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SEARCHING, SORTING, AND ASYMPTOTIC COMPLEXITY

Lecture 10
CS2110 – Fall 2015

Merge two adjacent sorted segments

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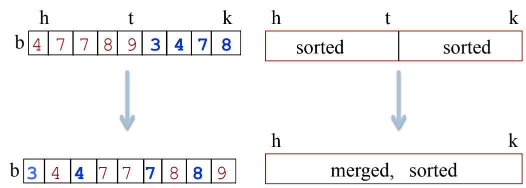
```
/* Sort b[h..k]. Precondition: b[h..t] and b[t+1..k] are sorted. */
public static merge(int[] b, int h, int t, int k) {
}
```



Merge two adjacent sorted segments

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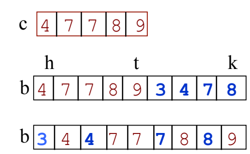
```
/* Sort b[h..k]. Precondition: b[h..t] and b[t+1..k] are sorted. */
public static merge(int[] b, int h, int t, int k) {
    Copy b[h..t] into another array c;
    Copy values from c and b[t+1..k] in ascending order into b[h..]
}
```



We leave you to write this method. Just move values from c and b[t+1..k] into b in the right order, from smallest to largest.

Runs in time linear in size of b[h..k].

Mergesort

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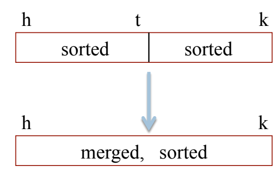
```
/** Sort b[h..k] */
public static void mergesort(int[] b, int h, int k]) {
    if (size b[h..k] < 2)
        return;
    int t= (h+k)/2;
    mergesort(b, h, t);
    mergesort(b, t+1, k);
    merge(b, h, t, k);
}
```



Mergesort

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```
/** Sort b[h..k] */
public static void mergesort(
        int[] b, int h, int k]) {
    if (size b[h..k] < 2)
        return;
    int t= (h+k)/2;
    mergesort(b, h, t);
    mergesort(b, t+1, k);
    merge(b, h, t, k);
}
```

Let n = size of b[h..k]

Merge: time proportional to n

Depth of recursion: log n

Can therefore shown (later) that time taken is proportional to n log n

But space is also proportional to n!

QuickSort versus MergeSort

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```
/** Sort b[h..k] */
public static void QS
      (int[] b, int h, int k) {
  if (k – h < 1) return;
  int j= partition(b, h, k);
  QS(b, h, j-1);
  QS(b, j+1, k);
}
```

```
/** Sort b[h..k] */
public static void MS
      (int[] b, int h, int k) {
  if (k  – h < 1) return;
  MS(b, h, (h+k)/2);
  MS(b, (h+k)/2 + 1, k);
  merge(b, h, (h+k)/2, k);
}
```

One processes the array then recurses.
One recurses then processes the array.

Readings, Homework

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- Textbook: Chapter 4
- Homework:
  - Recall our discussion of linked lists and A2.
  - What is the worst case complexity for appending an items on a linked list? For testing to see if the list contains X? What would be the best case complexity for these operations?
  - If we were going to talk about complexity (speed) for operating on a list, which makes more sense: worst-case, average-case, or best-case complexity? Why?

What Makes a Good Algorithm?

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Suppose you have two possible algorithms or ADT implementations that do the same thing; which is *better*?

What do we mean by *better*?

- Faster?
- Less space?
- Easier to code?
- Easier to maintain?
- Required for homework?

How do we measure time and space of an algorithm?

Basic Step: One "constant time" operation

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Basic step:

- Input/output of scalar value
- Access value of scalar variable, array element, or object field
- assign to variable, array element, or object field
- do one arithmetic or logical operation
- method call (not counting arg evaluation and execution of method body)

- If-statement: number of basic steps on branch that is executed
- Loop: (number of basic steps in loop body) * (number of iterations) –also bookkeeping
- Method: number of basic steps in method body (include steps needed to prepare stack-frame)

Counting basic steps in worst-case execution

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Linear Search

Let n = b.length

```
/** return true iff v is in b */
static boolean find(int[] b, int v) {
  for (int i = 0; i < b.length; i++) {
    if (b[i] == v) return true;
  }
  return false;
}
```

worst-case execution

basic step	# times executed
i= 0;	1
i < b.length	n+1
i++	n
b[i] == v	n
return true	0
return false	1
Total	3n + 3

We sometimes simplify counting by counting only important things.
Here, it's the number of array element comparisons b[i] == v.
That's the number of loop iterations: n.

Sample Problem: Searching

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Second solution:
Binary Search

inv:
b[0..h] <= v < b[k..]

```
/** b is sorted. Return h satisfying
    b[0..h] <= v < b[h+1..] */
static int bsearch(int[] b, int v) {
   int h= -1;
   int k= b.length;
   while (h+1 != k) {
      int e= (h+ k)/2;
      if (b[e] <= v)  h= e;
      else k= e;
   }
   return h;
}
```

Number of iterations (always the same):
~log b.length
Therefore,
log b.length
arrray comparisons

What do we want from a definition of "runtime complexity"?

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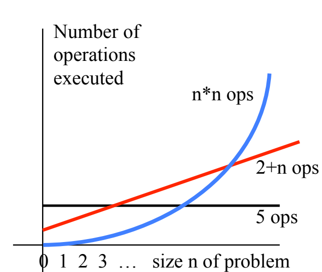

1. Distinguish among cases for large n, not small n

2. Distinguish among important cases, like
- n*n basic operations
- n basic operations
- log n basic operations
- 5 basic operations

3. Don't distinguish among trivially different cases.
- 5 or 50 operations
- n, n+2, or 4n operations

Definition of O(…)

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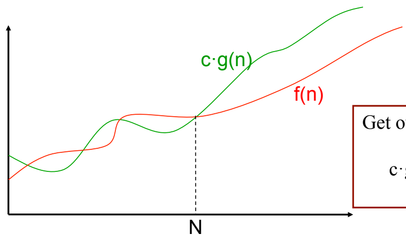
Formal definition: f(n) is O(g(n)) if there exist constants c and N such that for all $n \geq N$, $f(n) \leq c \cdot g(n)$



Graphical view

Get out far enough –for n >=N– c·g(n) is bigger than f(n)

What do we want from a definition of "runtime complexity"?

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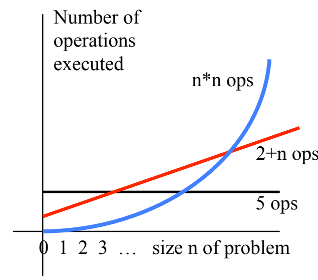

Formal definition: f(n) is O(g(n)) if there exist constants c and N such that for all n ≥ N, f(n) ≤ c·g(n)

Roughly, f(n) is O(g(n)) means that f(n) grows like g(n) or slower, to within a constant factor

Prove that $(n^2 + n)$ is $O(n^2)$

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Formal definition: f(n) is O(g(n)) if there exist constants c and N such that for all $n \geq N$, $f(n) \leq c \cdot g(n)$

Example: Prove that $(n^2 + n)$ is $O(n^2)$

Methodology:

Start with f(n) and slowly transform into c · g(n):

- Use = and <= and < steps
- At appropriate point, can choose N to help calculation
- At appropriate point, can choose c to help calculation

Prove that $(n^2 + n)$ is $O(n^2)$

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Formal definition: f(n) is O(g(n)) if there exist constants c and N such that for all $n \geq N$, $f(n) \leq c \cdot g(n)$

Example: Prove that $(n^2 + n)$ is $O(n^2)$

f(n)

= <definition of f(n)>

$n^2 + n$

<= <for n >= 1, n <= n²>

$n^2 + n^2$

= <arith>

$2*n^2$

= <choose g(n) = n²>

2*g(n)

Choose N = 1 and c = 2

Prove that 100 n + log n is O(n)

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Formal definition: f(n) is O(g(n)) if there exist constants c and N such that for all $n \geq N$, $f(n) \leq c \cdot g(n)$

f(n)

= <put in what f(n) is>

100 n + log n

<= <We know log n ≤ n for n ≥ 1>

100 n + n

= <arith>

101 n

= <g(n) = n>

101 g(n)

Choose N = 1 and c = 101

O(…) Examples

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Let $f(n) = 3n^2 + 6n - 7$
- f(n) is $O(n^2)$
- f(n) is $O(n^3)$
- f(n) is $O(n^4)$
- …

$p(n) = 4 n \log n + 34 n - 89$
- p(n) is O(n log n)
- p(n) is $O(n^2)$

$h(n) = 20 \cdot 2^n + 40n$
- h(n) is $O(2^n)$

a(n) = 34
- a(n) is O(1)

Only the *leading* term (the term that grows most rapidly) matters

If it's $O(n^2)$, it's also $O(n^3)$ etc! However, we always use the smallest one

Problem-size examples

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- Suppose a computer can execute 1000 operations per second; how large a problem can we solve?

alg	1 second	1 minute	1 hour
O(n)	1000	60,000	3,600,000
O(n log n)	140	4893	200,000
$O(n^2)$	31	244	1897
$3n^2$	18	144	1096
$O(n^3)$	10	39	153
$O(2^n)$	9	15	21

Commonly Seen Time Bounds

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O(1)	constant	excellent
O(log n)	logarithmic	excellent
O(n)	linear	good
O(n log n)	n log n	pretty good
$O(n^2)$	quadratic	OK
$O(n^3)$	cubic	maybe OK
$O(2^n)$	exponential	too slow

Worst-Case/Expected-Case Bounds

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May be difficult to determine time bounds for all imaginable inputs of size n

Simplifying assumption #4:
Determine number of steps for either
- worst-case or
- expected-case or average case

- Worst-case
  - Determine how much time is needed for the *worst possible* input of size n
- Expected-case
  - Determine how much time is needed *on average* for all inputs of size n

Simplifying Assumptions

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Use the size of the input rather than the input itself – n

Count the number of "basic steps" rather than computing exact time

Ignore multiplicative constants and small inputs (order-of, big-O)

Determine number of steps for either
- worst-case
- expected-case

These assumptions allow us to analyze algorithms effectively

Worst-Case Analysis of Searching

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Linear Search

```
// return true iff v is in b
static bool find (int[] b, int v) {
  for (int x : b) {
    if (x == v) return true;
  }
  return false;
}
```

worst-case time: O(#b)
Expected time O(#b)

#b = size of b

Binary Search

```
// Return h that satisfies
//     b[0..h] <= v < b[h+1..]
static bool bsearch(int[] b, int v {
  int h= -1;  int t= b.length;
  while ( h != t-1 ) {
    int  e= (h+t)/2;
    if (b[e] <= v)  h= e;
    else t= e;
  }
}
```

Always ~(log #b+1) iterations.
Worst-case and expected times: O(log #b)

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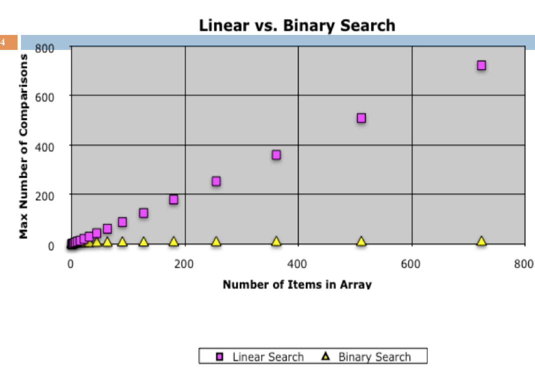

Analysis of Matrix Multiplication

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Multiply n-by-n matrices A and B:

Convention, matrix problems measured in terms of n, the number of rows, columns

- Input size is really $2n^2$, not n
- Worst-case time: $O(n^3)$
- Expected-case time: $O(n^3)$

```
for (i = 0; i < n; i++)
  for (j = 0; j < n; j++) {
    c[i][j] = 0;
    for (k = 0; k < n; k++)
      c[i][j] += a[i][k]*b[k][j];
  }
```

Remarks

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Once you get the hang of this, you can quickly zero in on what is relevant for determining asymptotic complexity

- Example: you can usually ignore everything that is not in the innermost loop. Why?

One difficulty:

- Determining runtime for recursive programs
  Depends on the depth of recursion

Why bother with runtime analysis?

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Computers so fast that we can do whatever we want using simple algorithms and data structures, right?

Not really – data-structure/algorithm improvements can be a *very big* win

Scenario:

- A runs in n^2 msec
- A' runs in $n^2/10$ msec
- B runs in 10 n log n msec

Problem of size $n=10^3$

- A: 10^3 sec ≈ 17 minutes
- A': 10^2 sec ≈ 1.7 minutes
- B: 10^2 sec ≈ 1.7 minutes

Problem of size $n=10^6$

- A: 10^9 sec ≈ 30 years
- A': 10^8 sec ≈ 3 years
- B: $2 \cdot 10^5$ sec ≈ 2 days

1 day = 86,400 sec ≈ 10^5 sec
1,000 days ≈ 3 years

Algorithms for the Human Genome

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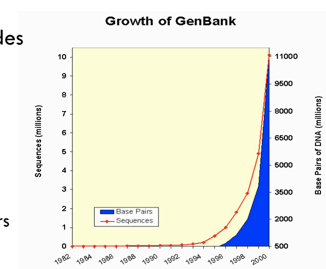
Human genome
= 3.5 billion nucleotides
~ 1 Gb

@1 base-pair instruction/μsec

- n^2 → 388445 years
- n log n → 30.824 hours
- n → 1 hour



Limitations of Runtime Analysis

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Big-O can hide a very large constant

- Example: selection
- Example: small problems

The specific problem you want to solve may not be the worst case

- Example: Simplex method for linear programming

Your program may not run often enough to make analysis worthwhile

- Example: one-shot vs. every day
- You may be analyzing and improving the wrong part of the program

- Very common situation
- Should use profiling tools

What you need to know / be able to do

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- Know the definition of f(n) is O(g(n))
- Be able to prove that some function f(n) is O(g(n)). The simplest way is as done on two slides above.
- Know worst-case and average (expected) case O(…) of basic searching/sorting algorithms: linear/binary search, partition alg of quicksort, insertion sort, selection sort, quicksort, merge sort.
- Be able to look at an algorithm and figure out its worst case O(…) based on counting basic steps or things like array-element swaps

Lower Bound for Comparison Sorting

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Goal: Determine minimum time *required* to sort *n* items

Note: we want *worst-case*, not *best-case* time

- Best-case doesn't tell us much. E.g. Insertion Sort takes O(n) time on already-sorted input
- Want to know *worst-case time* for *best possible* algorithm

- How can we prove anything about the *best possible* algorithm?
- Want to find characteristics that are common to *all* sorting algorithms
- Limit attention to *comparison-based algorithms* and try to count number of comparisons

Comparison Trees

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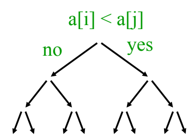
- Comparison-based algorithms make decisions based on comparison of data elements
- Gives a *comparison tree*
- If algorithm fails to terminate for some input, comparison tree is infinite
- Height of comparison tree represents *worst-case number of comparisons* for that algorithm
- Can show: *Any* correct comparison-based algorithm must make at least n log n comparisons in the worst case



Lower Bound for Comparison Sorting

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- Say we have a correct comparison-based algorithm
- Suppose we want to sort the elements in an array b[]
- Assume the elements of b[] are distinct
- Any permutation of the elements is initially possible
- When done, b[] is sorted
- But the algorithm could not have taken the same path in the comparison tree on different input permutations

Lower Bound for Comparison Sorting

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How many input permutations are possible? $n! \sim 2^{n \log n}$

For a comparison-based sorting algorithm to be correct, it must have at least that many leaves in its comparison tree

To have at least $n! \sim 2^{n \log n}$ leaves, it must have height at least n log n (since it is only binary branching, the number of nodes at most doubles at every depth)

Therefore its longest path must be of length at least n log n, and that is its worst-case running time

Mergesort

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```
/** Sort b[h..k] */
public static mergesort(
  int[] b, int h, int k]) {
    if (size b[h..k] < 2)
        return;
    int t= (h+k)/2;
    mergesort(b, h, t);
    mergesort(b, t+1, k);
    merge(b, h, t, k);
}
```

Runtime recurrence

T(n): time to sort array of size n

$T(1) = 1$

$T(n) = 2T(n/2) + O(n)$

Can show by induction that

T(n) is O(n log n)

Alternatively, can see that T(n) is O(n log n) by looking at tree of recursive calls