Friday is Halloween.
Why did I receive a

Christmas card on
Halloween?

SHORTEST PATHS

Lecture 19

CS2110 – Fall2014

Readings?

- Read chapter 28

Shortest Paths in Graphs

Problem of finding shortest (min-cost) path in a graph occurs often

- Find shortest route between Ithaca and West Lafayette, IN
- Result depends on notion of cost
  - Least mileage… or least time… or cheapest
  - Perhaps, expends the least power in the butterfly while flying fastest
  - Many "costs" can be represented as edge weights

Every time you use googlemaps to find directions you are using a shortest-path algorithm

Dijkstra's shortest-path algorithm

Edsger Dijkstra, in an interview in 2010 (*CACM*):

… the algorithm for the shortest path, which I designed in about 20 minutes. One morning I was shopping in Amsterdam with my young fiance, and tired, we sat down on the cafe terrace to drink a cup of coffee, and I was just thinking about whether I could do this, and I then designed the algorithm for the shortest path. As I said, it was a 20-minute invention. [Took place in 1956]

Dijkstra, E.W. A note on two problems in Connexion with graphs. *Numerische Mathematik* 1, 269–271 (1959).

Visit http://www.dijkstrascry.com for all sorts of information on Dijkstra and his contributions. As a historical record, this is a gold mine.

Dijkstra's shortest-path algorithm

Dijsktra describes the algorithm in English:

- When he designed it in 1956, most people were programming in assembly language!
- Only *one* high-level language: Fortran, developed by John Backus at IBM and not quite finished.

No theory of order-of-execution time —topic yet to be developed. In paper, Dijsktra says, "my solution is preferred to another one … "the amount of work to be done seems considerably less."

Dijkstra, E.W. A note on two problems in Connexion with graphs. *Numerische Mathematik* 1, 269–271 (1959).

1968 NATO Conference on Software Engineering, Garmisch, Germany

Term “software engineering” coined for this conference

1968 NATO Conference on Software Engineering, Garmisch, Germany

Marktoberdorf Summer School, Germany, 1998

(Each year, ~100 PhD students from around the world would get two weeks of lectures by CS faculty.

Dijkstra's shortest path algorithm

The n (> 0) nodes of a graph numbered 0..n-1.

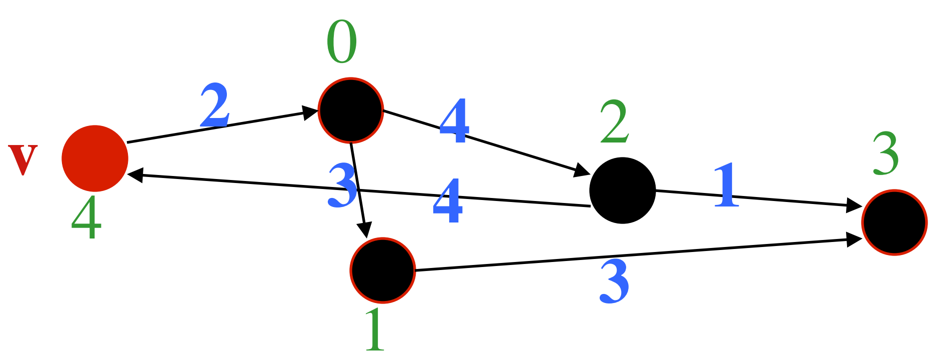
Each edge has a positive weight.

weight(v1, v2) is the weight of the edge from node v1 to v2.

Some node v be selected as the *start* node.

Calculate length of shortest path from v to each node.

Use an array L[0..n-1]: for **each** node w, store in L[w] the length of the shortest path from v to w.



L[0] = 2
L[1] = 5
L[2] = 6
L[3] = 7
L[4] = 0

Dijkstra's shortest path algorithm

Develop algorithm, not just present it.

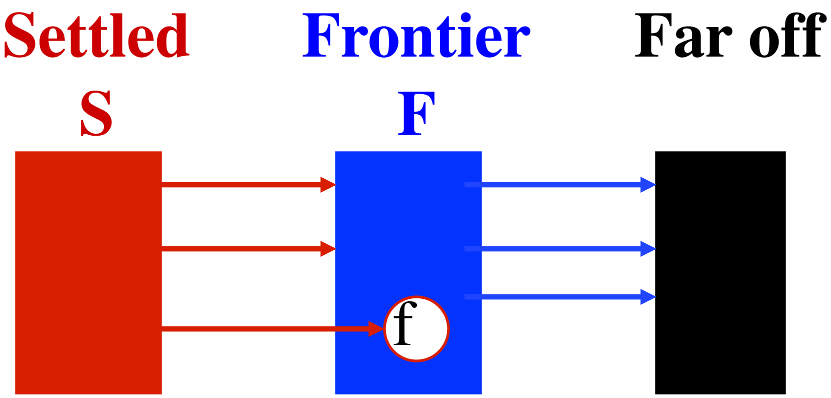
Need to show you the state of affairs —the relation among all variables— just before each node i is given its final value L[i].

This relation among the variables is an *invariant*, because it is always true.

Because each node i (except the first) is given its final value L[i] during an iteration of a loop, the *invariant* is called a *loop invariant*.

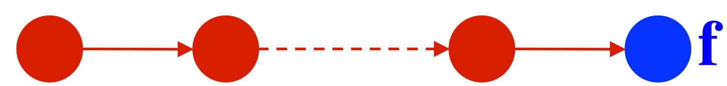
L[0] = 2
L[1] = 5
L[2] = 6
L[3] = 7
L[4] = 0

The loop invariant

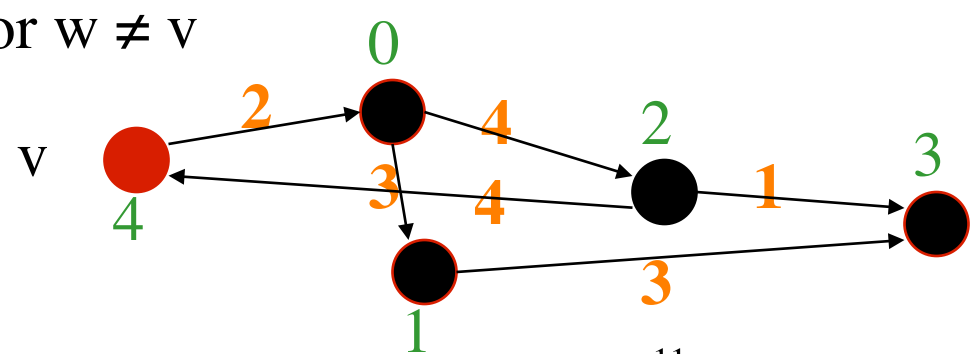


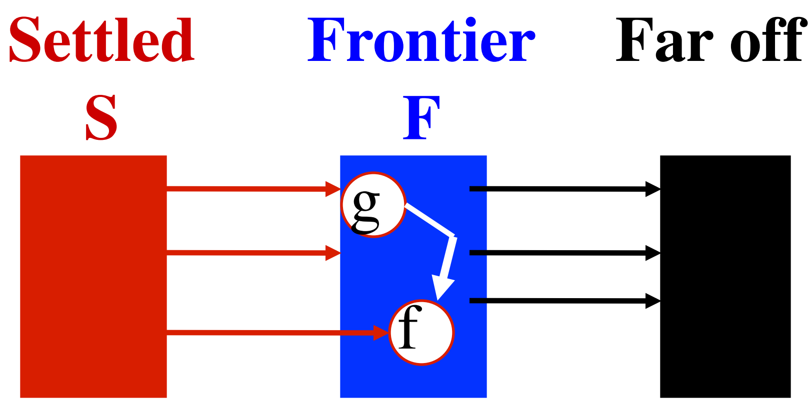
(edges leaving the black set and edges from the blue to the red set are not shown)

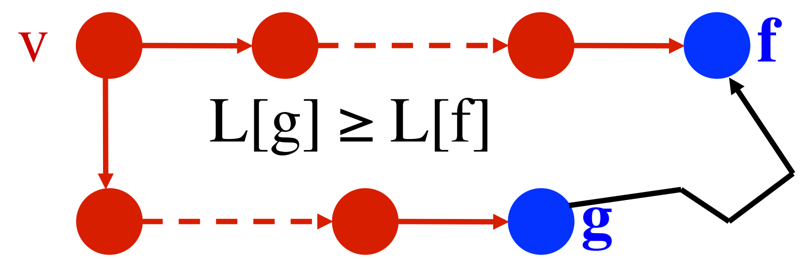
1. For a Settled node s, L[s] is length of shortest v → s path.
2. All edges leaving S go to F.
3. For a Frontier node f, L[f] is length of shortest v → f path using only red nodes (except for f)



4. For a Far-off node b, L[b] = ∞
5. L[v] = 0, L[w] > 0 for w ≠ v







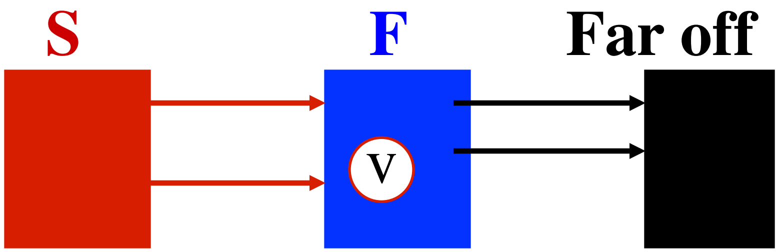
1. **For a Settled node s**, **L[s]** is length of shortest v → r path.
2. **All edges leaving S go to F.**
3. **For a Frontier node f, L[f]** is length of shortest v → f path using only Settled nodes (except for f).
4. **For a Far-off node b, L[b] = ∞. 5.** L[v] = 0, L[w] > 0 for w ≠ v

Theorem. For a node **f** in **F** with minimum L value (over nodes in **F**), **L[f]** is the length of the shortest path from **v** to **f**.

Case 1: **v** is in **S**.

Case 2: **v** is in **F**. Note that L[v] is 0; it has minimum L value

The algorithm



For all w, L[w]= ∞; L[v]= 0;
F= { v }; S= { };

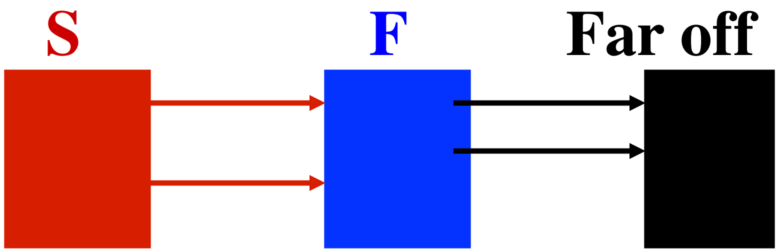
1. **For s, L[s]** is length of shortest v→ s path.
2. **Edges leaving S go to F**.
3. **For f, L[f]** is length of shortest v → f path using red nodes (except for f).
4. **For b in Far off, L[b] = ∞**
5. L[v] = 0, L[w] > 0 for w ≠ v

Theorem: For a node **f** in **F** with min L value, L[f] is shortest path length

Loopy question 1:

How does the loop start? What is done to truthify the invariant?

The algorithm



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5. L[v] = 0, L[w] > 0 for w ≠ v

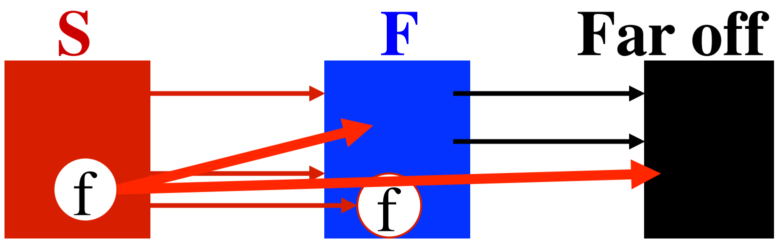
Theorem: For a node **f** in **F** with min L value, L[f] is shortest path length

For all w, L[w]= ∞;   L[v]= 0;
F= { v }; S= { };
while F ≠ {} {

}

Loopy question 2:
When does loop stop? When is array L completely calculated?

The algorithm



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4. **For b, L[b] = ∞**
5. L[v] = 0, L[w] > 0 for w ≠ v

Theorem: For a node **f** in **F** with min L value, L[f] is shortest path length

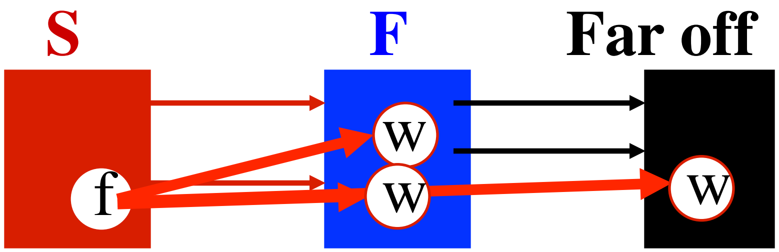
For all w, L[w]= ∞; L[v]= 0;
F= { v }; S= { };
while F ≠ {} {
    f= node in F with min L value;
    Remove f from F, add it to S;

}

Loopy question 3:

How is progress toward termination accomplished?

The algorithm



1. **For s, L[s]** is length of shortest v → s path.
2. **Edges leaving S go to F**.
3. **For f, L[f]** is length of shortest v → f path using red nodes (except for f).
4. **For b, L[b] = ∞**
5. L[v] = 0, L[w] > 0 for w ≠ v

Theorem: For a node **f** in **F** with min L value, L[f] is shortest path length

For all w, L[w]= ∞;   L[v]= 0;
F= { v }; S= { };
while   F ≠ {} {
  f= node in F with min L value;
  Remove f from F, add it to S;
  for each edge (f,w) {
    if (L[w] is ∞) add w to F;

    if (L[f] + weight (f,w) < L[w])
      L[w]= L[f] + weight(f,w);
  }
}

Algorithm is finished

Loopy question 4:

How is the invariant maintained?

About implementation

1. No need to implement **S**.
2. Implement **F** as a min-heap.
3. Instead of ∞, use
   Integer.MAX_VALUE.

For all w, L[w]= ∞; L[v]= 0;
F= { v }; ~~S= { };~~
while F ≠ {} {
  f= node in F with min L value;
  Remove f from F, ~~add it to S;~~
  for each edge (f,w) {
   ~~**if** (L[w] is ∞) add w to F;~~
   ~~**if** (L[f] + weight (f,w) < L[w])~~
   ~~L[w]= L[f] + weight(f,w);~~
  }
}

if (L[w] == Integer.MAX_VAL) {
  L[w]= L[f] + weight(f,w);
  add w to F;
} **else** L[w]= Math.min(L[w],
      L[f] + weight(f,w));

Execution time

n nodes, reachable from v. e ≥ n-1 edges
n–1 ≤ e ≤ n*n

```
For all w, L[w]= ∞;  L[v]= 0;              O(n)
F= { v };                                  O(1)
while F ≠ {} {                             O(n)
  f= node in F with min L value;           O(n)
  Remove f from F;                         O(n log n)
  for each edge (f,w) {                    O(n + e)
    if (L[w] == Integer.MAX_VAL) {         O(e)
        L[w]= L[f] + weight(f,w);          O(n-1)
        add w to F;                        O(n log n)
    }
    else L[w]=                             O((e-(n-1)) log n)
        Math.min(L[w], L[f] + weight(f,w));
  }
}
```

outer loop:
n iterations.
Condition evaluated n+1 times.

inner loop:
e iterations.
Condition evaluated n + e times.

Complete graph: $O(n^2 \log n)$. Sparse graph: $O(n \log n)$