1

SPANNING TREES

Lecture 20
CS2110 – Fall 2014

Spanning Trees

2

- Definitions
- Minimum spanning trees
- 3 greedy algorithms (incl. Kruskal's & Prim's)
- Concluding comments:
  - Greedy algorithms
  - Travelling salesman problem

Undirected Trees

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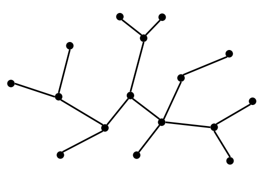
- An undirected graph is a *tree* if there is exactly one simple path between any pair of vertices



Facts About Trees

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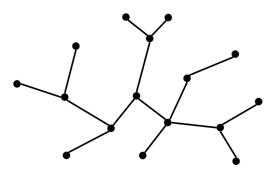
- |E| = |V| – 1
- connected
- no cycles

In fact, any two of these properties imply the third, and imply that the graph is a tree



Spanning Trees

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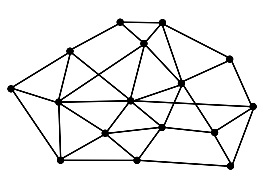
A *spanning tree* of a connected undirected graph (V,E) is a subgraph (V,E') that is a tree



Spanning Trees

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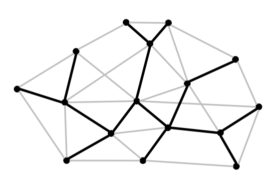
A *spanning tree* of a connected undirected graph (V,E) is a subgraph (V,E') that is a tree

- Same set of vertices V
- E' ⊆ E
- (V,E') is a tree



Spanning Trees: Examples

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http://mathworld.wolfram.com/SpanningTree.html

Finding a Spanning Tree

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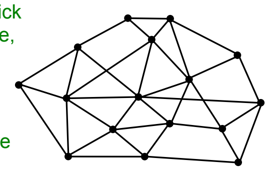
A subtractive method

- Start with the whole graph – it is connected
- If there is a cycle, pick an edge on the cycle, throw it out – the graph is still connected (why?)
- Repeat until no more cycles



Finding a Spanning Tree

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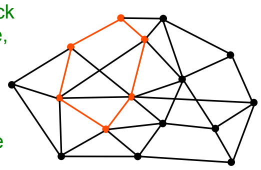

Finding a Spanning Tree

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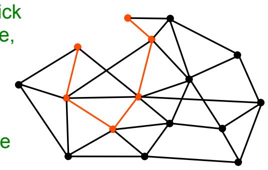
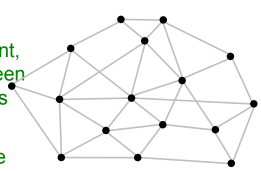


Finding a Spanning Tree

11

An additive method

- Start with no edges – there are no cycles
- If more than one connected component, insert an edge between them – still no cycles (why?)
- Repeat until only one component



Finding a Spanning Tree

12

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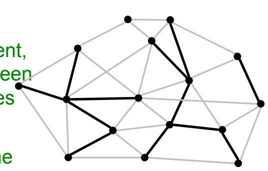

Finding a Spanning Tree

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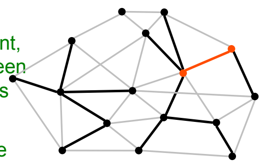

Finding a Spanning Tree

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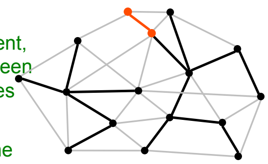

Finding a Spanning Tree

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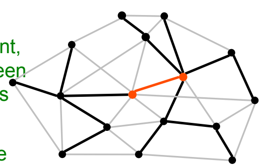

Finding a Spanning Tree

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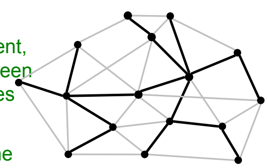

Minimum Spanning Trees

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- Suppose edges are weighted, and we want a spanning tree of *minimum cost* (sum of edge weights)
- Some graphs have exactly one minimum spanning tree. Others have multiple trees with the same cost, any of which is a minimum spanning tree

Minimum Spanning Trees

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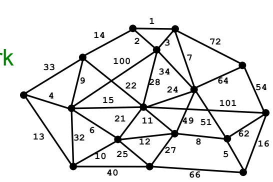
- Suppose edges are weighted, and we want a spanning tree of *minimum cost* (sum of edge weights)
- Useful in network routing & other applications
- For example, to stream a video



3 Greedy Algorithms

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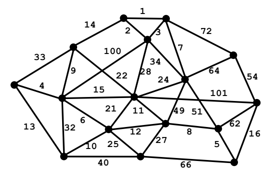
A. Find a max weight edge – if it is on a cycle, throw it out, otherwise keep it



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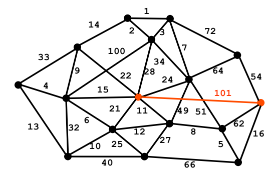

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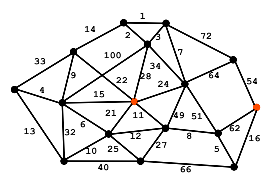

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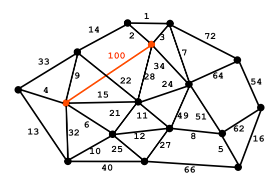

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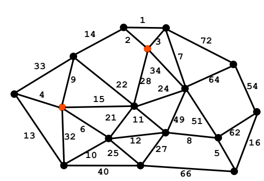

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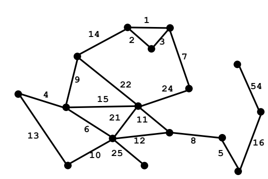

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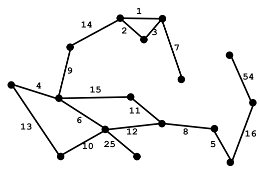

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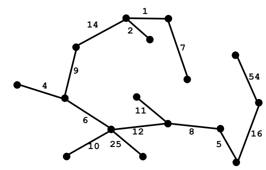

3 Greedy Algorithms

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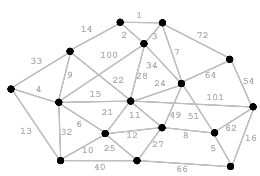
B. Find a min weight edge – if it forms a cycle with edges already taken, throw it out, otherwise keep it

Kruskal's algorithm



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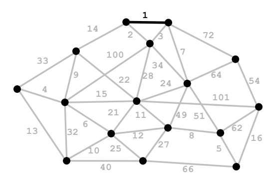

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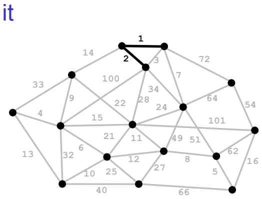

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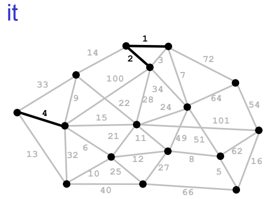

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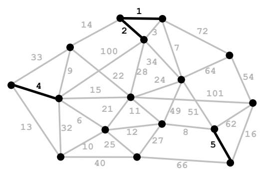

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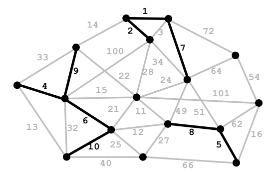

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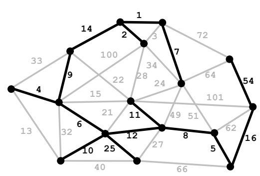

3 Greedy Algorithms

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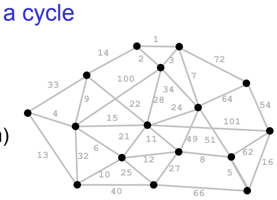
C. Start with any vertex, add min weight edge extending that connected component that does not form a cycle

Prim's algorithm
(reminiscent of Dijkstra's algorithm)



3 Greedy Algorithms

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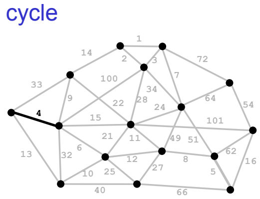

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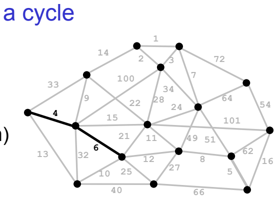

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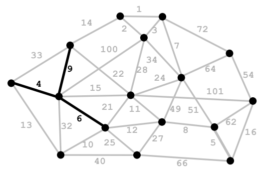

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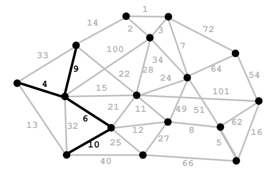

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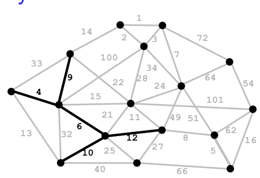

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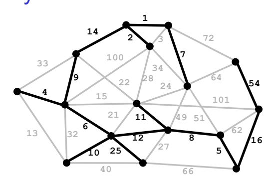

3 Greedy Algorithms

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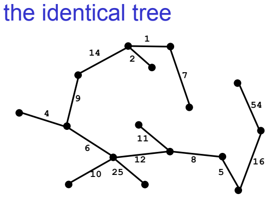
- When edge weights are all distinct, or if there is exactly one minimum spanning tree, the 3 algorithms all find the identical tree



Prim's Algorithm

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```
prim(s) {
   D[s] = 0; //start vertex
   D[i] = ∞ for all i≠s;
   while (some vertices are unmarked) {
      v = unmarked vertex with smallest D;
      mark v;
      for (each w adj to v)
         D[w] = min(D[w], c(v,w));
   }
}
```

- $O(n^2)$ for adj matrix
  - While-loop is executed n times
  - For-loop takes O(n) time

- O(m + n log n) for adj list
  - Use a PQ
  - Regular PQ produces time O(n + m log m)
  - Can improve to O(m + n log n) using a fancier heap

Application of MST

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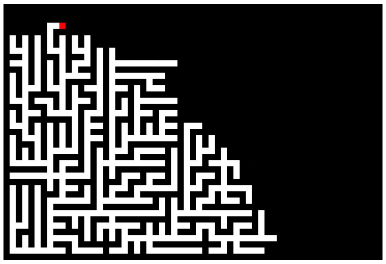
- Maze generation using Prim's algorithm



The generation of a maze using Prim's algorithm on a randomly weighted grid graph that is 30x20 in size.
http://en.wikipedia.org/wiki/File:MAZE_30x20_Prim.ogv

More complicated maze generation

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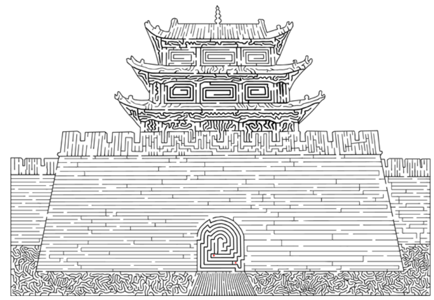

http://www.cgl.uwaterloo.ca/~csk/projects/mazes/

Greedy Algorithms

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- These are examples of Greedy Algorithms
- The Greedy Strategy is an algorithm design technique
  - Like Divide & Conquer
- Greedy algorithms are used to solve optimization problems
  - The goal is to find the *best* solution
- Works when the problem has the greedy-choice property
  - A global optimum can be reached by making locally optimum choices
- Example: Change Making Problem
  - Given an amount of money, find the smallest number of coins to make that amount
- Solution: Use a Greedy Algorithm
  - Give as many large coins as you can
- This greedy strategy produces the optimum number of coins for the US coin system
- Different money system ⇒ greedy strategy may fail
  - Example: old UK system

Similar Code Structures

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```
while (some vertices are
          unmarked) {
   v = best unmarked vertex
   mark v;
   for (each w adj to v)
        update D[w];
}
```

- Breadth-first-search (bfs)
  - best: next in queue
  - update: D[w] = D[v]+1
- Dijkstra's algorithm
  - best: next in priority queue
  - update: D[w] = min(D[w], D[v]+c(v,w))
- Prim's algorithm
  - best: next in priority queue
  - update: D[w] = min(D[w], c(v,w))

here c(v,w) is the v→w edge weight

Traveling Salesman Problem

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- Given a list of cities and the distances between each pair, what is the shortest route that visits each city exactly once and returns to the origin city?
  - The true TSP is very hard (NP complete)… for this we want the <u>perfect</u> answer in all cases, and can't revisit.
  - Most TSP algorithms start with a spanning tree, then "evolve" it into a TSP solution. Wikipedia has a lot of information about packages you can download…