String s) {
   int h=  0;  int k=  s.length() – 1;
   // invariant: s[0..h-1] is reverse of s[k+1..]
   while (h < k) {
       if (s.charAt(h) != s.charAt(k))
           return false;
       h=  h+1;  k=  k-1;
   }
   return true;
}
```

Loop invariant — invariant because it's true before/ after each loop iteration



Engineering principle

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Break a project up into parts, making them as independent as possible. When the parts are constructed, put them together.

Each part can be understood by itself, without mentioning the others.

Reason for introducing loop invariants

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```
Given c >= 0, store b^c in x
z= 1;  x= b;  y=  c;
while (y != 0) {
    if (y is even) {
       x= x*x;  y= y/2;
    } else {
       z= z*x;  y= y - 1;
    }
}
{z  =  b^c}
```

Algorithm to compute b^c.

Can't understand any piece of it without understanding all.
In fact, only way to get a handle on it is to execute it on some test case.

Need to understand initialization without looking at any other code.
Need to understand condition y != 0 without looking at loop body
Etc.

Invariant: is true before and after each iteration

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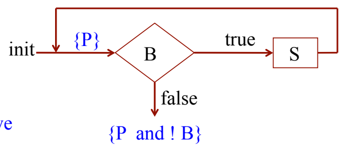
```
initialization;
// invariant P
while (B) {S}
```



Upon termination, we know P true, B false

"invariant" means unchanging. Loop invariant: an assertion —a true-false statement— that is true before and after each iteration of the loop —every time B is to be evaluated.

Help us understand each part of loop without looking at all other parts.

Simple example to illustrate methodology

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```
Store sum of 0..n in s
Precondition: n >= 0
// { n >= 0}
k= 1; s= 0;
// inv:  s = sum of 0..k-1 &&
//        0 <= k <= n+1
while (k <= n) {
    s= s + k;
    k= k + 1;
}
{s = sum of  0..n}
```

First loopy question.
Does it start right?
Does initialization make invariant true?

Yes!
```
    s = sum of 0..k-1
=     <substitute initialization>
    0 = sum of 0..1-1
=     <arithmetic>
    0 = sum of 0..0
```

We understand initialization without looking at any other code

Simple example to illustrate methodology

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```
Store sum of 0..n in s
Precondition: n >= 0
// { n >= 0}
k= 1; s= 0;
// inv:  s = sum of 0..k-1 &&
//        0 <= k <= n+1
while (k <= n) {
    s= s + k;
    k= k + 1;
}
{s = sum of  0..n}
```

Second loopy question.
Does it stop right?
Upon termination, is postcondition true?

Yes!
```
    inv  &&  ! k <= n
=>     <look at inv>
    inv  &&  k = n+1
=>     <use inv>
    s =  sum of 0..n+1-1
```

We understand that postcondition is true without looking at init or repetend

Simple example to illustrate methodology

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```
Store sum of 0..n in s
Precondition: n >= 0
// { n >= 0}
k= 1; s= 0;
// inv:  s = sum of 0..k-1 &&
//        0 <= k <= n+1
while (k <= n) {
    s= s + k;
    k= k + 1;
}
{s = sum of  0..n}
```

Third loopy question.
Progress?
Does the repetend make progress toward termination?

Yes! Each iteration increases k, and when it gets larger than n, the loop terminates

We understand that there is no infinite looping without looking at init and focusing on ONE part of the repetend.

Simple example to illustrate methodology

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```
Store sum of 0..n in s
Precondition: n >= 0
// { n >= 0}
k= 1; s= 0;
// inv: s = sum of 0..k-1 &&
//        0 <= k <= n+1
while (k <= n) {
    s= s + k;
    k= k + 1;
}
{s = sum of  0..n}
```

Fourth loopy question.
Invariant maintained by each iteration?

Is this property true?
  {inv && k <= n} repetend {inv}

Yes!

```
{s = sum of 0..k-1}
s= s + k;
{s =  sum of 0..k}
k= k+1;
{s = sum of 0..k-1}
```

4 loopy questions to ensure loop correctness

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```
{precondition Q}
init;
// invariant P
while (B) {
    S
}
{R}
```

Four loopy questions: if answered yes, algorithm is correct.

First loopy question;
Does it start right?
Is {Q} init {P}   true?

Second loopy question:
Does it stop right?
Does P && ! B imply R?

Third loopy question:
Does repetend make progress?
Will B eventually become false?

Fourth loopy question:
Does repetend keep invariant true?
Is {P && ! B} S {P} true?

Note on ranges m..n

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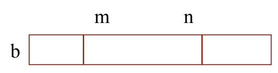
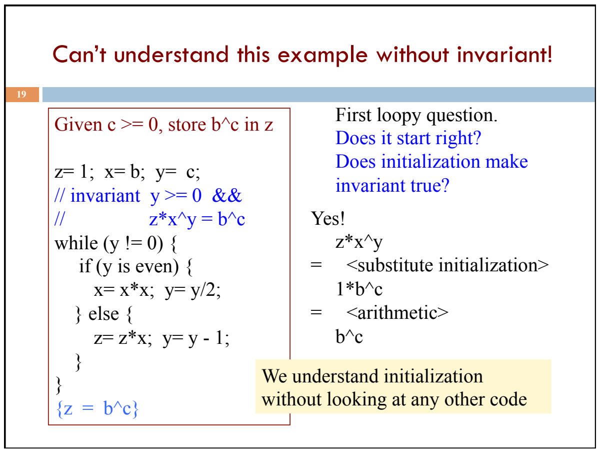
Range m..n contains n+1–m ints: m, m+1, ..., n
(Think about this as "Follower (n+1) minus First (m)")

2..4 contains 2, 3, 4:	that is 4 + 1 – 2 = 3 values
2..3 contains 2, 3:	that is 3 + 1 – 2 = 2 values
2..2 contains 2:	that is 2 + 1 – 2 = 1 value
2..1 contains :	that is 1 + 1 – 2 = 0 values

Convention: notation m..n implies that m <= n+1
Assume convention even if it is not mentioned!
If m is 1 larger than n, the range has 0 values

array segment b[m..n]:





Can't understand this example without invariant!

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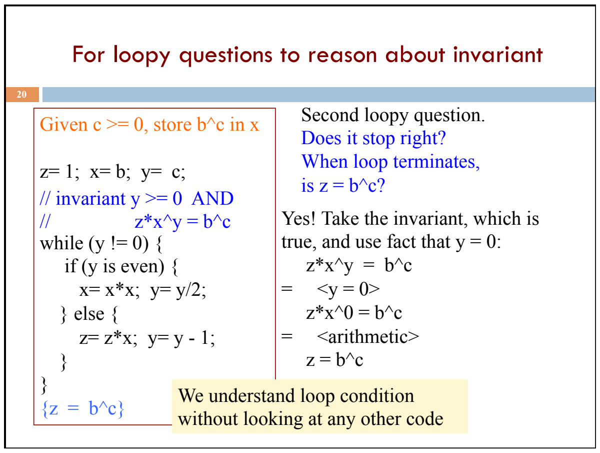
```
Given c >= 0, store b^c in z

z= 1;  x= b;  y= c;
// invariant  y >= 0  &&
//                z*x^y = b^c
while (y != 0) {
    if (y is even) {
       x= x*x;  y= y/2;
    } else {
       z= z*x;  y= y - 1;
    }
}
{z  =  b^c}
```

First loopy question.
Does it start right?
Does initialization make invariant true?

Yes!
```
    z*x^y
=    <substitute initialization>
    1*b^c
=    <arithmetic>
    b^c
```

We understand initialization without looking at any other code



For loopy questions to reason about invariant

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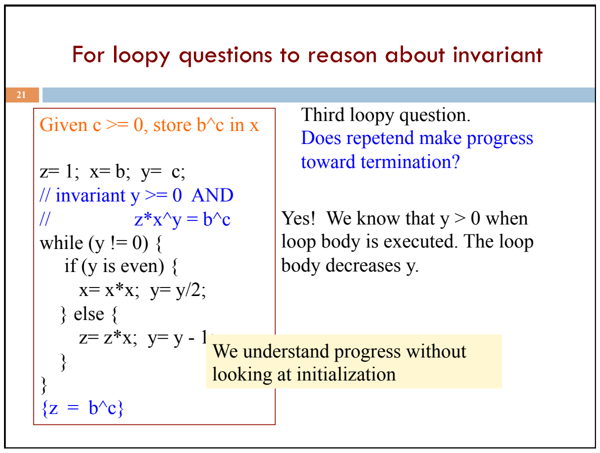
```
Given c >= 0, store b^c in x

z= 1;  x= b;  y= c;
// invariant y >= 0  AND
//                z*x^y = b^c
while (y != 0) {
    if (y is even) {
       x= x*x;  y= y/2;
    } else {
       z= z*x;  y= y - 1;
    }
}
{z  =  b^c}
```

Second loopy question.
Does it stop right?
When loop terminates, is z = b^c?

Yes! Take the invariant, which is true, and use fact that y = 0:
```
    z*x^y  =  b^c
=     <y = 0>
    z*x^0 = b^c
=     <arithmetic>
    z = b^c
```

We understand loop condition without looking at any other code



For loopy questions to reason about invariant

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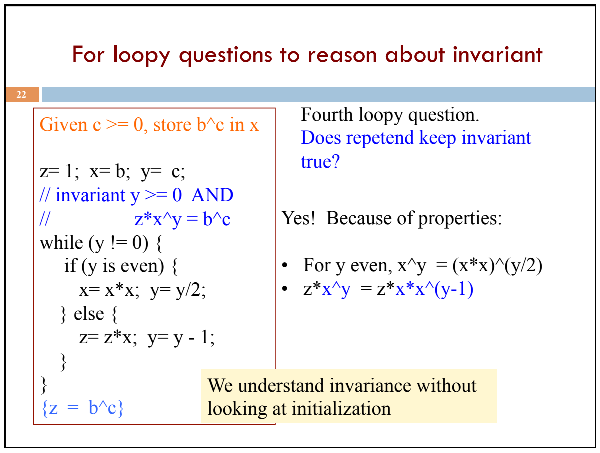
```
Given c >= 0, store b^c in x

z= 1;  x= b;  y= c;
// invariant y >= 0  AND
//                z*x^y = b^c
while (y != 0) {
    if (y is even) {
       x= x*x;  y= y/2;
    } else {
       z= z*x;  y= y - 1;
    }
}
{z  =  b^c}
```

Third loopy question.
Does repetend make progress toward termination?

Yes! We know that y > 0 when loop body is executed. The loop body decreases y.

We understand progress without looking at initialization



For loopy questions to reason about invariant

22

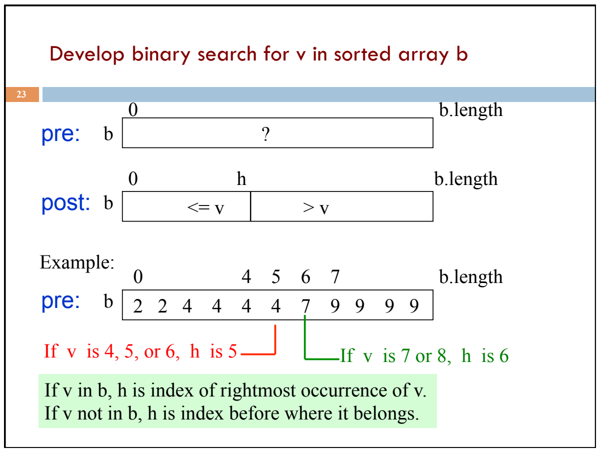
```
Given c >= 0, store b^c in x

z= 1;  x= b;  y= c;
// invariant y >= 0  AND
//                z*x^y = b^c
while (y != 0) {
    if (y is even) {
       x= x*x;  y= y/2;
    } else {
       z= z*x;  y= y - 1;
    }
}
{z  =  b^c}
```

Fourth loopy question.
Does repetend keep invariant true?

Yes! Because of properties:

- For y even, x^y = (x*x)^(y/2)
- z*x^y = z*x*x^(y-1)

We understand invariance without looking at initialization



Develop binary search for v in sorted array b

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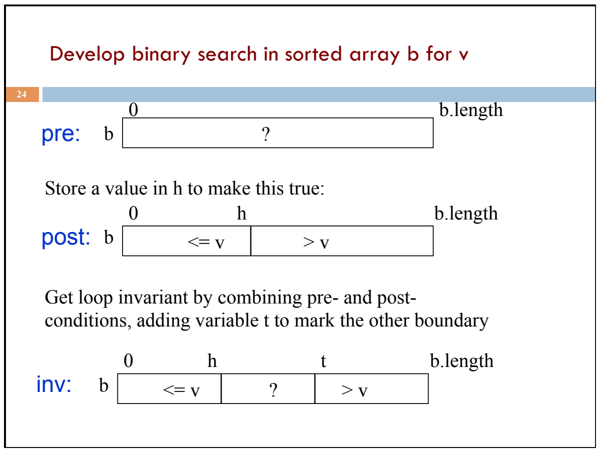
```
          0                                  b.length
pre:   b |                ?                 |

          0           h                      b.length
post:  b |    <= v    |       > v           |

Example:  0           4   5   6   7          b.length
pre:   b | 2  2  4  4  4  4  7  9  9  9  9  |
```

If v is 4, 5, or 6, h is 5

If v is 7 or 8, h is 6

If v in b, h is index of rightmost occurrence of v.
If v not in b, h is index before where it belongs.



Develop binary search in sorted array b for v

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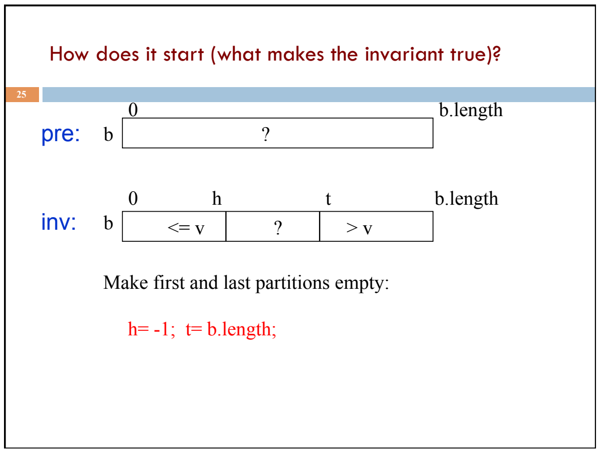
```
          0                                  b.length
pre:   b |                ?                 |
```

Store a value in h to make this true:

```
          0           h                      b.length
post:  b |    <= v    |       > v           |
```

Get loop invariant by combining pre- and post-conditions, adding variable t to mark the other boundary

```
          0          h          t            b.length
inv:   b |   <= v   |    ?     |    > v     |
```



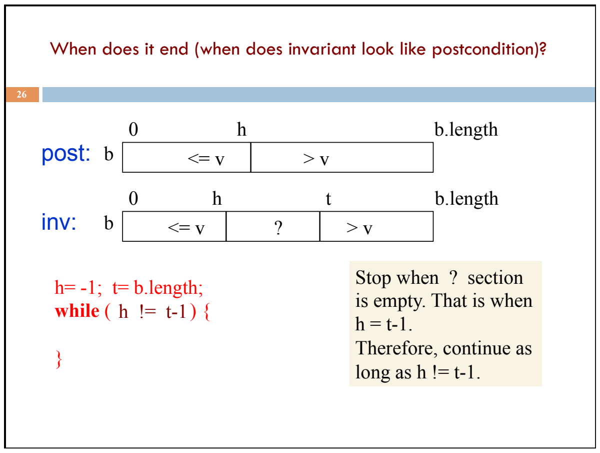
How does it start (what makes the invariant true)?

```
         0                          b.length
pre:  b |             ?            |

         0       h         t        b.length
inv:  b |  <= v  |    ?    |  > v  |
```

Make first and last partitions empty:

```
h= -1;  t= b.length;
```



When does it end (when does invariant look like postcondition)?

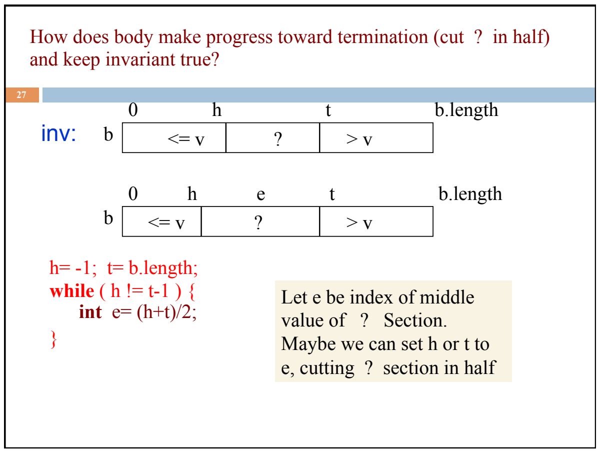
```
         0          h               b.length
post: b |   <= v    |     > v      |

         0       h         t        b.length
inv:  b |  <= v  |    ?    |  > v  |
```

```
h= -1;  t= b.length;
while ( h  !=  t-1 ) {

}
```

Stop when ? section is empty. That is when h = t-1.
Therefore, continue as long as h != t-1.



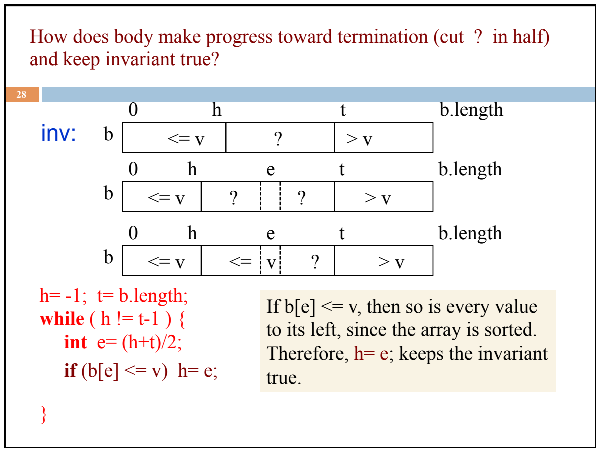
How does body make progress toward termination (cut ? in half) and keep invariant true?

```
         0       h         t        b.length
inv:  b |  <= v  |    ?    |  > v  |

         0     h      e    t        b.length
      b | <= v |      ?    |  > v  |
```

```
h= -1;  t= b.length;
while ( h != t-1 ) {
    int  e= (h+t)/2;
}
```

Let e be index of middle value of ? Section.
Maybe we can set h or t to e, cutting ? section in half



How does body make progress toward termination (cut ? in half) and keep invariant true?

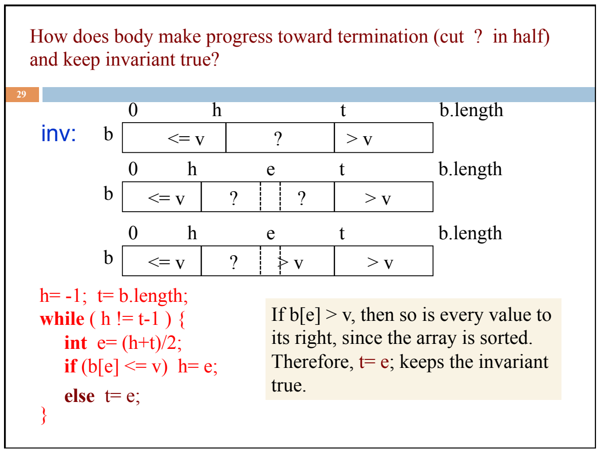
```
         0       h         t        b.length
inv:  b |  <= v  |    ?    |  > v  |

         0     h     e     t        b.length
      b | <= v |  ?  | |  ?  |  > v  |

         0     h     e     t        b.length
      b | <= v | <=  |v|  ?  |  > v  |
```

```
h= -1;  t= b.length;
while ( h != t-1 ) {
    int  e= (h+t)/2;
    if (b[e] <= v)  h= e;
}
```

If b[e] <= v, then so is every value to its left, since the array is sorted.
Therefore, h= e; keeps the invariant true.



How does body make progress toward termination (cut ? in half) and keep invariant true?

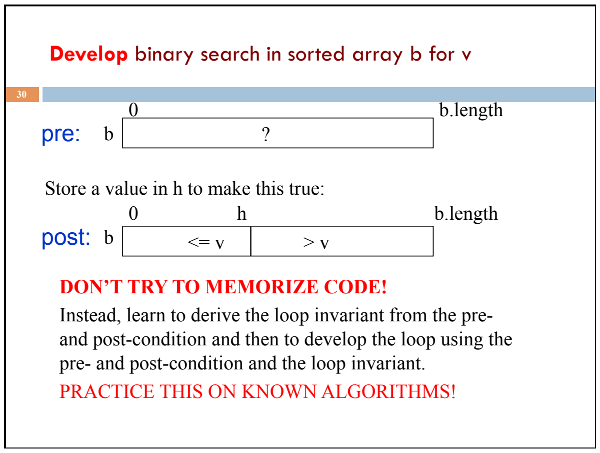
```
         0       h          t       b.length
inv:  b |  <= v  |    ?     | > v  |

         0     h     e     t        b.length
      b | <= v |  ?  | |  ?  |  > v  |

         0     h     e     t        b.length
      b | <= v |  ?  |> v  |  > v   |
```

```
h= -1;  t= b.length;
while ( h != t-1 ) {
    int  e= (h+t)/2;
    if (b[e] <= v)  h= e;
    else  t= e;
}
```

If b[e] > v, then so is every value to its right, since the array is sorted.
Therefore, t= e; keeps the invariant true.



Develop binary search in sorted array b for v

```
         0                          b.length
pre:  b |             ?            |
```

Store a value in h to make this true:

```
         0          h               b.length
post: b |   <= v    |     > v      |
```

DON'T TRY TO MEMORIZE CODE!

Instead, learn to derive the loop invariant from the pre- and post-condition and then to develop the loop using the pre- and post-condition and the loop invariant.

PRACTICE THIS ON KNOWN ALGORITHMS!

Processing arrays from beg to end (or end to beg)

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Many loops process elements of an array b (or a String, or any list) in order: b[0], b[1], b[2], …

If the postcondition is

R: b[0..b.length-1] has been processed

Then in the beginning, nothing has been processed, i.e.

b[0..-1] has been processed

After k iterations, k elements have been processed:

P: b[0..k-1] has been processed

```
                  0            k              b.length
invariant P:  b | processed  | not processed |
```

Processing arrays from beg to end (or end to beg)

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```
Task: Process b[0..b.length-1]
k= 0;
{inv P}
while (  k != b.length   ) {
   Process b[k];     // maintain invariant
   k= k + 1;         // progress toward termination
}
{R: b[0..b.length-1]  has been processed}
```

Replace b.length in postcondition by fresh variable k to get invariant
b[0..k-1] has been processed

or draw it as a picture

```
            0            k              b.length
inv P:  b | processed  | not processed |
```

Processing arrays from beg to end (or end to beg)

33

```
Task: Process b[0..b.length-1]
k= 0;
{inv P}
while (  k != b.length   ) {
   Process b[k];     // maintain invariant
   k= k + 1;         // progress toward termination
}
{R: b[0..b.length-1]  has been processed}
```

Most loops that process the elements of an array in order will have this loop invariant and will look like this.

```
            0            k              b.length
inv P:  b | processed  | not processed |
```

Count the number of zeros in b. Start with last program and refine it for this task

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```
Task: Set s to the number of 0's in b[0..b.length-1]
k= 0;   s= 0;
{inv P}
while (  k != b.length   ) {
   if (b[k] == 0)  s= s + 1;
   k= k + 1;         // progress toward termination
}
{R: s = number of 0's in b[0..b.length-1]}
```

```
            0               k              b.length
inv P:  b | s = # 0's here | not processed |
```

Be careful. Invariant may require processing elements in reverse order!

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This invariant forces processing from beginning to end

```
            0            k              b.length
inv P:  b | processed  | not processed |
```

This invariant forces processing from end to beginning

```
            0                k          b.length
inv P:  b | not processed  | processed |
```

Process elements from end to beginning

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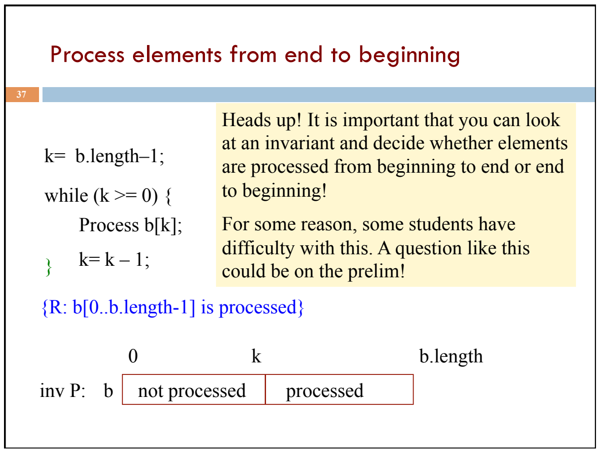
```
k= b.length–1;             // how does it start?
while (k >= 0) {           // how does it end?
    Process b[k];          // how does it maintain invariant?
    k= k – 1;              // how does it make progress?
}
{R: b[0..b.length-1] is processed}
```

```
            0                k          b.length
inv P:  b | not processed  | processed |
```



Process elements from end to beginning

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```
k= b.length–1;
while (k >= 0) {
      Process b[k];
}     k= k – 1;
```

Heads up! It is important that you can look at an invariant and decide whether elements are processed from beginning to end or end to beginning!

For some reason, some students have difficulty with this. A question like this could be on the prelim!

{R: b[0..b.length-1] is processed}

```
                0               k                        b.length
inv P:   b  | not processed | processed      |
```