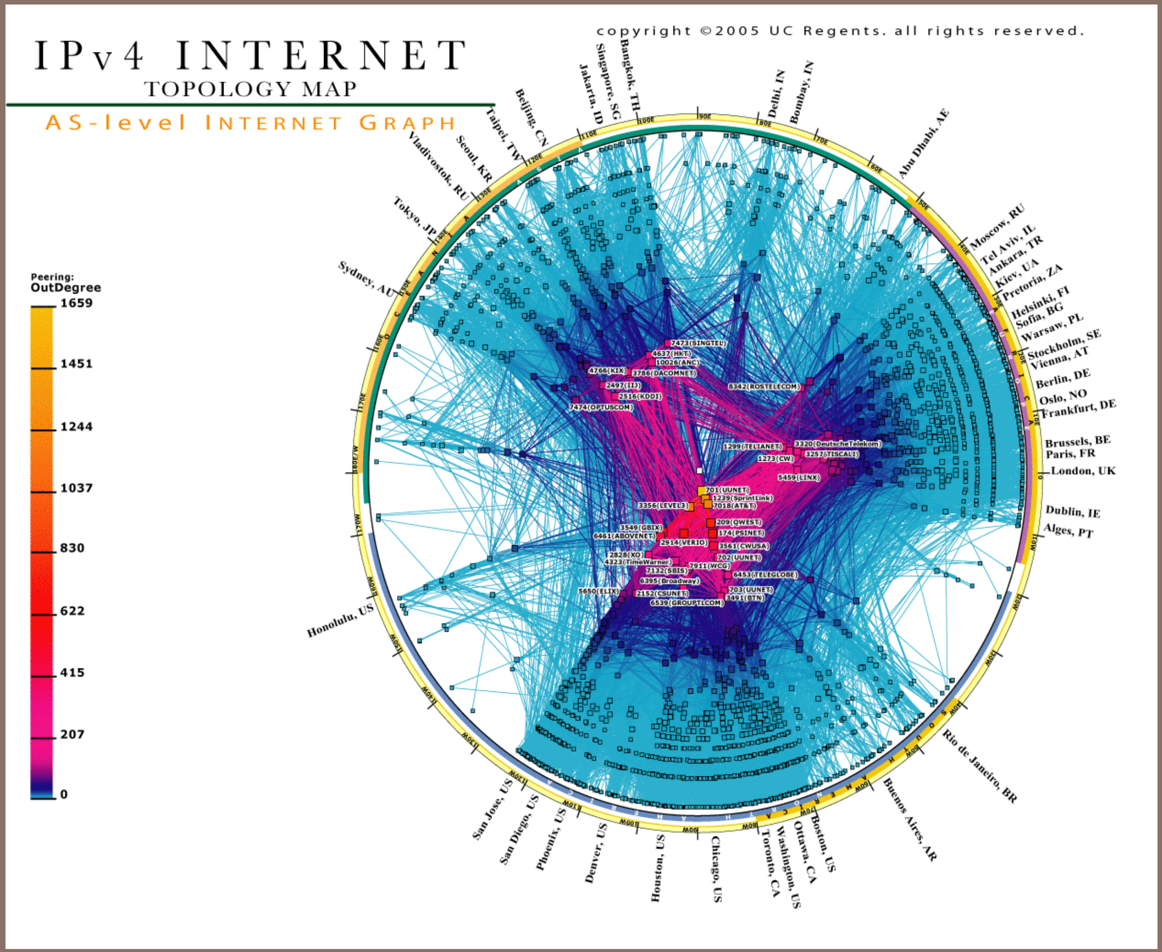
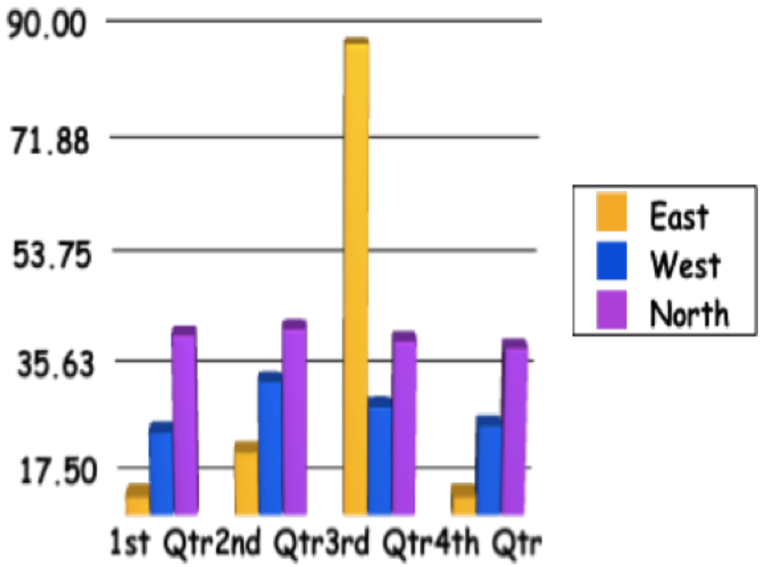


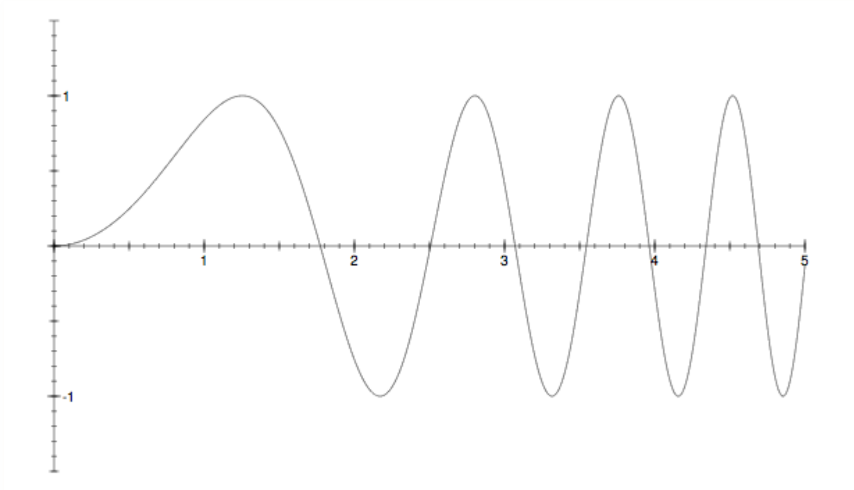
GRAPHS

Lecture 17

CS2110 – Spring 2014

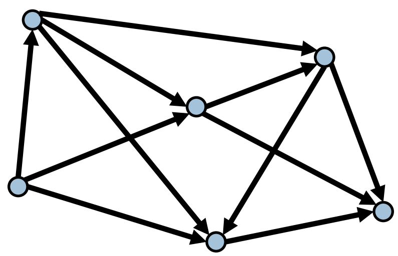
These are not Graphs

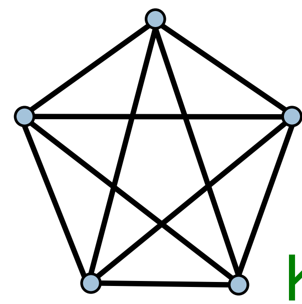




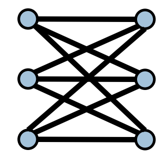
...not the kind we mean, anyway

These are Graphs

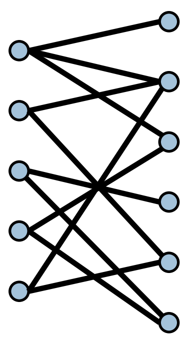


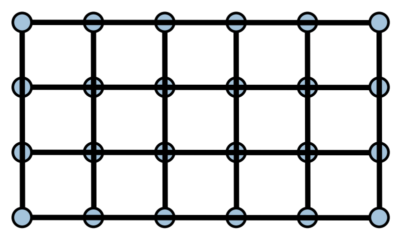


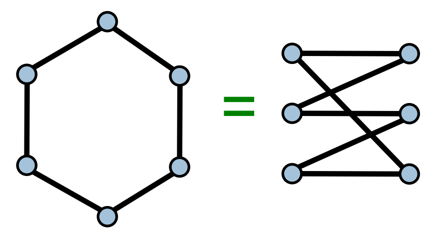
K_5



$K_{3,3}$







Applications of Graphs

- Communication networks
- The internet is a huge graph
- Routing and shortest path problems
- Commodity distribution (flow)
- Traffic control
- Resource allocation
- Geometric modeling
- ...

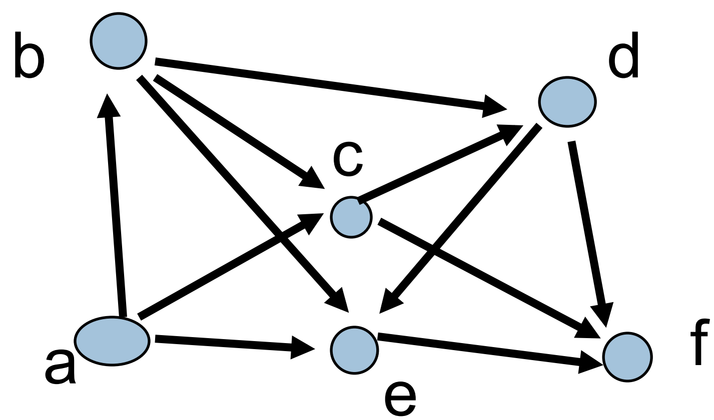
Graph Definitions

- A directed graph (or digraph) is a pair (V, E) where
  - V is a set
  - E is a set of ordered pairs (u,v) where u,v $\in$ V
    - Sometimes require u $\neq$ v (i.e. no self-loops)

- An element of V is called a vertex (pl. vertices) or node
- An element of E is called an edge or arc

- |V| is the size of V, often denoted by n
- |E| is size of E, often denoted by m

Example Directed Graph (Digraph)



V = {a,b,c,d,e,f}

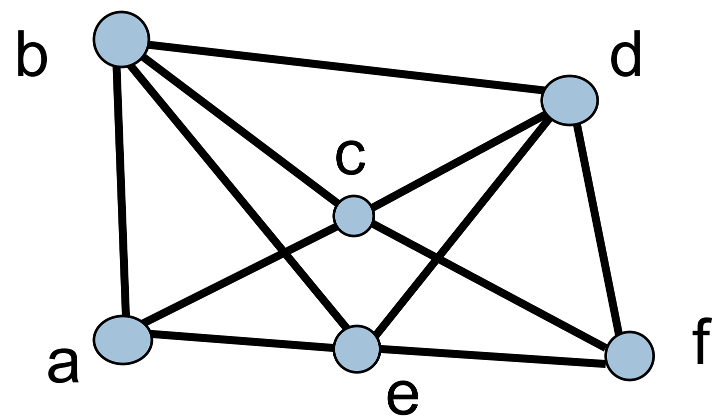
E = {(a,b), (a,c), (a,e), (b,c), (b,d), (b,e), (c,d), (c,f), (d,e), (d,f), (e,f)}

|V| = 6, |E| = 11

Example *Undirected* Graph

An *undirected graph* is just like a directed graph, except the edges are *unordered pairs* (*sets*) {u,v}

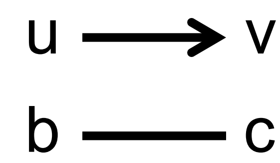
Example:



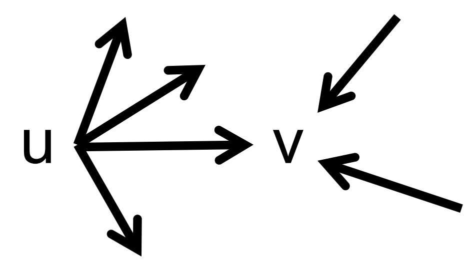
V = {a,b,c,d,e,f}
E = {{a,b}, {a,c}, {a,e}, {b,c}, {b,d}, {b,e}, {c,d}, {c,f}, {d,e}, {d,f}, {e,f}}

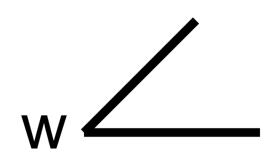
Some Graph Terminology

- u is the source , v is the sink of (u,v)
- u, v, b, c are the endpoints of (u,v) and (b, c)
- u, v are adjacent nodes. b, c are adjacent nodes



- outdegree of u in directed graph:
  number of edges for which u is source
- indegree of v in directed graph:
  number of edges for which v is sink
- degree of vertex w in undirected graph:
  number of edges of which w is an endpoint

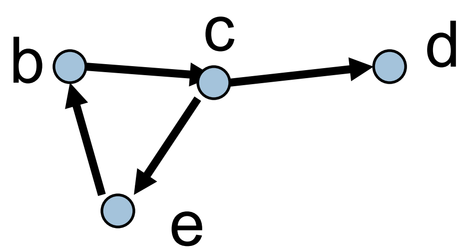




outdegree of u: 4 indegree of v: 3 degree of w: 2

More Graph Terminology

- **path:** sequence of adjacent vertexes
- **length of path**: number of edges
- **simple path**: no vertex is repeated



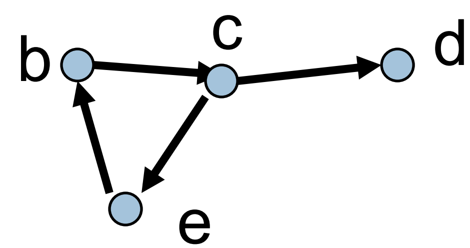
simple path of length 2: (b, c, d)

simple path of length 0: (b)

not a simple path: (b, c, e, b, c, d)

More Graph Terminology

- **cycle**: path that ends at its beginning
- **simple cycle**: only repeated vertex is its beginning/end
- **acyclic graph**: graph with no cycles
- **dag**: directed acyclic graph



cycles: (b, c, e, b)   (b, c, e, b, c, e, b)

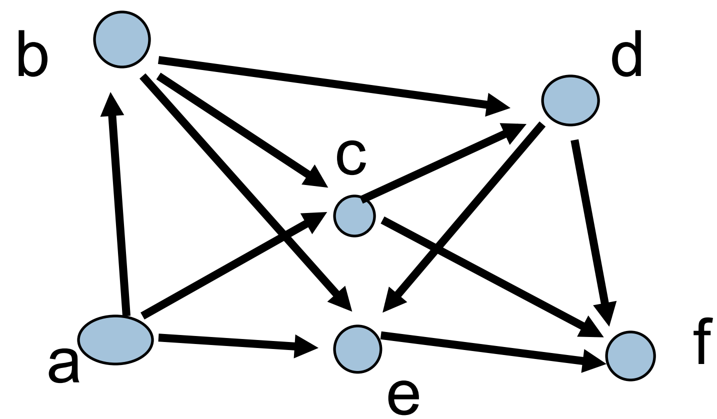
simple cycle: (c, e, b, c)

graph shown is not a dag

Question: is (d) a cycle?

No. A cycle must have at least one edge

Is this a dag?

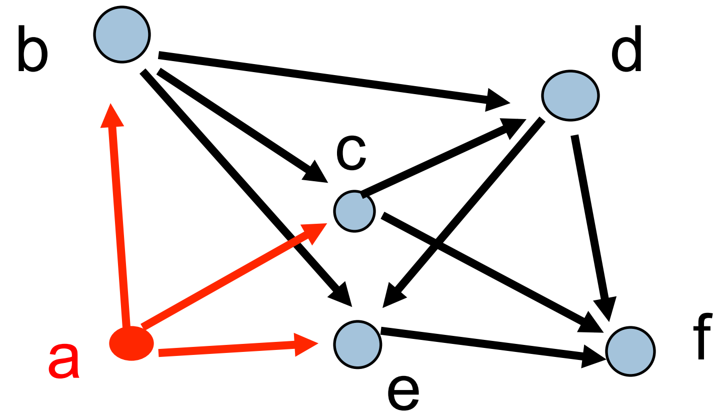


- Intuition: A dag has a vertex with indegree 0. Why?

- This idea leads to an algorithm:

  A digraph is a dag if and only if one can iteratively delete indegree-0 vertices until the graph disappears

Is this a dag?



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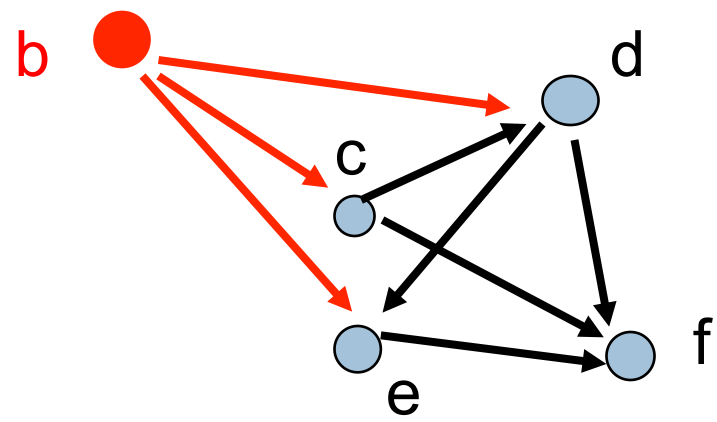

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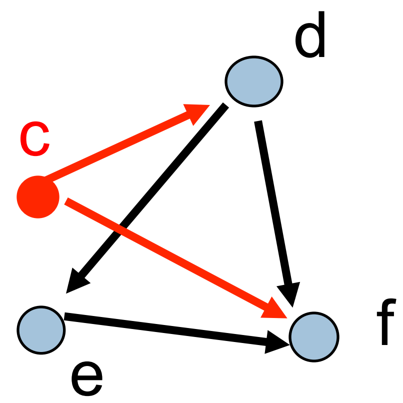

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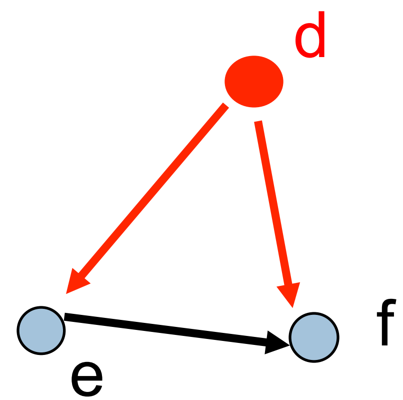

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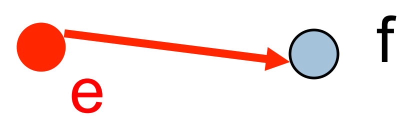

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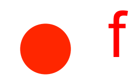
Is this a dag?

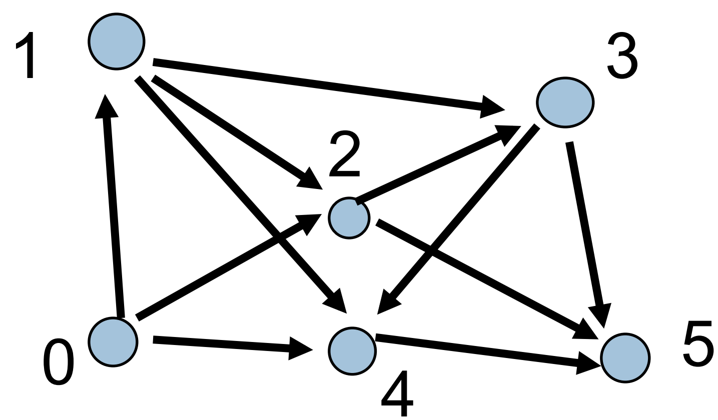


- Intuition: A dag has a vertex with indegree 0. Why?
- This idea leads to an algorithm:

  A digraph is a dag if and only if one can iteratively delete indegree-0 vertices until the graph disappears

Topological Sort

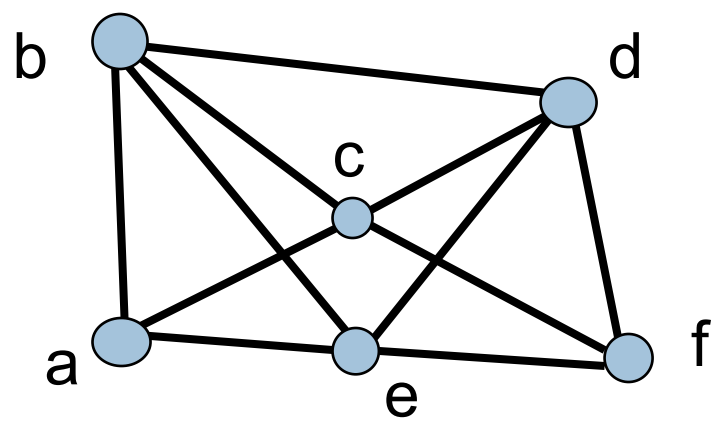
- We just computed a topological sort of the dag
  - This is a numbering of the vertices such that all edges go from lower- to higher-numbered vertices



- Useful in job scheduling with precedence constraints

Graph Coloring

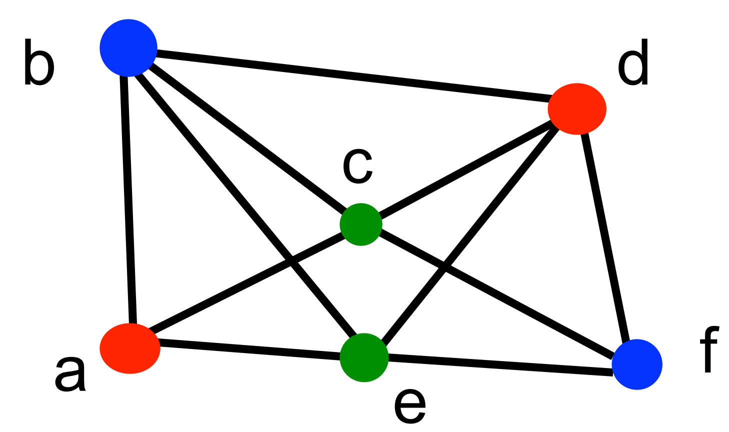
Coloring of an undirected graph: an assignment of a color to each node such that no two adjacent vertices get the same color



How many colors are needed to color this graph?

Graph Coloring

A coloring of an undirected graph: an assignment of a color to each node such that no two adjacent vertices get the same color

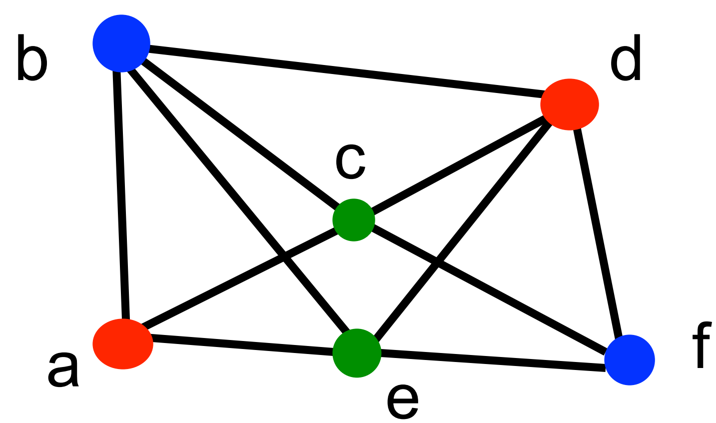


How many colors are needed to color this graph?

3

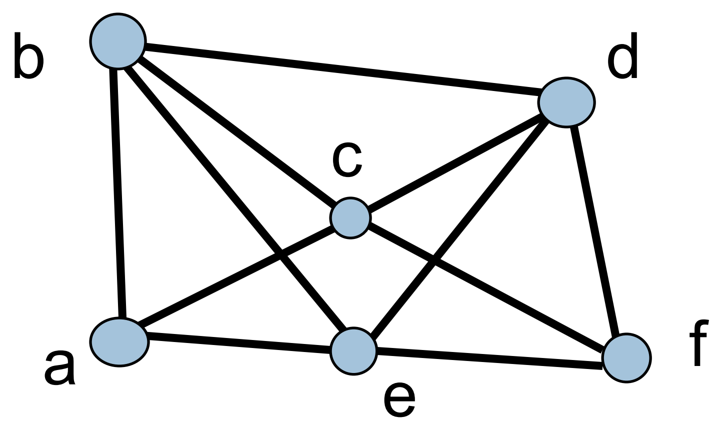
An Application of Coloring

- Vertices are jobs
- Edge (u,v) is present if jobs u and v each require access to the same shared resource, so they cannot execute simultaneously
- Colors are time slots to schedule the jobs
- Minimum number of colors needed to color the graph = minimum number of time slots required



Planarity

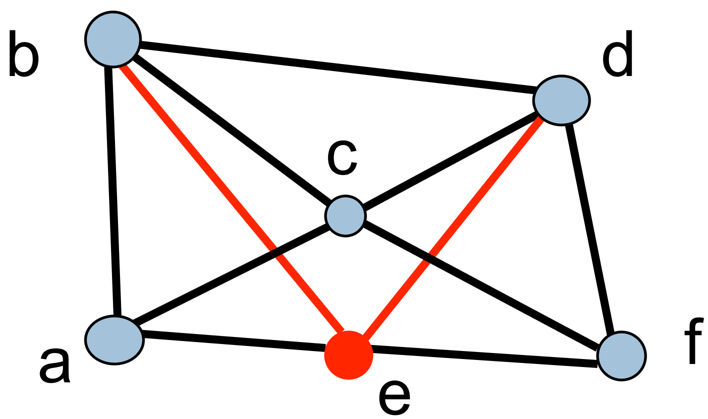
A graph is planar if it can be embedded in the plane with no edges crossing

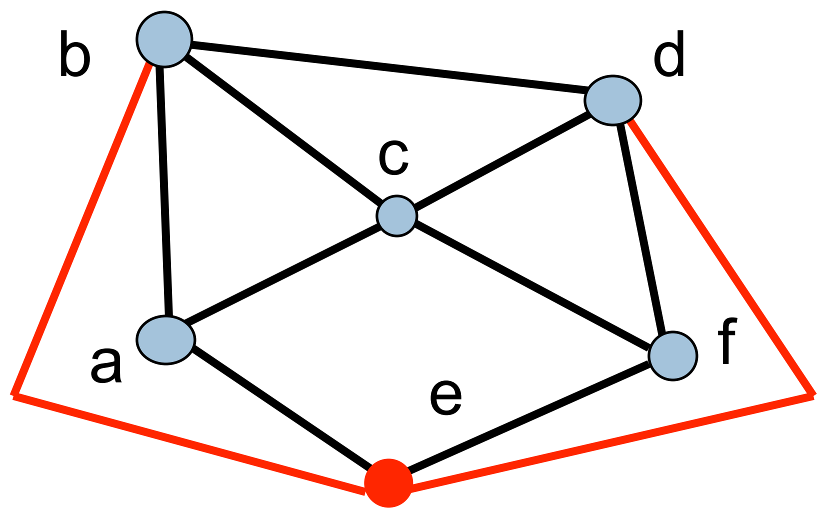


Is this graph planar?

Planarity

A graph is planar if it can be embedded in the plane with no edges crossing



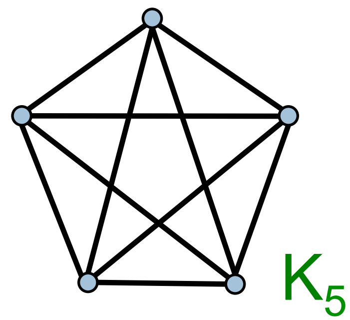


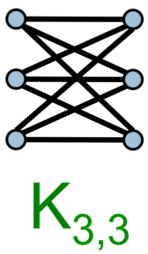
Is this graph planar?

YES

Detecting Planarity

Kuratowski's Theorem





A graph is planar if and only if it does not contain a copy of K_5 or $K_{3,3}$ (possibly with other nodes along the edges shown)

Detecting Planarity

Early 1970's John Hopcroft spent time at Stanford, talked to grad student Bob Tarjan (now at Princeton). Together, they developed a linear-time algorithm to test a graph for planarity. Significant achievement.

Won Turing Award

The Four-Color Theorem

Every planar graph is 4-colorable

(Appel & Haken, 1976)

Interesting history. “Proved” in about 1876 and published, but ten years later, a mistake was found. It took 90 more years for a proof to be found.

Countries are nodes; edge between them if they have a common boundary. You need 5 colors to color a map —water has to be blue!

The Four-Color Theorem

Every planar graph is 4-colorable

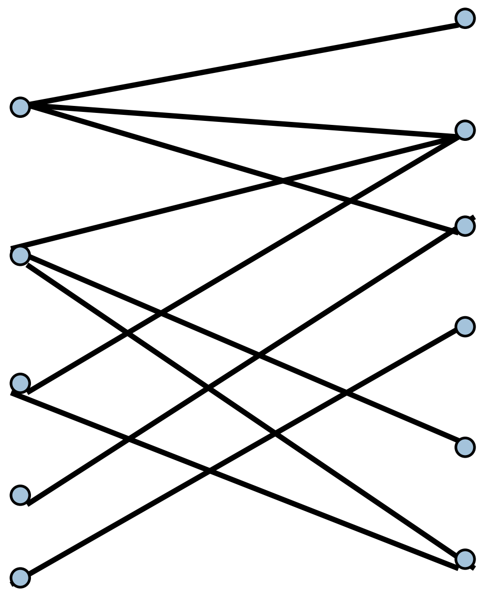
(Appel & Haken, 1976)

Proof rests on a lot of computation! A program checks thousands of "configurations", and if none are colorable, theorem holds.

Program written in assembly language. Recursive, contorted, to make it efficient. Gries found an error in it but a "safe kind": it might say a configuration was colorable when it wasn't.

Bipartite Graphs

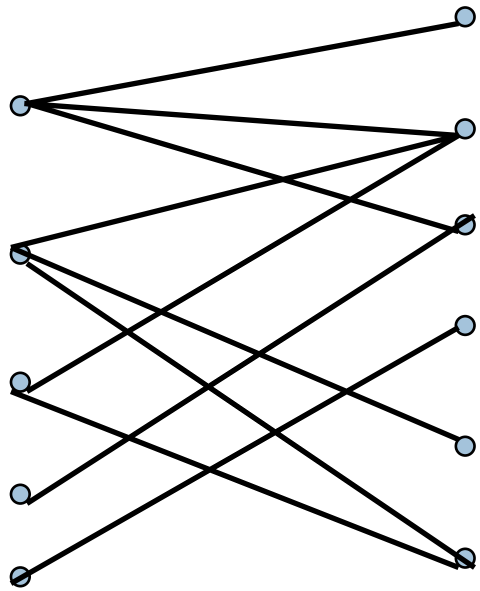
A directed or undirected graph is bipartite if the vertices can be partitioned into two sets such that all edges go between the two sets



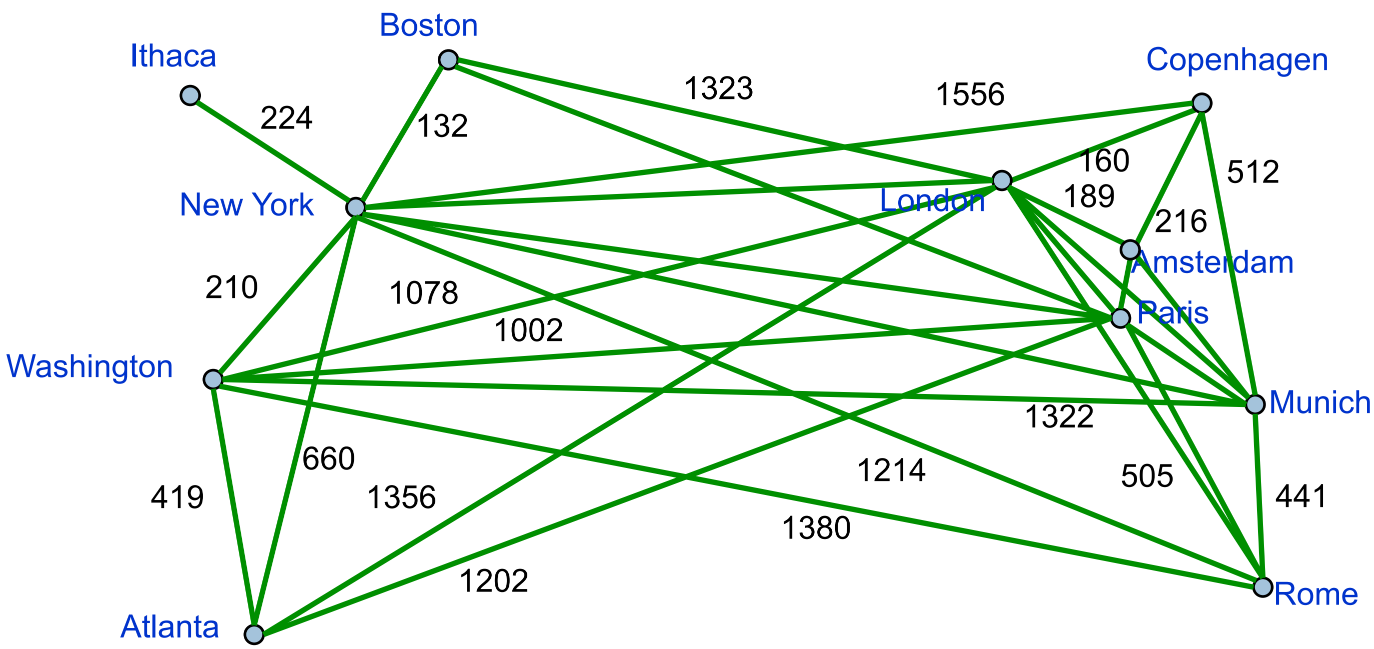
Bipartite Graphs

The following are equivalent

- G is bipartite
- G is 2-colorable
- G has no cycles of odd length

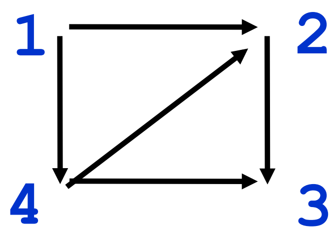


Traveling Salesperson

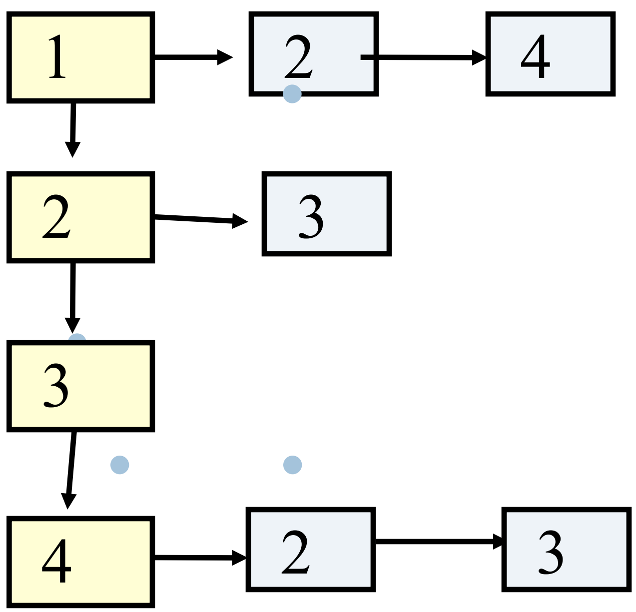


Find a path of minimum distance that visits every city

Representations of Graphs



Adjacency List



Adjacency Matrix

	1	2	3	4
1	0	1	0	1
2	0	0	1	0
3	0	0	0	0
4	0	1	1	0

Adjacency Matrix or Adjacency List?

n: number of vertices

m: number of edges

d(u): outdegree of u

Adjacency Matrix

- Uses space $O(n^2)$
- Can iterate over all edges in time $O(n^2)$
- Can answer "Is there an edge from u to v?" in O(1) time
- Better for dense graphs (lots of edges)

- Adjacency List
  - Uses space O(m+n)
  - Can iterate over all edges in time O(m+n)
  - Can answer "Is there an edge from u to v?" in O(d(u)) time
  - Better for sparse graphs (fewer edges)

Graph Algorithms

- Search
  - depth-first search
  - breadth-first search
- Shortest paths
  - Dijkstra's algorithm
- Minimum spanning trees
  - Prim's algorithm
  - Kruskal's algorithm

Depth-First Search

- Follow edges depth-first starting from an arbitrary vertex r, using a stack to remember where you came from
- When you encounter a vertex previously visited, or there are no outgoing edges, retreat and try another path
- Eventually visit all vertices reachable from r
- If there are still unvisited vertices, repeat
- O(m) time

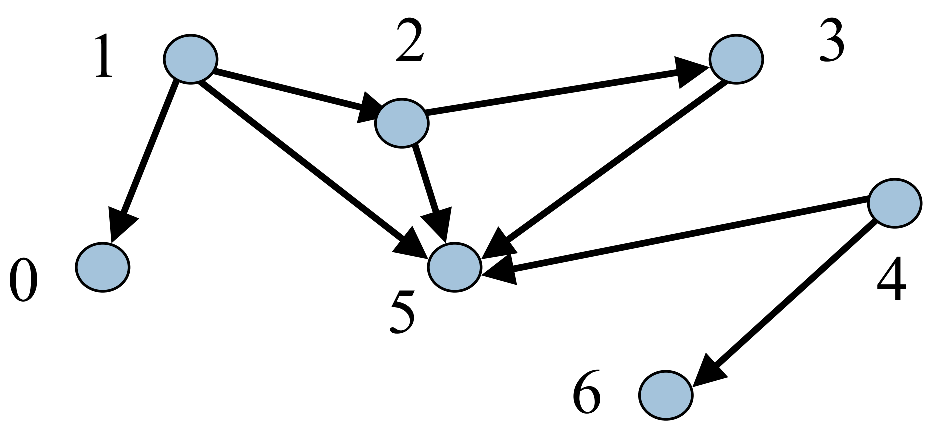
Difficult to understand!
Let's write a recursive procedure

Depth-First Search

boolean[] visited;

node u is visited means: visited[u] is true
To visit u means to: set visited[u] to true

Node u is REACHABLE from node v if there is a path (u, …, v) in which all nodes of the path are unvisited.



Suppose all nodes are unvisited.

The nodes that are REACHABLE from node 1 are 1, 0, 2, 3, 5

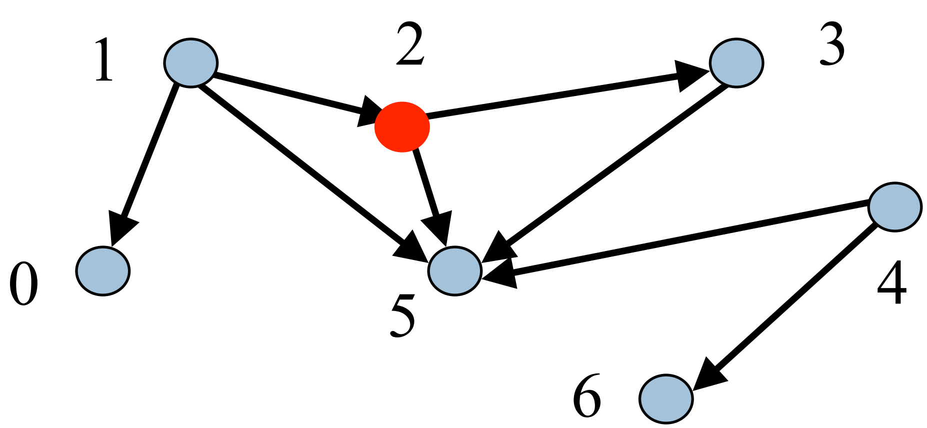
The nodes that are REACHABLE from 4 are 4, 5, 6.

Depth-First Search

boolean[] visited;

To "visit" a node u: set visited[u] to true.

Node u is REACHABLE from node v if there is a path (u, …, v) in which all nodes of the path are unvisited.



Suppose 2 is already visited, others unvisited.

The nodes that are REACHABLE from node 1 are 1, 0, 5

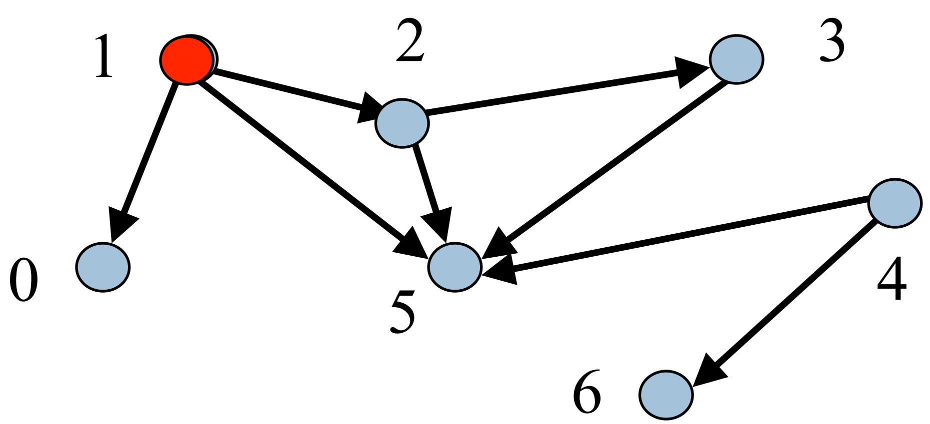
The nodes that are REACHABLE from 4 are 4, 5, 6.

Depth-First Search


```
/** Node u is unvisited. Visit all nodes
    that are REACHABLE from u. */
public static void dfs(int u) {
    visited[u]= true;


}
```

Let u be 1
The nodes that are REACHABLE from node 1 are
1, 0, 2, 3, 5

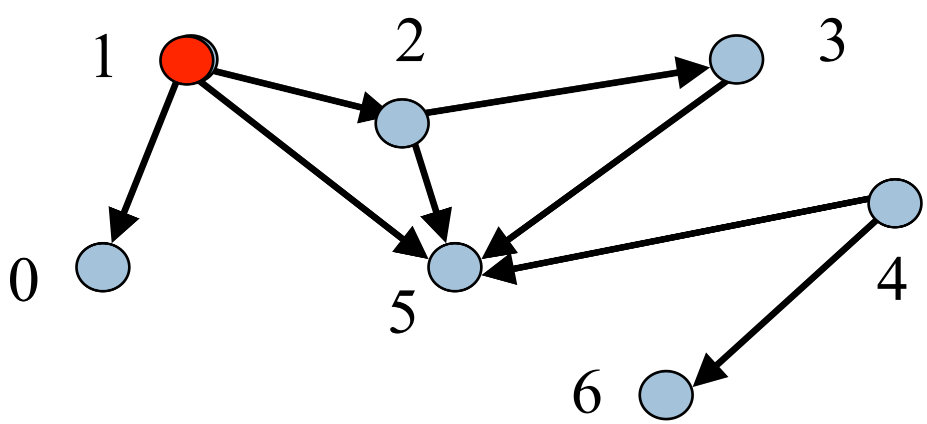


Depth-First Search


```
/** Node u is unvisited. Visit all nodes
    that are REACHABLE from u. */
public static void dfs(int u) {

    visited[u]= true;

    for each edge (u, v)
        if v is unvisited then dfs(v);
}
```



Let u be 1
The nodes to be visited are
0, 2, 3, 5

Have to do dfs on all unvisited neighbors of u

Depth-First Search


```
/** Node u is unvisited. Visit all nodes
    that are REACHABLE from u. */
public static void dfs(int u) {

    visited[u]= true;

    for each edge (u, v)
        if v is unvisited then dfs(v);
}
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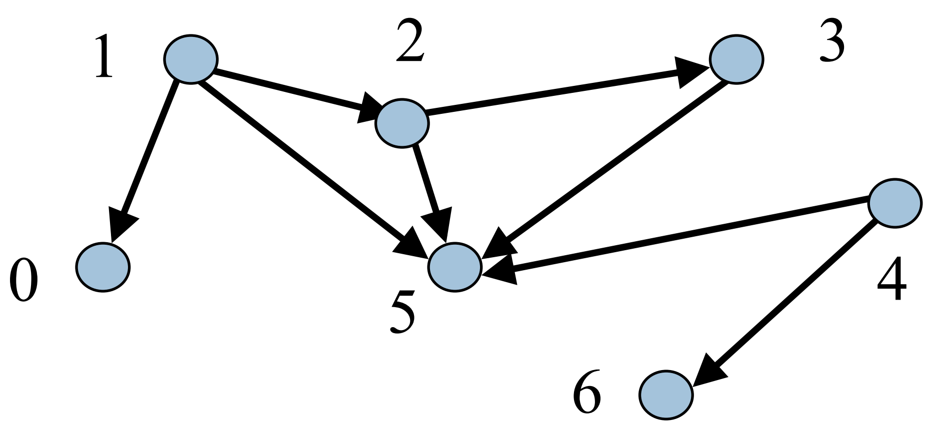


Let u be 1
The nodes to be visited are
0, 2, 3, 5

Suppose the for each loop visits neighbors in numerical order. Then dfs(1) visits the nodes in this order:
1, 0, 2, 3, 5

Depth-First Search


```
/** Node u is unvisited. Visit all nodes
    that are REACHABLE from u. */
public static void dfs(int u) {
     visited[u]= true;
     for each edge (u, v)
         if v is unvisited then dfs(v);
}
```

That's all there is to the basic dfs. You may have to change it to fit a particular situation.

Example: There may be a different way (other than array visited) to know whether a node has been visited

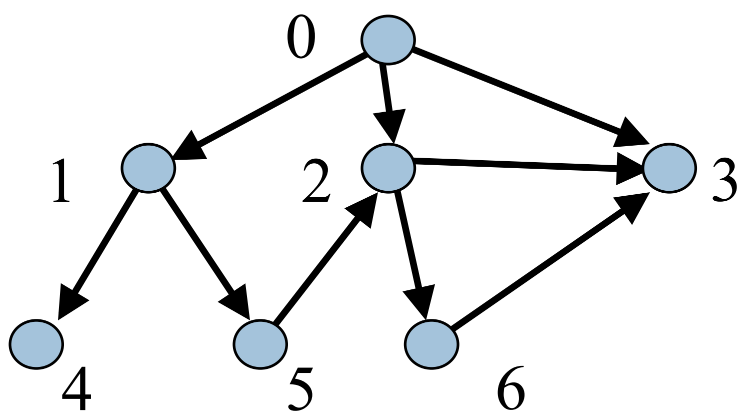
Example: Instead of using recursion, use a loop and maintain the stack yourself.

Breadth-First Search (BFS)

BFS visits all neighbors first before visiting their neighbors. It goes level by level.

Use a queue instead of a stack

- stack: last-in, first-out (LIFO)
- queue: first-in, first-out (FIFO)



dfs(0) visits in this order:
0, 1, 4, 5, 2, 3, 6

bfs(0) visits in this order:
0,1, 2, 3, 4, 5, 6

Breadth-first not good for the Bfly: too much flying back and forth

Summary

- We have seen an introduction to graphs and will return to this topic on Thursday
  - Definitions
  - Testing for a dag
  - Depth-first and breadth-first search