Recitation 11

Analysis of Algorithms and Inductive Proofs

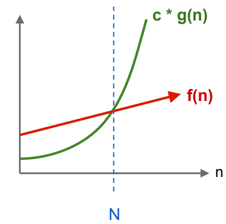
Review: Big O definition

f(n) is O(g(n))

iff

There exists $c > 0$ and $N > 0$ such that:

f(n) $\leq$ **c * g(n)** for $n \geq N$



Example: n+6 is O(n)

n + 6 ---this is f(n)
<= <if 6 <= n, write as>
n + n
= <arith>
2*n
<choose c = 2>
= c*n ---this is c * g(n)

So choose c = 2 and N = 6

f(n) is O(g(n)): There exist $c > 0$, $N > 0$ such that:
f(n) $\leq$ **c * g(n)** for $n \geq N$

Review: Big O

Is used to classify algorithms by how they respond to changes in input size n.

Important vocabulary:

- Constant time: O(1)
- Logarithmic time: O(log n)
- Linear time: O(n)
- Quadratic time: $O(n^2)$
- Exponential time: $O(2^n)$

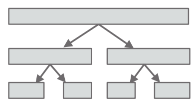
Review: Big O

1. log(n) + 20	is	O(log(n))	(logarithmic)
2. n + log(n)	is	O(n)	(linear)
3. n/2 and 3*n	are	O(n)	
4. n * log(n) + n	is	n * log(n)	
5. $n^2 + 2*n + 6$	is	$O(n^2)$	(quadratic)
6. $n^3 + n^2$	is	$O(n^3)$	(cubic)
7. 2^n + n5	is	$O(2^n)$	(exponential)

Merge Sort

Runtime of merge sort

```
/** Sort b[h..k]. */
public static void mS(Comparable[] b, int h, int k) {
      if (h >= k) return;
      int e= (h+k)/2;
      mS(b, h, e);
      mS(b, e+1, k);
      merge(b, h, e, k);
}
```



mS is **mergeSort** for readability

Runtime of merge sort

```
/** Sort b[h..k]. */
public static void mS(Comparable[] b, int h, int k) {
      if (h >= k) return;
      int e= (h+k)/2;
      mS(b, h, e);
      mS(b, e+1, k);
      merge(b, h, e, k);
}
```

- We will *count* the number of comparisons mS makes
- Use `T(n)` for the number of array element comparisons that mS makes on an array segment of size *n*

mS is **mergeSort** for readability

Runtime of merge sort

```
/** Sort b[h..k]. */
public static void mS(Comparable[] b, int h, int k) {
      if (h >= k) return;          T(0) = 0
      int e= (h+k)/2;              T(1) = 0
      mS(b, h, e);
      mS(b, e+1, k);
      merge(b, h, e, k);
}
```

Use `T(n)` for the number of array element comparisons that mergeSort makes on an array of size *n*

Runtime of merge sort

```
/** Sort b[h..k]. */
public static void mS(Comparable[] b, int h, int k) {
      if (h >= k) return;
      int e= (h+k)/2;
      mS(b, h, e);        T(e+1-h) comparisons = T(n/2)
      mS(b, e+1, k);      T(k-e)   comparisons = T(n/2)
      merge(b, h, e, k);        How long does merge take?
}
```


Runtime of merge

pseudocode for merge

```
/** Pre: b[h..e] and b[e+1..k] are already sorted */
merge(Comparable[] b, int h, int e, int k)
   Copy both segments
   While both copies are non-empty
      Compare the first element of each segment
      Set the next element of b to the smaller value
      Remove the smaller element from its segment
```

One comparison, one add, one remove

k-h loops must empty one segment

Runtime is O(k-h)

Runtime of merge sort

```
/** Sort b[h..k]. */
public static void mS(Comparable[] b, int h, int k) {
      if (h >= k) return;
      int e= (h+k)/2;
      mS(b, h, e);        T(e+1-h) comparisons = T(n/2)
      mS(b, e+1, k);      T(k-e)   comparisons = T(n/2)
      merge(b, h, e, k);        O(k-h)   comparisons = O(n)
}
```

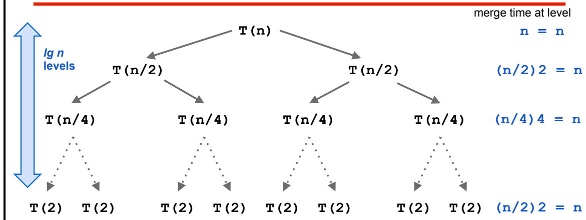
Recursive Case:
T(n) = 2T(n/2) + O(n)

Runtime

We determined that
    T(1) = 0
    T(n) = 2T(n/2) + n for n > 1

We will prove that
    T(n) = n $\log_2$ n (or **n lg n** for short)

Recursion tree



lg n levels * n comparisons is O(n log n)

Proof by induction

To prove **T(n) = n lg n**,
we can assume true for smaller values of **n** (like recursion)

T(n) = 2T(n/2) + n
    = 2(n/2)lg(n/2) + n
    = n(lg n - lg 2) + n ← Property of logarithms
    = n(lg n - 1) + n ← $\log_2 2 = 1$
    = n lg n - n + n
    = n lg n

Heap Sort

Heap Sort

Very simple idea:
1. Turn the array into a max-heap
2. Pull each element out

```
/** Sort b */
public static void heapSort(Comparable[] b) {
    heapify(b);
    for (int i= b.length-1; i >= 0; i--) {
        b[i]= poll(b, i);
    }
}
```


Heap Sort

```
/** Sort b */
public static void heapSort(Comparable[] b) {
    heapify(b);
    for (int i= b.length-1; i >= 0; i--) {
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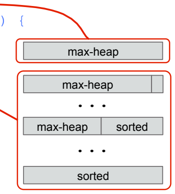

Why does it have to be a max-heap?

Heap Sort runtime

```
/** Sort b */
public static void heapSort(Comparable[] b) {
    heapify(b);                                  ← O(n lg n)
    for (int i= b.length-1; i >= 0; i--) {      ← loops n times
        b[i]= poll(b, i);                        ← O(lg n)
    }
}
```

Total runtime:

O(n lg n) + n*O(lg n) = O(n lg n)