RECURSION

Lecture 7

CS2110 – Spring 2015

Overview references to sections in text

☐ Note: We've covered everything in JavaSummary.pptx!

☐ What is recursion? 7.1-7.39   slide 1-7

☐ Base case   7.1-7.10 slide 13

☐ How Java stack frames work 7.8-7.10 slide 28-32

NEXT WEEK IS FEBRUARY BREAK
1. No lecture on Tuesday.
2. No CS2111 on Tuesday.
3. No recitation/discussion sections on Tuesday/Wednesday
4. See you in lecture next Thursday

A little about generics –used in A3

public class LinkedList<E> { …}   // E is a type parameter

/** Values in d1 can be ANY objects —String, JFrame, etc. */
LinkedList d1= **new** LinkedList();

…
String x= ((String) d1.getFirst()).getValueOf(); // cast is needed

/** The values in d2 are only objects of class String */
LinkedList<String> d2= **new** LinkedList<String>();

…
String s= d2.getFirst().getValueOf(); // no cast is needed

What does generic mean?

From Merriam-Webster online:

ge·ner·ic *adjective*

a : relating or applied to or descriptive of all members of a genus, species, class, or group **:** common to or characteristic of a whole group or class **:** typifying or subsuming **:** not specific or individual

generic applies to that which characterizes every individual in a category or group and may suggest further that what is designated may be thought of as a clear and certain classificatory criterion

# Sum the digits in a non-negative integer

```
/** return sum of digits in n.
 * Precondition:  n >= 0 */
public static int sum(int n) {
    if (n < 10) return n;



    // { n has at least two digits }
    // return first digit + sum of rest
    return sum(n/10) +  n%10 ;
}
```

sum calls itself!

E.g. sum(7) = 7

E.g. sum(8703) = sum(870) + 3;

Two issues with recursion

```
/** return sum of digits in n.
 * Precondition:  n >= 0 */
public static int sum(int n) {
    if (n < 10) return n;


    // { n has at least two digits }
    // return first digit + sum of rest
    return sum(n/10)  +  n%10  + ;
}
```

sum calls itself!

1. Why does it work? How does the method executed?

2. How do we understand a given recursive method, or how do we write/develop a recursive method?

Stacks and Queues

stack grows
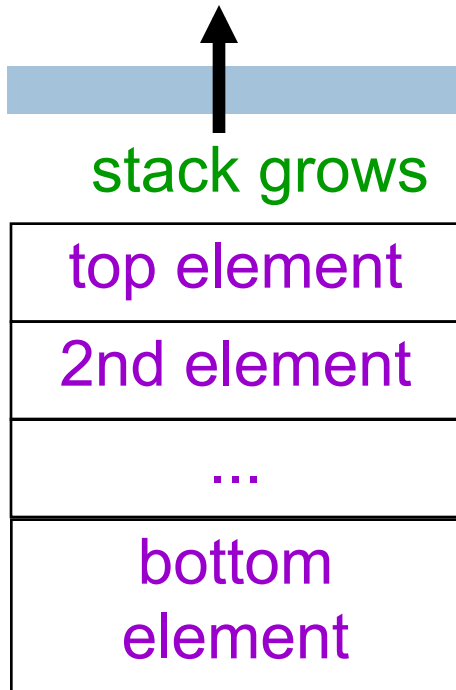
| top element |
| 2nd element |
| ... |
| bottom element |

| first   second   ...   last |

Americans wait in a line, the Brits wait in a queue !

Stack: list with (at least) two basic ops:

* Push an element onto its top
* Pop (remove) top element

Last-In-First-Out (LIFO)

Like a stack of trays in a cafeteria

Queue: list with (at least) two basic ops:

* Append an element
* Remove first element

First-In-First-Out (FIFO)

Stack Frame

A "frame" contains information about a method call:

At runtime, Java maintains a stack that contains frames for all method calls that are being executed but have not completed.

a frame

| local variables |
| parameters |
| return info |

Method call: push a frame for call on stack, assign argument values to parameters, execute method body. Use the frame for the call to reference local variables, parameters.

End of method call: pop its frame from the stack; if it is a function, leave the return value on top of stack.

Example: Sum the digits in a non-negative integer

```
public static int sum(int n) {
    if (n < 10) return n;
    return sum(n/10) + n%10;
}

public static void main(
        String[] args) {
    int r= sum(824);
    System.out.println(r);
}
```

Frame for method in the system
that calls method main

frame:

n ___
return info

frame:

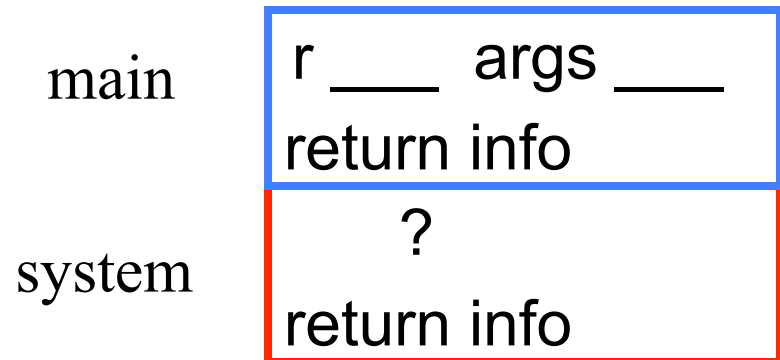| r ___ args ___ |
| --- |
| return info |

frame:

?
return info

Example: Sum the digits in a non-negative integer

```java
public static int sum(int n) {
    if (n < 10) return n;
    return sum(n/10)  +  n%10;
}


public static void main(…) {
  int r= sum(824);
  System.out.println(r);
}
```

Frame for method in the system
that calls method main: main is
then called

main

system

r ___	args ___
return info	
?	
return info	

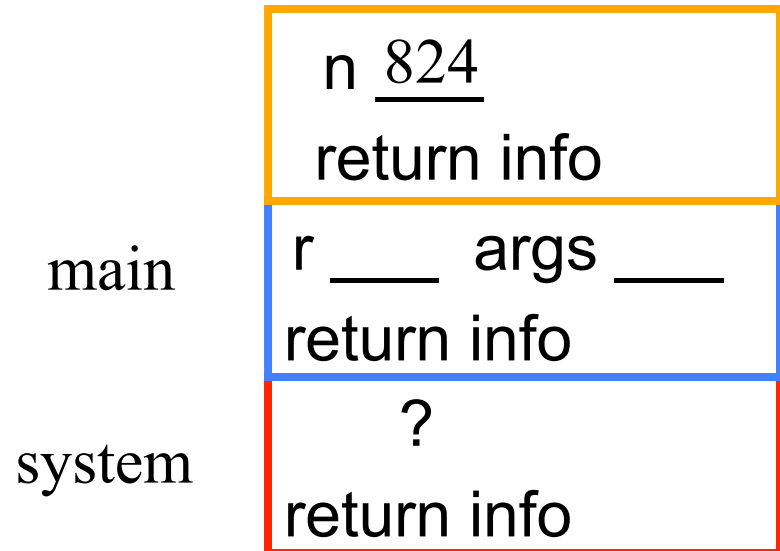# Example: Sum the digits in a non-negative integer

```
public static int sum(int n) {
    if (n < 10) return n;
    return sum(n/10) + n%10 ;
}

public static void main(…) {
    int r= sum(824);
    System.out.println(r);
}
```

Method main calls sum:

main

system

n 824
return info
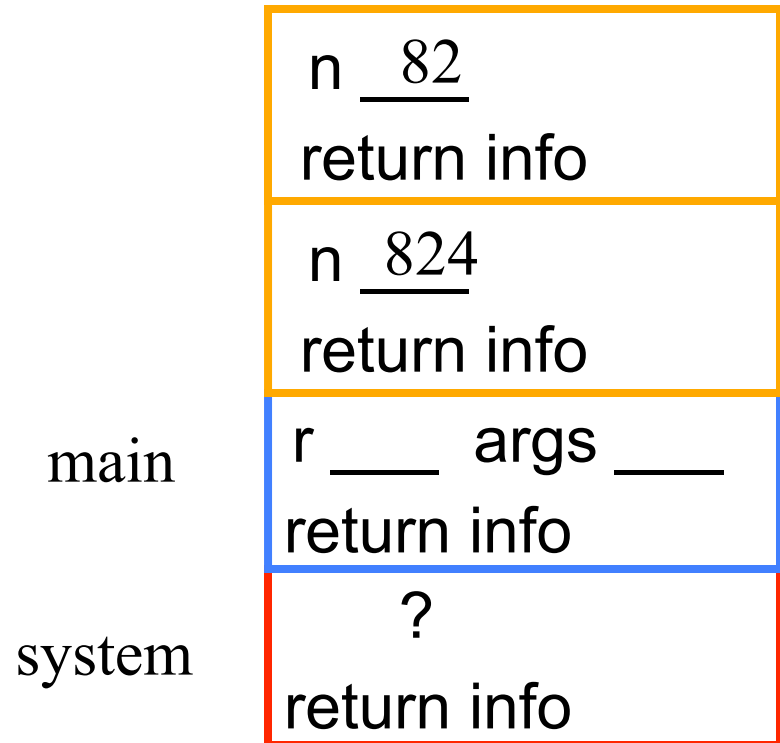
r ___   args ___
return info

?
return info

Example: Sum the digits in a non-negative integer

```
public static int sum(int n) {
    if (n < 10) return n;
    return sum(n/10)  +  n%10;
}

public static void main(…) {
    int r= sum(824);
    System.out.println(r);
}
```

n >= 10, sum calls sum:

| n 82 |
| return info |

| n 824 |
| return info |

main

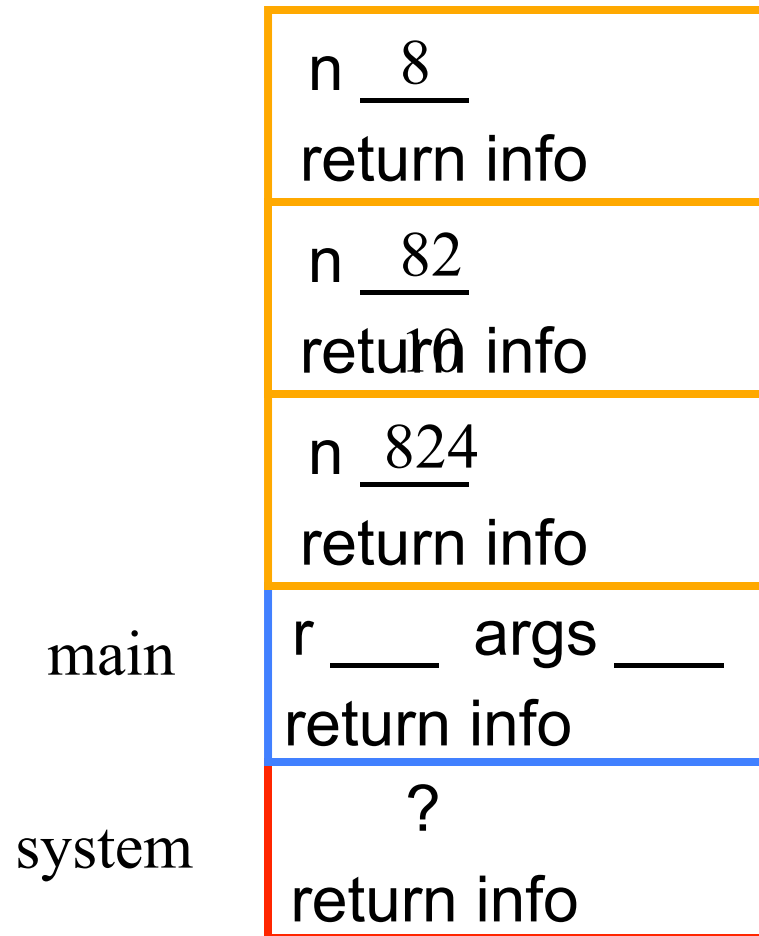| r ___   args ___ |
| return info |

system

| ? |
| return info |

Example: Sum the digits in a non-negative integer

```
public static int sum(int n) {
    if (n < 10) return n;
    return sum(n/10)  +  n%10;
}


public static void main(…) {
    int r= sum(824);
    System.out.println(r);
}
```

n >= 10. sum calls sum:

n __8__
return info

n __82__
return info

n __824__
return info

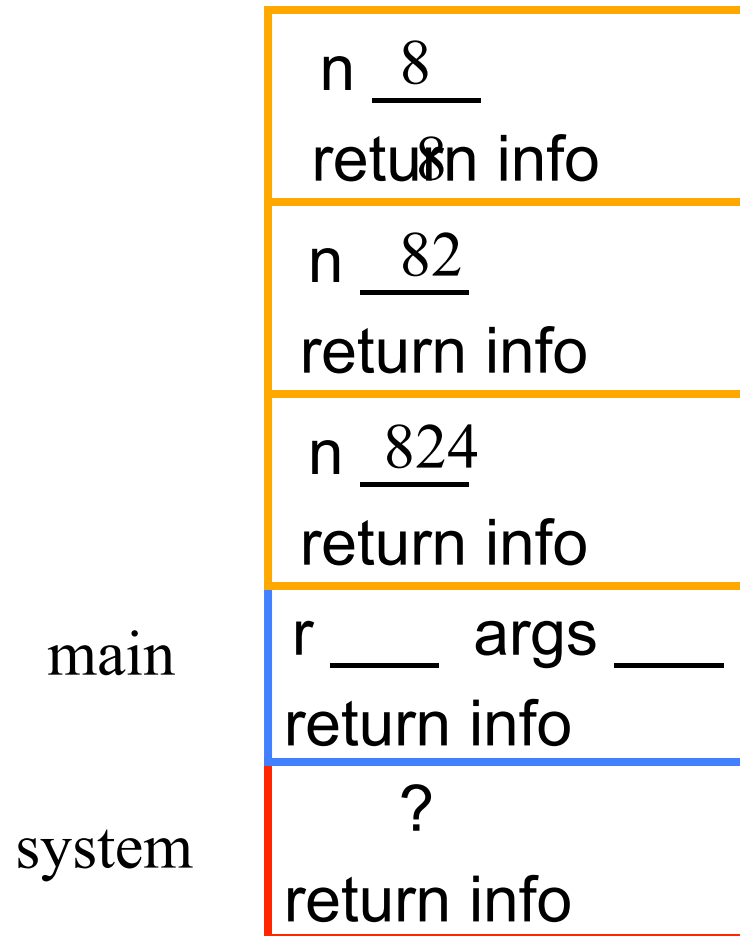main   r ___ args ___
return info

system   ?
return info

Example: Sum the digits in a non-negative integer

```
public static int sum(int n) {
    if (n < 10) return n;
    return sum(n/10) + n%10;
}

public static void main(…) {
    int r= sum(824);
    System.out.println(r);
}
```

n < 10, sum stops: frame is
popped and n is put on stack:

|	
n __8__	
retu__8__n info	
n __82__	
return info	
n __824__	
return info	
r ____ args ____	
return info	
?	
return info	

main

system

Example: Sum the digits in a non-negative integer

```
public static int sum(int n) {
    if (n < 10) return n;
    return sum(n/10) + n%10;
}

public static void main(…) {
    int r= sum(824);
    System.out.println(r);
}
```

main

8

| n $\frac{82}{10}$ |
| return info |

| n 824 |
| return info |

| r ___   args ___ |
| return info |

| ? |
| return info |

Using return value 8, stack computes
8 + 2 = 10, pops frame from stack,
puts return value 10 on stack
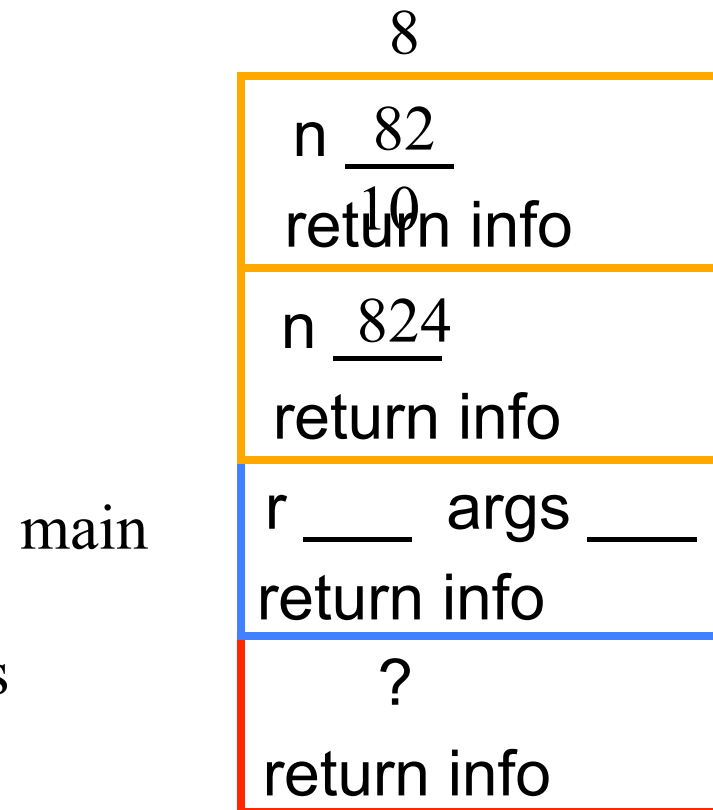
Example: Sum the digits in a non-negative integer

```
public static int sum(int n) {
    if (n < 10) return n;
    return sum(n/10)  +  n%10;
}

public static void main(…) {
    int r= sum(824);
    System.out.println(r);
}
```

main

10

| n 824 |
| return info |

| n 824 |
| return info |

| r ___   args ___ |
| return info |

| ? |
| return info |

Using return value 10, stack computes
10 + 4 = 14, pops frame from stack,
puts return value 14 on stack
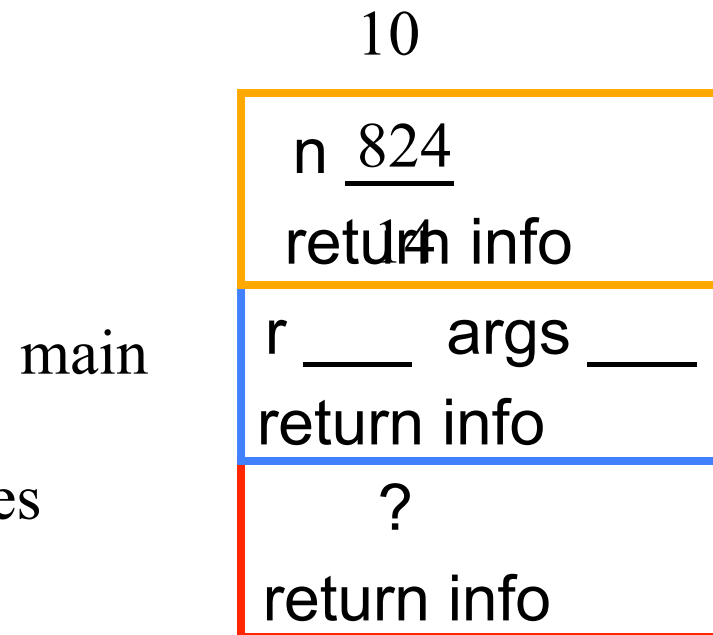
Example: Sum the digits in a non-negative integer

```
public static int sum(int n) {
    if (n < 10) return n;
    return sum(n/10)  +  n%10;
}

public static void main(…) {
    int r= sum(824);
    System.out.println(r);
}
```

Using return value 14, main stores
14 in r and removes 14 from stack

14

main

r 14	args __
return info	

?

return info
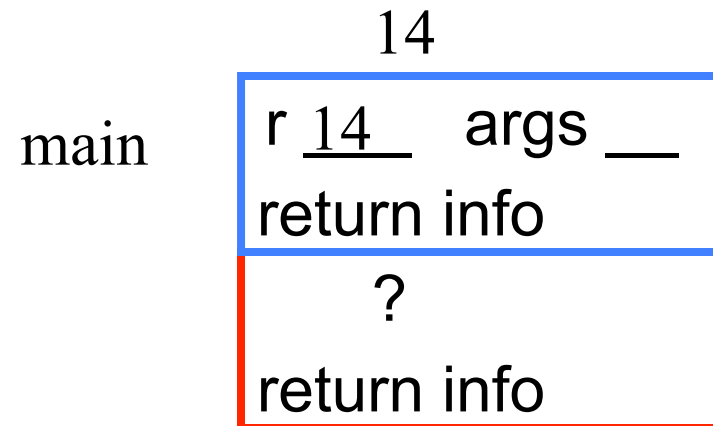
Summary of method call execution

Memorize this!

- 1. A frame for a call contains parameters, local variables, and other information needed to properly execute a method call.

- 2. To execute a method call: push a frame for the call on the stack, assign arg values to pars, and execute method body.

  When executing method body, look in frame for call for parameters and local variables.

  When method body finishes, pop frame from stack and (for a function) push the return value on the stack.

- For function call: When control given back to call, it pops the return value and uses it as the value of the function call.

Questions about local variables

```
public static void m(…) {

   …

    while (…) {

      int d= 5;

      …

    }

}
```

```
public static void m(…) {

    int d;

    …

    while (…) {

       d= 5;

       …

    }

}
```

In a call m(),
when is local variable d created and when is it destroyed?
Which version of procedure m do you like better? Why?

Recursion is used extensively in math

Math definition of n factorial

E.g. $3! = 3*2*1 = 6$

$0! = 1$

$n! = n * (n-1)!$ for $n > 0$

Math definition of b c for c >= 0

$b^0 = 1$

$b^c = b * b^{c-1}$ for $c > 0$

Lots of things defined recursively:
expression, grammars, trees, ….
We will see such things later

Easy to make math definition into a Java function!

```
public static int fact(int n) {
    if (n == 0) return 1;

    return n * fact(n-1);
}
```

Two views of recursive methods

☐ **How are calls on recursive methods executed?**
We saw that. Use this only to gain understanding / assurance that recursion works

☐ **How do we understand a recursive method — know that it satisfies its specification? How do we write a recursive method?**
This requires a totally different approach. Thinking about how the method gets executed will confuse you completely! We now introduce this approach.

Understanding a recursive method

Step 1. Have a precise spec!

Step 2. Check that the method works in the base case(s): Cases where the parameter is small enough that the result can be computed simply and without recursive calls.

If n < 10, then n consists of a single digit. Looking at the spec, we see that that digit is the required sum.

```java
/** = sum of digits of n.
  * Precondition:  n >= 0 */
public static int sum(int n) {
    if (n < 10) return n;

    // n has at least two digits
    return sum(n/10)  +  n%10 ;
}
```

Understanding a recursive method

Step 1. Have a precise spec!

Step 2. Check that the method works in the base case(s).

Step 3. Look at the recursive case(s). In your mind, replace each recursive call by what it does according to the method spec and verify that the correct result is then obtained.

```
/** =  sum of digits of n.
 * Precondition:  n >= 0 */
public static int sum(int n) {
    if (n < 10) return n;

    // n has at least two digits
    return sum(n/10)  +  n%10 ;
}
```

return sum(n/10) + n%10;

return (sum of digits of n/10) + n%10;   // e.g. n = 843

Understanding a recursive method

Step 1. Have a precise spec!

Step 2. Check that the method works in the base case(s).

Step 3. Look at the recursive case(s). In your mind, replace each recursive call by what it does acc. to the spec and verify correctness.

```
/** =  sum of digits of n.
  * Precondition:  n >= 0 */
public static int sum(int n) {
    if (n < 10) return n;

    // n has at least two digits
    return sum(n/10)  +  n%10 ;
}
```

Step 4. (No infinite recursion) Make sure that the args of recursive calls are in some sense smaller than the pars of the method.

$n/10 < n$

Understanding a recursive method

Step 1. Have a precise spec!   Important! Can't do step 3 without it

Step 2. Check that the method works in the base case(s).

Step 3. Look at the recursive case(s). In your mind, replace each recursive call by what it does according to the spec and verify correctness.

Once you get the hang of it, this is what makes recursion easy! This way of thinking is based on math induction, which we will see later in the course.

Step 4. (No infinite recursion) Make sure that the args of recursive calls are in some sense smaller than the pars of the method

Writing a recursive method

Step 1. Have a precise spec!

Step 2. Write the base case(s): Cases in which no recursive calls are needed Generally, for "small" values of the parameters.

Step 3. Look at all other cases. See how to define these cases in terms of smaller problems of the same kind. Then implement those definitions, using recursive calls for those smaller problems of the same kind. Done suitably, point 4 is automatically satisfied.

Step 4. (No infinite recursion) Make sure that the args of recursive calls are in some sense smaller than the pars of the method

Examples of writing recursive functions

For the rest of the class, we demo writing recursive functions using the approach outlined below. The java file we develop will be placed on the course webpage some time after the lecture.

Step 1. Have a precise spec!

Step 2. Write the base case(s).

Step 3. Look at all other cases. See how to define these cases in terms of smaller problems of the same kind. Then implement those definitions, using recursive calls for those smaller problems of the same kind.

The Fibonacci Function

Mathematical definition:

fib(0) = 0 ← two base cases!

fib(1) = 1 ←

fib(n) = fib(n − 1) + fib(n − 2), n ≥ 2

Fibonacci sequence: 0, 1, 1, 2, 3, 5, 8, 13,
…

Fibonacci (Leonardo Pisano) 1170-1240?

Statue in Pisa, Italy
Giovanni Paganucci
1863

```
/** = fibonacci(n). Pre: n >= 0 */
static int fib(int n) {
   if (n <= 1) return n;
   // { 1 < n }
   return fib(n-2) + fib(n-1);
}
```

Example: Is a string a palindrome?

```
/** = "s is a palindrome" */
public static boolean isPal(String s) {
    if (s.length() <= 1)
        return true;

    // { s has at least 2 chars }
    int n= s.length()-1;
    return s.charAt(0) == s.charAt(n)  &&  isPal(s.substring(1, n));
}
```

Substring from s[1] to s[n-1]

isPal("racecar") returns true
isPal("pumpkin") returns false

Example: Count the e's in a string

```
/** =  number of times c occurs in s */
public static int countEm(char c, String s) {
   if (s.length() == 0) return 0;

   // { s has at least 1 character }
   if (s.charAt(0) != c)
      return countEm(c, s.substring(1));

   // { first character of s is c}
   return 1 + countEm (c, s.substring(1));
}
```

substring s[1..],
i.e. s[1], …,
s(s.length()-1)

- countEm('e', "it is easy to see that this has many e's") = 4
- countEm('e', "Mississippi") = 0

Computing a^n for $n >= 0$

Power computation:

- $a^0 = 1$
- If $n != 0$, $a^n = a * a^{n-1}$
- If $n != 0$ and even, $a^n = (a*a)^{n/2}$

Java note: For ints x and y, x/y is the integer part of the quotient

Judicious use of the third property gives a logarithmic algorithm, as we will see

Example: $3^8 = (3*3) * (3*3) * (3*3) * (3*3) = (3*3)^4$

Computing a^n for n >= 0

Power computation:

- $a^0 = 1$

- If n != 0, $a^n = a * a^{n-1}$

- If n != 0 and even, $a^n = (a*a)^{n/2}$

```
/** = a**n. Precondition: n >= 0 */
static int power(int a, int n) {
    if (n == 0) return 1;
    if (n%2 == 0) return power(a*a, n/2);
    return a * power(a, n-1);
}
```

Tiling Elaine's kitchen

Kitchen in Gries's house is 8 x 8. A refrigerator sits on one of the 1 x 1 squares

His wife, Elaine, wants the kitchen tiled with el-shaped tiles – every square except where the refrigerator sits should be tiled.

/** tile a 2^n by 2^n kitchen with 1
    square filled. */
public static void tile(int n)

We abstract away keeping track
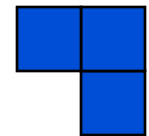of where the filled square is, etc.
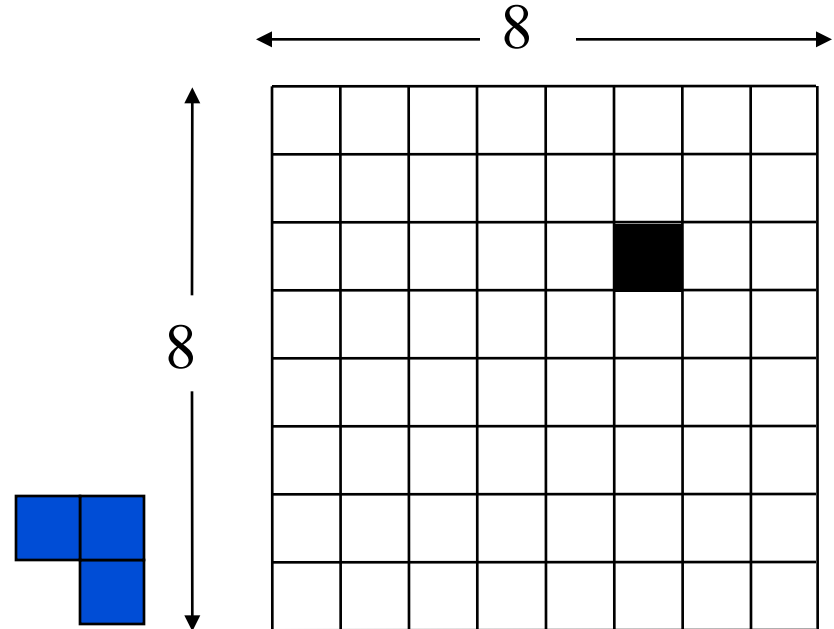
Tiling Elaine's kitchen

/** tile a 2^n by 2^n kitchen with 1
    square filled. */
public static void tile(int n) {

   if (n == 0) return;

}

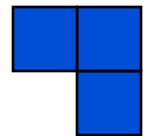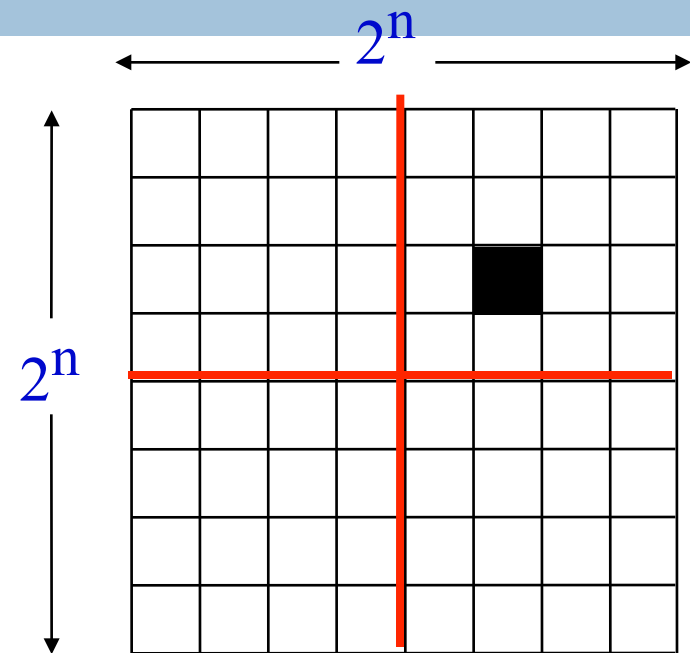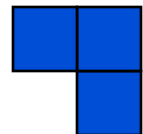We generalize to a 2^n by 2^n kitchen
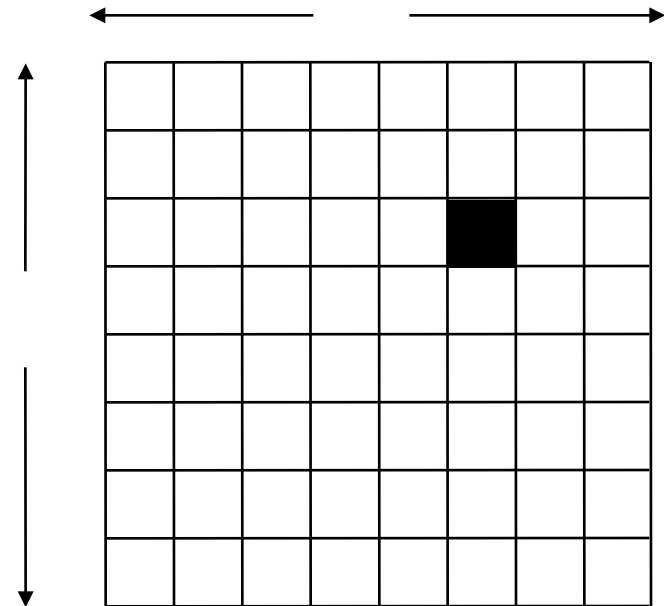
Base case?

Tiling Elaine's kitchen

$$2^n$$

/** tile a 2^n by 2^n kitchen with 1
   square filled. */
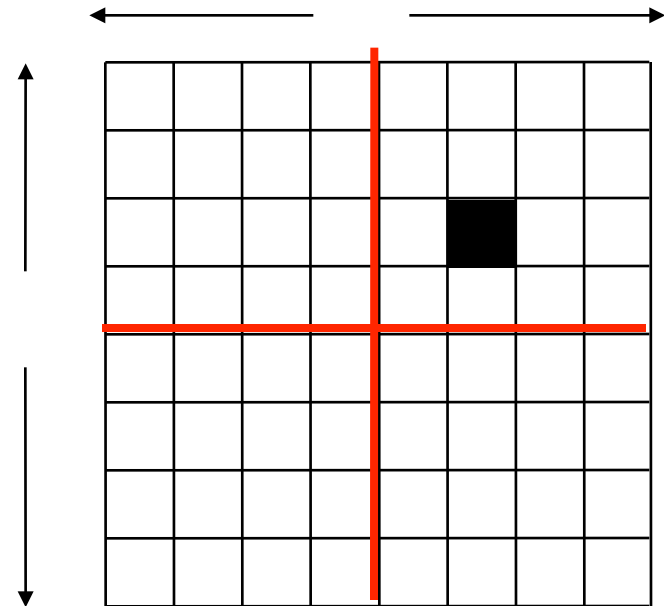public static void tile(int n) {

   if (n == 0) return;

$$2^n$$

}

n > 0. What can we do to get kitchens of size 2^{n-1} by 2^{n-1}

Tiling Elaine's kitchen

/** tile a 2^n by 2^n kitchen with 1
    square filled. */
public static void tile(int n) {

    if (n == 0) return;

}

We can tile the upper-right 2^{n-1} by 2^{n-1} kitchen recursively.
But we can't tile the other three because they don't have a filled square.
What can we do? Remember, the idea is to tile the kitchen!

Tiling Elaine's kitchen

/** tile a 2^n by 2^n kitchen with 1
    square filled. */
public static void tile(int n) {

    if (n == 0) return;
    Place one tile so that each kitchen
    has one square filled;

    Tile upper left kitchen recursively;
    Tile upper right kitchen recursively;
    Tile lower left kitchen recursively;
    Tile lower right kitchen recursively;
}
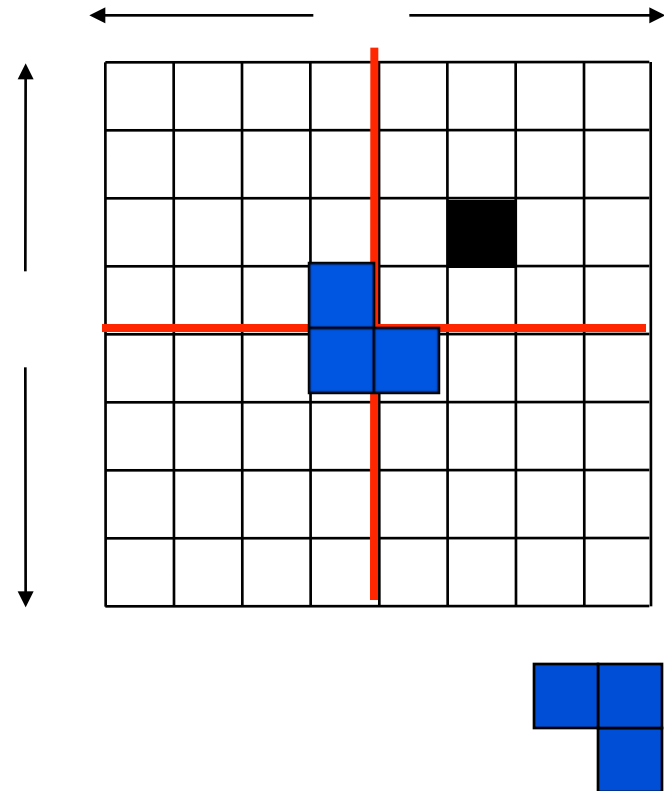
Conclusion

Recursion is a convenient and powerful way to define functions

Problems that seem insurmountable can often be solved in a "divide-and-conquer" fashion:

- Reduce a big problem to smaller problems of the same kind, solve the smaller problems
- Recombine the solutions to smaller problems to form solution for big problem