BINARY SEARCH AND LOOP INVARIANTS

Lecture 12A
CS2110 – Spring 2014

Develop binary search in sorted array b for v

pre: b

0	b.length
?	

post: b

0	h	b.length
<= v	> v	

Example:

pre: b

0				4	5	6	7				b.length
2	2	4	4	4	4	7	9	9	9	9	

If v is 4, 5, or 6, h is 5

If v is 7 or 8, h is 6

If v in b, h is index of rightmost occurrence of v.
If v not in b, h is index before where it belongs.

Develop binary search in sorted array b for v

	0		b.length
pre: b	?		

	0	h	b.length
post: b	<= v	> v	

Better than Binary search in last lecture because it

(1) Finds not a random occurrence of v but the rightmost one. Useful in some situations

(2) If v is not in b, it gives useful information: it belongs between b[h] and b[h+1]

(3) Works also when array is empty!

Develop binary search in sorted array b for v

pre:

	0		b.length
b	?		

Store a value in h to make this true:

post:

	0	h	b.length
b	<= v	> v	

Get loop invariant by combining pre- and post-conditions, adding variable t to mark the other boundary

inv:

	0	h	t	b.length
b	<= v	?	> v	

How does it start (what makes the invariant true)?

pre:

	0		b.length
b	?		

inv:

	0	h	t	b.length
b	<= v	?	> v	

Make first and last partitions empty:

h= -1; t= b.length;

When does it end (when does invariant look like postcondition)?

post: b

0	h	b.length
<= v	> v	

inv: b

0	h	t	b.length
<= v	?	> v	

```
h= -1;  t= b.length;
while ( h  !=  t-1 ) {

}
```

Stop when ? section is empty. That is when h = t-1.
Therefore, continue as long as h != t-1.

How does body make progress toward termination (cut ? in half) and keep invariant true?

inv:

	0	h	t	b.length
b	<= v	?	> v	

	0	h	e	t	b.length
b	<= v	?		> v	

```
h= -1;  t= b.length;
while ( h != t-1 ) {
    int  e= (h+t)/2;
}
```

Let e be index of middle value of ? Section.
Maybe we can set h or t to e, cutting ? section in half

How does body make progress toward termination (cut ? in half) and keep invariant true?

inv:

0	h	t	b.length
b: <= v	?	> v	

0	h	e	t	b.length
b: <= v	?	?	> v	

0	h	e	t	b.length
b: <= v	<= v	?	> v	

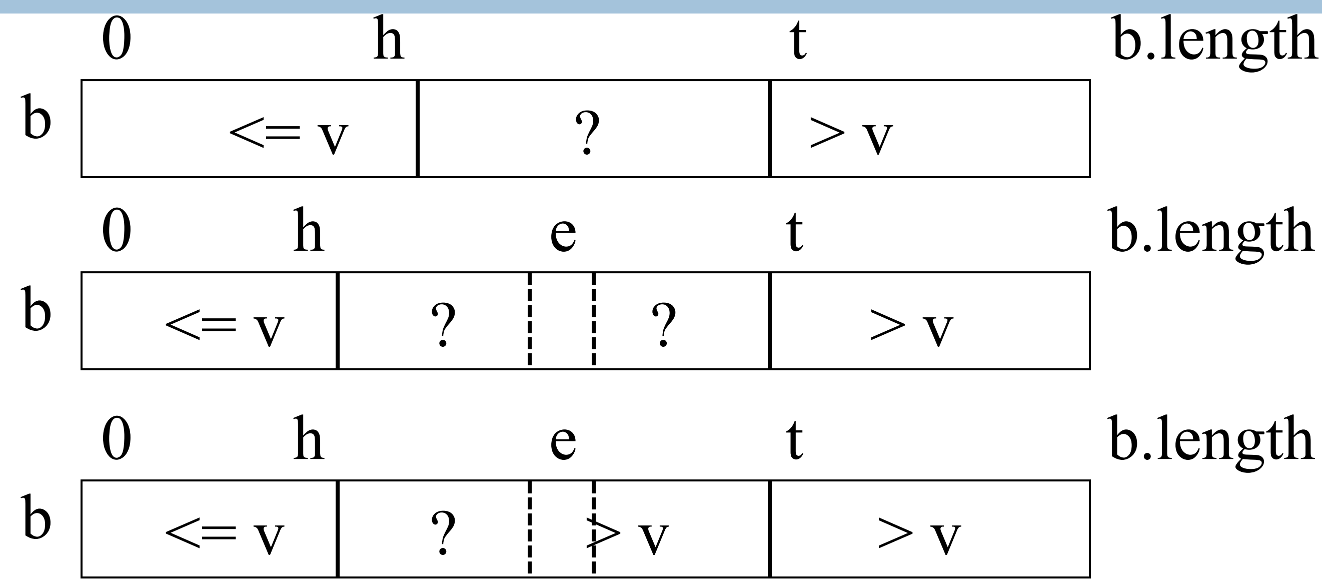
```
h= -1;  t= b.length;
while ( h != t-1 ) {
    int  e= (h+t)/2;
    if (b[e] <= v)  h= e;

}
```

If b[e] <= v, then so is every value to its left, since the array is sorted. Therefore, h= e; keeps the invariant true.

How does body make progress toward termination (cut ? in half) and keep invariant true?

inv:

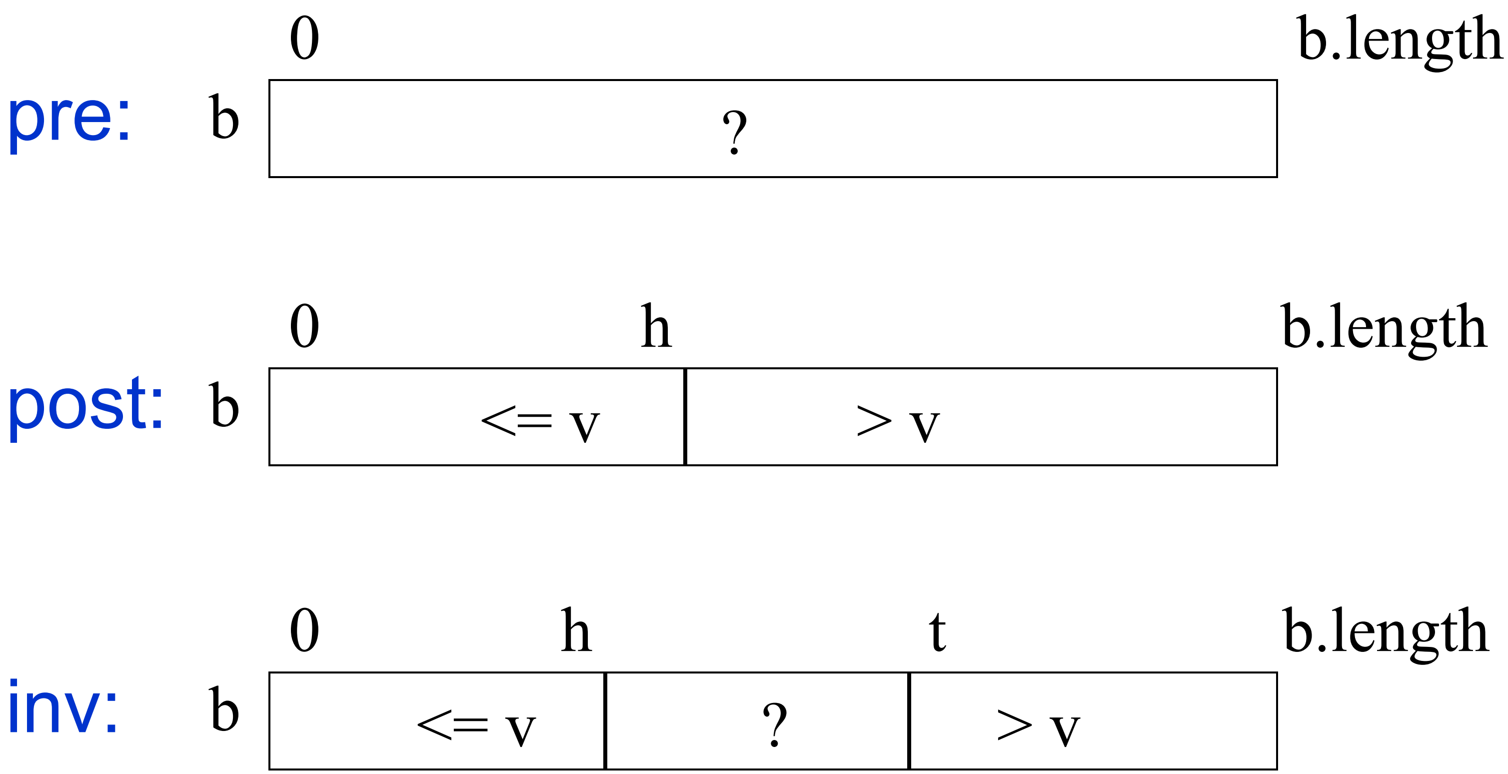



```
h= -1;  t= b.length;
while ( h != t-1 ) {
    int  e= (h+t)/2;
    if (b[e] <= v)  h= e;
    else  t= e;
}
```

If b[e] > v, then so is every value to its right, since the array is sorted. Therefore, t= e; keeps the invariant true.

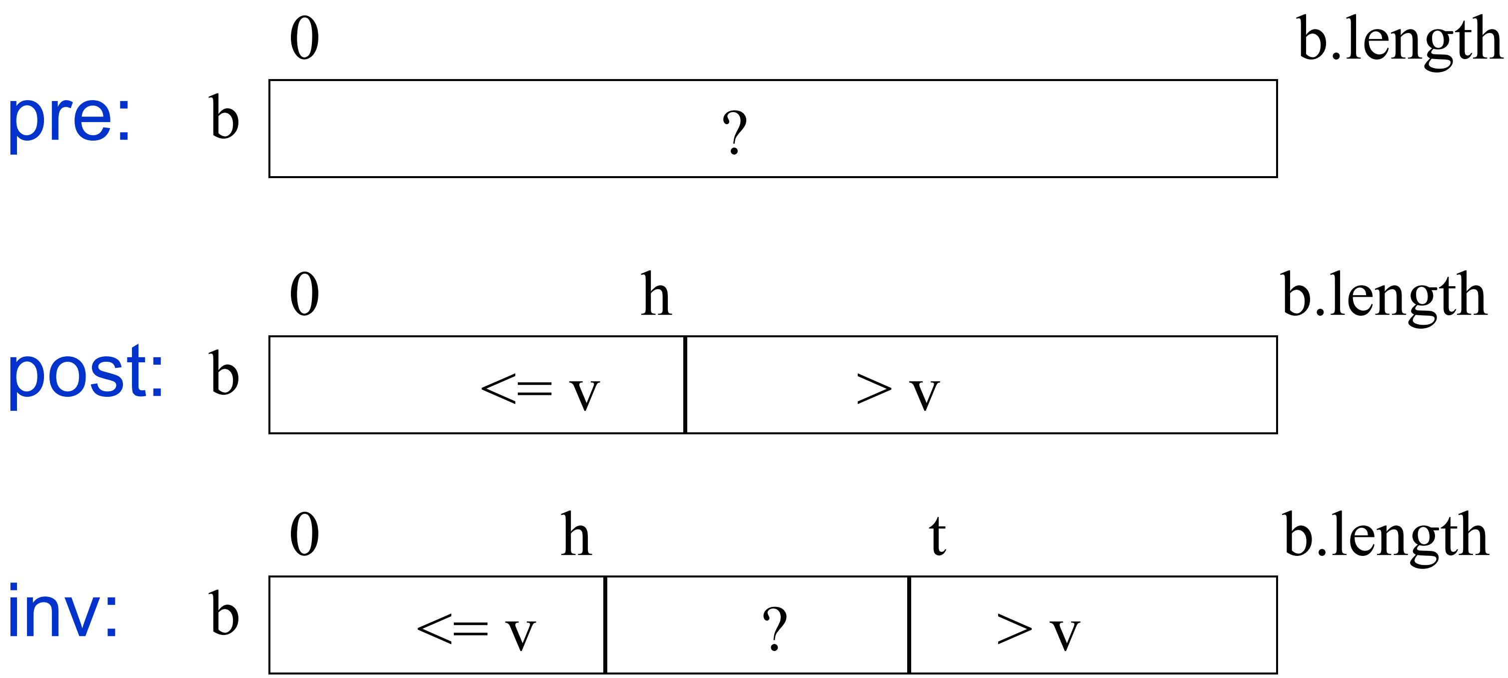
Loop invariants

We used the concept of a loop invariant in developing algorithms to reverse a linked list and do a binary search on a sorted array.



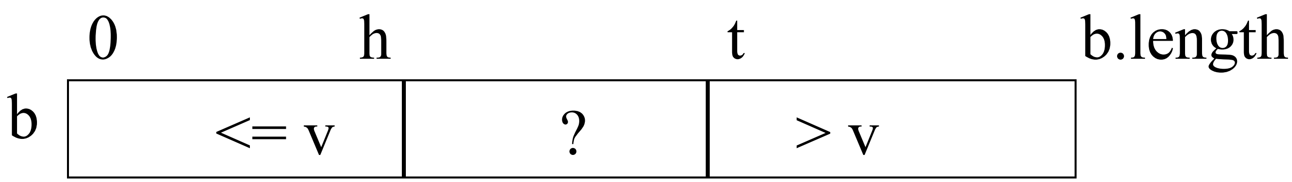
Loop invariant: Important part of every formal system for proving loops correct.

Extremely useful tool in *developing* a loop. Create (first draft of) invariant from pre- and post-conditions, then develop the parts of the loop from precondition, postcondition, invariant.



Loop invariant: Important part of every formal system for proving loops correct.

Invariant can be written in English, mathematics, diagrams, or mixtures of these. The important points are precision, clarity.

inv: 

inv: b[0..h] <= v < b[t..b.length-1]

inv: b[0..h] <= v < b[t..]

inv: everything in b[0..h] is at most v,
everything in b[t..] is greater than v

About notation b[h..k]. b[h..k] has k+1–h elements

h	h+1	h+2	h+3

[h..h+3] 4 elements

[h..h+2] 3 elements

[h..h+1] 2 elements

[h..h] 1 element

[h..h-1] How many elements?

Use the formula: 0!

Convention: The notation b[h..k] is used only when h <= k+1.
For example, b[0..-2] is not allowed.

When h = k+1, b[h..k] denotes the empty segment starting at b[h].

Developing loop from pre, post, inv: 4 loopy questions


```
// pre
init

// inv
while ( b ) {
    // inv && b
    Ensure inv remains true;
    progress
   // inv
}
// inv && ! b
// post
```

1. How does it start? What init makes invariant true?

2. When can it stop? Choose b so that inv && !b implies post

4. How do we make sure invariant is maintained?

3. How does body make progress toward termination?