SEARCHING, SORTING, AND ASYMPTOTIC COMPLEXITY

Lecture 13

CS2110 – Fall 2014

Prelim 1

- Tuesday, March 11. 5:30pm or 7:30pm.

- The review sheet is on the website,
- There will be a review session on Sunday 1-3.
- If you have a conflict, meaning you cannot take it at 5:30 or at 7:30, they contact me (or Maria Witlox) with your issue.

Readings, Homework

- Textbook: Chapter 4
- Homework:
  - Recall our discussion of linked lists from two weeks ago.
  - What is the <u>worst</u> case complexity for appending N items on a linked list? For testing to see if the list contains X? What would be the <u>best</u> case complexity for these operations?
  - If we were going to talk about O() complexity for a list, which of these makes more sense: worst, average or best-case complexity? Why?

What Makes a Good Algorithm?

- Suppose you have two possible algorithms or data structures that basically do the same thing; which is *better*?

- Well… what do we mean by *better*?
  - Faster?
  - Less space?
  - Easier to code?
  - Easier to maintain?
  - Required for homework?

- How do we measure time and space for an algorithm?

Sample Problem: Searching

- Determine if *sorted* array b contains integer v
- First solution: Linear Search (check each element)

```
/** return true iff v is in b */
static boolean find(int[] b, int v) {
   for (int i = 0; i < b.length; i++) {
      if (b[i] == v) return true;
   }
   return false;
}
```

Doesn't make use of fact that b is sorted.

```
static boolean find(int[] b, int v) {
   for (int x : b) {
      if (x == v) return true;
   }
   return false;
}
```

Sample Problem: Searching

Second solution: *Binary Search*

Still returning true iff v is in a

Keep true: all occurrences of v are in b[low..high]

```
static boolean find (int[] a, int v) {
    int low= 0;
    int high= a.length - 1;
    while (low <= high) {
        int mid = (low + high)/2;
        if (a[mid] == v) return true;
        if (a[mid] < v)
            low= mid + 1;
        else  high= mid - 1;
    }
    return false;
}
```

Linear Search vs Binary Search

Which one is better?

- Linear: easier to program
- Binary: faster… isn't it?

How do we measure speed?

- Experiment?
- Proof?
- What inputs do we use?

- Simplifying assumption #1: Use *size* of input rather than input itself
  - For sample search problem, input size is n where n is array size

- Simplifying assumption #2: Count number of "*basic steps*" rather than computing exact times

One Basic Step = One Time Unit

Basic step:

- Input/output of scalar value
- Access value of scalar variable, array element, or object field
- assign to variable, array element, or object field
- do one arithmetic or logical operation
- method invocation (not counting arg evaluation and execution of method body)

- For conditional: number of basic steps on branch that is executed
- For loop: (number of basic steps in loop body) * (number of iterations)
- For method: number of basic steps in method body (include steps needed to prepare stack-frame)

Runtime vs Number of Basic Steps

Is this cheating?

- The runtime is not the same as number of basic steps
- Time per basic step varies depending on computer, compiler, details of code…

Well … yes, in a way

- But the number of basic steps is *proportional* to the actual runtime

Which is better?

- n or n^2 time?
- 100 n or n^2 time?
- 10,000 n or n^2 time?

As n gets large, multiplicative constants become less important

Simplifying assumption #3: Ignore multiplicative constants

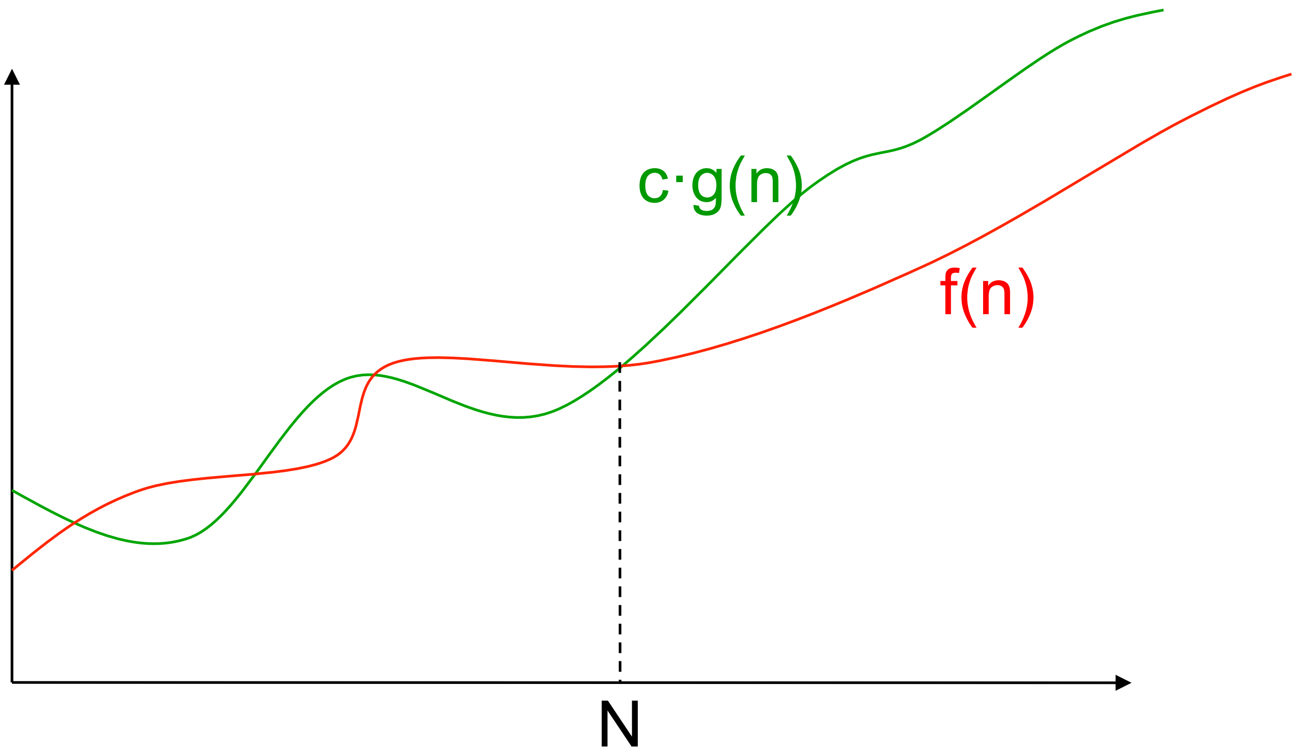
Using Big-O to Hide Constants

- We say f(n) is *order of* g(n) if f(n) is bounded by a constant times g(n)
- Notation: f(n) is O(g(n))
- Roughly, f(n) is O(g(n)) means that f(n) grows like g(n) or slower, to within a constant factor
- "Constant" means fixed and independent of n

- Example: $(n^2 + n)$ is $O(n^2)$
- We know $n \leq n^2$ for $n \geq 1$
- So $n^2 + n \leq 2\,n^2$ for $n \geq 1$
- So by definition, $n^2 + n$ is $O(n^2)$ for c=2 and N=1

Formal definition: f(n) is O(g(n)) if there exist constants c and N such that for all $n \geq N$, $f(n) \leq c \cdot g(n)$

A Graphical View



To prove that f(n) is O(g(n)):

- Find N and c such that $f(n) \leq c\, g(n)$ for all n > N
- Pair (c, N) is a *witness pair* for proving that f(n) is O(g(n))

Big-O Examples

Claim: $100\ n + \log n$ is $O(n)$

We know $\log n \leq n$ for $n \geq 1$

So $100\ n + \log n \leq 101\ n$

for $n \geq 1$

So by definition,

$100\ n + \log n$ is $O(n)$

for $c = 101$ and $N = 1$

Claim: $\log_B n$ is $O(\log_A n)$

since

$\log_B n = (\log_B A)(\log_A n)$

Question: Which grows faster: n or $\log n$?

Big-O Examples

Let $f(n) = 3n^2 + 6n - 7$

- $f(n)$ is $O(n^2)$
- $f(n)$ is $O(n^3)$
- $f(n)$ is $O(n^4)$
- …

$g(n) = 4\ n \log n + 34\ n - 89$

- $g(n)$ is $O(n \log n)$
- $g(n)$ is $O(n^2)$

$h(n) = 20 \cdot 2^n + 40n$

$h(n)$ is $O(2^n)$

$a(n) = 34$

- $a(n)$ is $O(1)$

Only the *leading* term (the term that grows most rapidly) matters

Problem-Size Examples

- Consisider a computing device that can execute 1000 operations per second; how large a problem can we solve?

	1 second	1 minute	1 hour
n	1000	60,000	3,600,000
$n \log n$	140	4893	200,000
n^2	31	244	1897
$3n^2$	18	144	1096
n^3	10	39	153
2^n	9	15	21

Commonly Seen Time Bounds

$O(1)$	constant	excellent
$O(\log n)$	logarithmic	excellent
$O(n)$	linear	good
$O(n \log n)$	n log n	pretty good
$O(n^2)$	quadratic	OK
$O(n^3)$	cubic	maybe OK
$O(2^n)$	exponential	too slow

Worst-Case/Expected-Case Bounds

May be difficult to determine time bounds for all imaginable inputs of size n

Simplifying assumption #4:
Determine number of steps for either

- worst-case or
- expected-case or average case

- Worst-case
  - Determine how much time is needed for the *worst possible* input of size n

- Expected-case
  - Determine how much time is needed *on average* for all inputs of size n

Simplifying Assumptions

Use the size of the input rather than the input itself – n

Count the number of "basic steps" rather than computing exact time

Ignore multiplicative constants and small inputs (order-of, big-O)

Determine number of steps for either

- worst-case
- expected-case

These assumptions allow us to analyze algorithms effectively

Worst-Case Analysis of Searching

Linear Search

```
// return true iff v is in b
static bool find (int[] b, int v) {
   for (int x : b) {
      if (x == v) return true;
   }
   return false;
}
```

worst-case time: O(n)

Binary Search

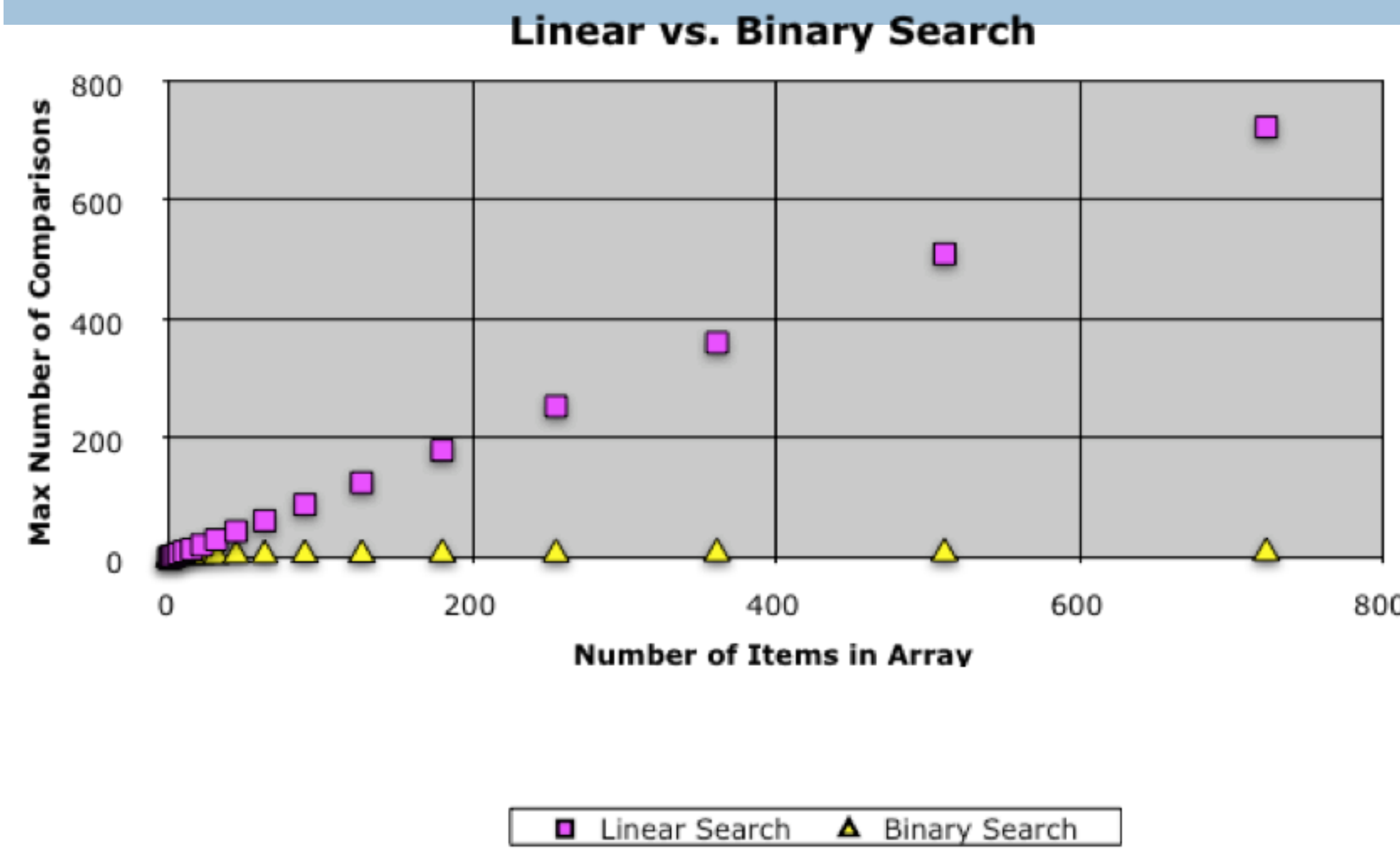
```
// Return h that satisfies
//      b[0..h] <= v < b[h+1..]
static bool bsearch(int[] b, int v {
   int h= -1;  int t= b.length;
   while ( h != t-1 ) {
       int  e= (h+t)/2;
       if (b[e] <= v)  h= e;
       else t= e;
   }
}
```

Always takes ~(log n+1) iterations.

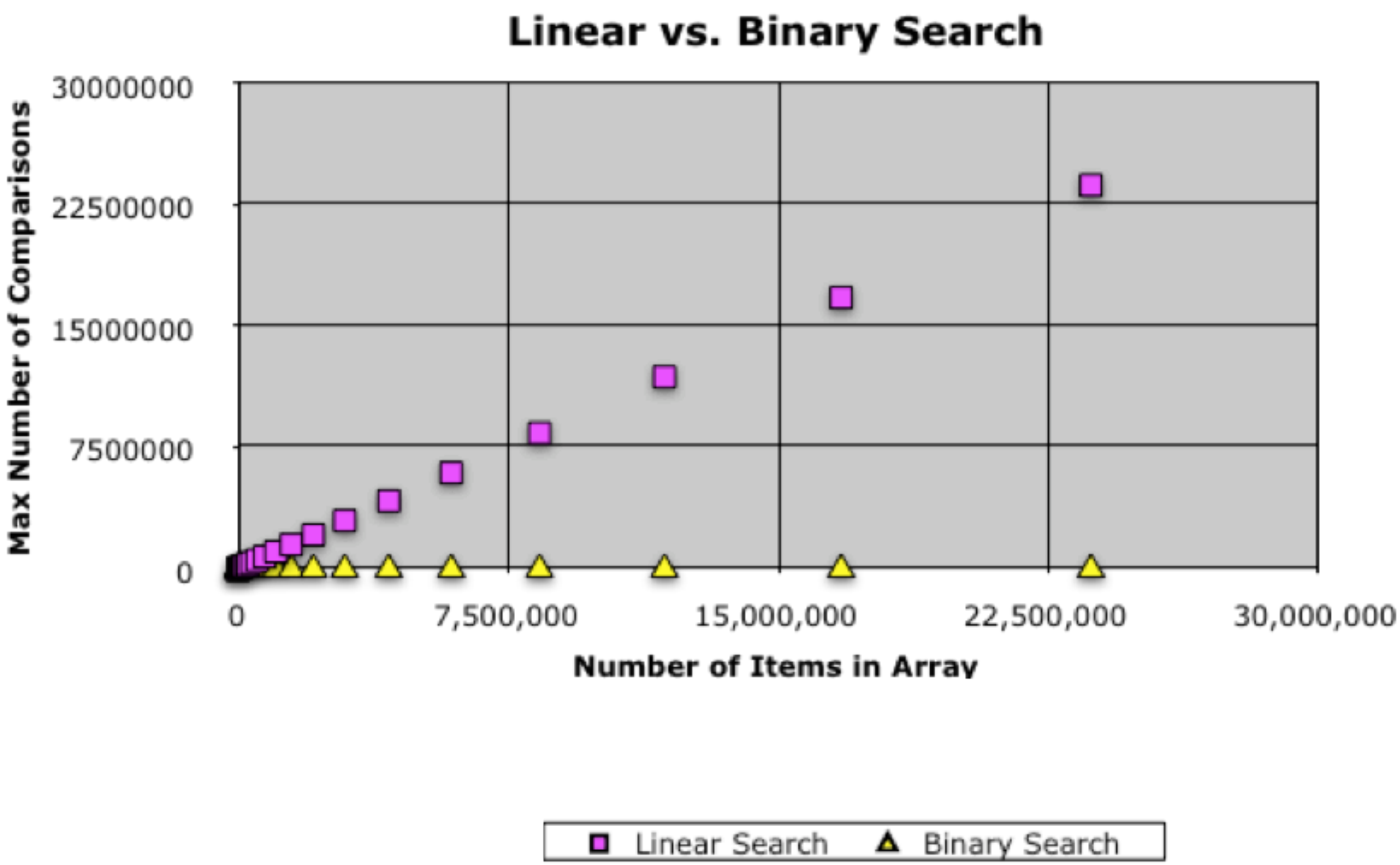
Worst-case and expected times:

O(log n)

Comparison of linear and binary search



Comparison of linear and binary search



Analysis of Matrix Multiplication

Multiply n-by-n matrices A and B:

Convention, matrix problems measured in terms of n, the number of rows, columns

- Input size is really $2n^2$, not n
- Worst-case time: $O(n^3)$
- Expected-case time: $O(n^3)$

```
for (i = 0; i < n; i++)
   for (j = 0; j < n; j++) {
      c[i][j] = 0;
      for (k = 0; k < n; k++)
         c[i][j] += a[i][k]*b[k][j];
   }
```

Remarks

Once you get the hang of this, you can quickly zero in on what is relevant for determining asymptotic complexity

- Example: you can usually ignore everything that is not in the innermost loop. Why?

One difficulty:

- Determining runtime for recursive programs
  Depends on the depth of recursion

Why Bother with Runtime Analysis?

Computers so fast that we can do whatever we want using simple algorithms and data structures, right?

Not really – data-structure/ algorithm improvements can be a *very big* win

Scenario:

- A runs in n^2 msec
- A' runs in $n^2/10$ msec
- B runs in 10 n log n msec

Problem of size $n=10^3$

- A: 10^3 sec ≈ 17 minutes
- A': 10^2 sec ≈ 1.7 minutes
- B: 10^2 sec ≈ 1.7 minutes

Problem of size $n=10^6$

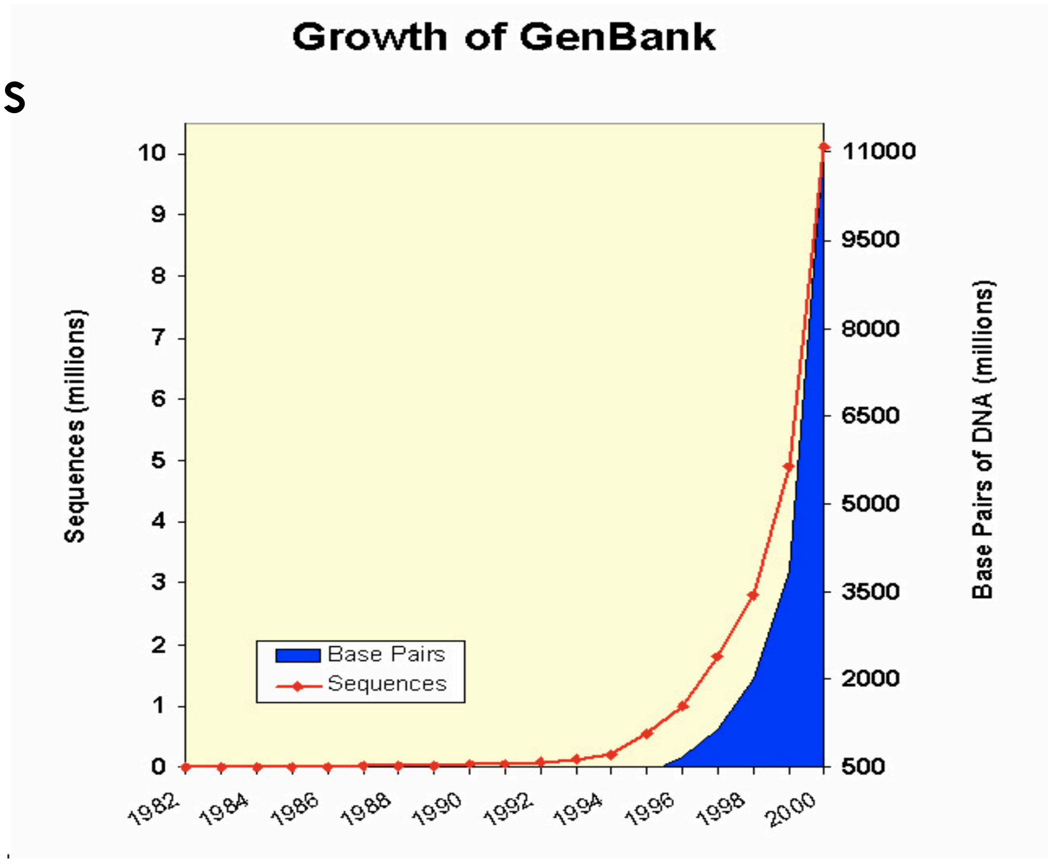
- A: 10^9 sec ≈ 30 years
- A': 10^8 sec ≈ 3 years
- B: $2 \cdot 10^5$ sec ≈ 2 days

1 day = 86,400 sec ≈ 10^5 sec

1,000 days ≈ 3 years

Algorithms for the Human Genome

Human genome
= 3.5 billion nucleotides
~ 1 Gb



@1 base-pair instruction/μsec

- $n^2 \rightarrow$ 388445 years
- $n \log n \rightarrow$ 30.824 hours
- $n \rightarrow$ 1 hour

Limitations of Runtime Analysis

Big-O can hide a very large constant

- Example: selection
- Example: small problems

The specific problem you want to solve may not be the worst case

- Example: Simplex method for linear programming

Your program may not be run often enough to make analysis worthwhile

- Example: one-shot vs. every day
- You may be analyzing and improving the wrong part of the program

- Very common situation
- Should use profiling tools

Summary

- Asymptotic complexity
  - Used to measure of time (or space) required by an algorithm
  - Measure of the *algorithm*, not the *problem*
- Searching a sorted array
  - Linear search: O(n) worst-case time
  - Binary search: O(log n) worst-case time
- Matrix operations:
  - Note: n = number-of-rows = number-of-columns
  - Matrix-vector product: $O(n^2)$ worst-case time
  - Matrix-matrix multiplication: $O(n^3)$ worst-case time
- More later with sorting and graph algorithms