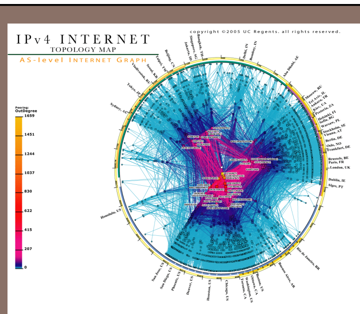


GRAPHS

Lecture 19
CS2110 – Fall 2013

Time to do A4, Recursion

2

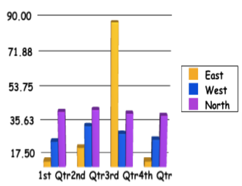
Histogram:
[00:02): 17
[02:04): 102
[04:06): 134
[06:08): 50
[08:10): 29
[10:12): 8
[12:14): 6
[14:16): 2
[16:18): 1
[18:30): 5

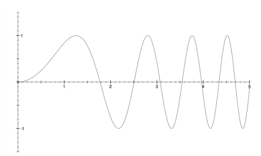
max: 28.45
av: 5.2
median: 4.5

If you took more than 6-7 hours for this assignment, you may have been wasting your time.

A certain amount of "floundering", just trying things, is good. But after a point, it just wastes time. Seek help if after an hour on one of the recursion problems you are stuck.

These are not Graphs

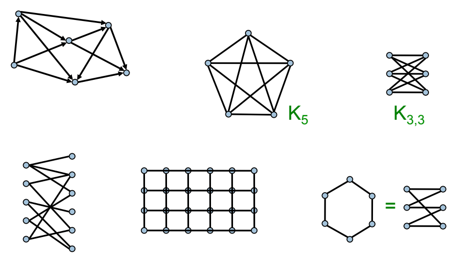




...not the kind we mean, anyway

These are Graphs

4



Applications of Graphs

5

- Communication networks
- The internet is a huge graph
- Routing and shortest path problems
- Commodity distribution (flow)
- Traffic control
- Resource allocation
- Geometric modeling
- ...

Graph Definitions

6

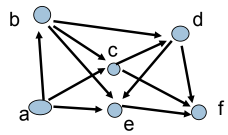
- A directed graph (or digraph) is a pair (V, E) where
  - V is a set
  - E is a set of ordered pairs (u,v) where $u,v \in V$
    - Sometimes require $u \neq v$ (i.e. no self-loops)

- An element of V is called a vertex (pl. vertices) or node
- An element of E is called an edge or arc

- |V| is the size of V, often denoted by n
- |E| is size of E, often denoted by m

Example Directed Graph (Digraph)

7



V = {a,b,c,d,e,f}
E = {(a,b), (a,c), (a,e), (b,c), (b,d), (b,e), (c,d),
(c,f), (d,e), (d,f), (e,f)}

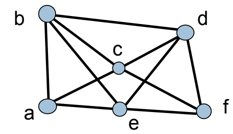
|V| = 6, |E| = 11

Example *Undirected* Graph

8

An *undirected graph* is just like a directed graph, except the edges are *unordered pairs* (*sets*) {u,v}

Example:

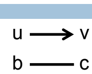


V = {a,b,c,d,e,f}
E = {{a,b}, {a,c}, {a,e}, {b,c}, {b,d}, {b,e}, {c,d}, {c,f},
{d,e}, {d,f}, {e,f}}

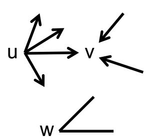
Some Graph Terminology

9

- u is the source , v is the sink of (u,v)
- u, v, b, c are the endpoints of (u,v) and (b, c)
- u, v are adjacent nodes. b, c are adjacent nodes



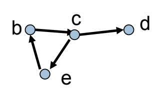
- outdegree of u in directed graph: number of edges for which u is source
- indegree of v in directed graph: number of edges for which v is sink
- degree of vertex w in undirected graph: number of edges of which w is an endpoint



outdegree of u: 4 indegree of v: 3 degree of w: 2

More Graph Terminology

- path: sequence of adjacent vertexes
- length of path: number of edges
- simple path: no vertex is repeated



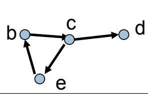
simple path of length 2: (b, c, d)

simple path of length 0: (b)

not a simple path: (b, c, e, b, c, d)

More Graph Terminology

- cycle: path that ends at its beginning
- simple cycle: only repeated vertex is its beginning/end
- acyclic graph: graph with no cycles
- dag: directed acyclic graph



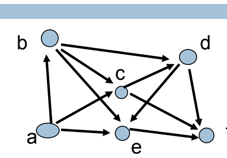
cycles: (b, c, e, b) (b, c, e, b, c, e, b)
simple cycle: (c, e, b, c)
graph shown is not a dag

Question: is (d) a cycle?

No. A cycle must have at least one edge

Is this a dag?

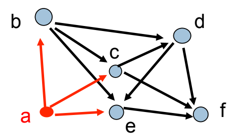
12



- Intuition: A dag has a vertex with indegree 0. Why?
- This idea leads to an algorithm:
  A digraph is a dag if and only if one can iteratively delete indegree-0 vertices until the graph disappears

Is this a dag?

13

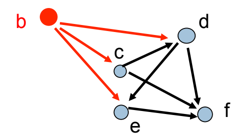


- Intuition: A dag has a vertex with indegree 0. Why?
- This idea leads to an algorithm:

  A digraph is a dag if and only if one can iteratively delete indegree-0 vertices until the graph disappears

Is this a dag?

14



- Intuition: A dag has a vertex with indegree 0. Why?
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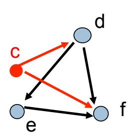
Is this a dag?

15

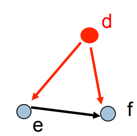


- Intuition: A dag has a vertex with indegree 0. Why?
- This idea leads to an algorithm:

  A digraph is a dag if and only if one can iteratively delete indegree-0 vertices until the graph disappears

Is this a dag?

16



- Intuition: A dag has a vertex with indegree 0. Why?
- This idea leads to an algorithm:

  A digraph is a dag if and only if one can iteratively delete indegree-0 vertices until the graph disappears

Is this a dag?

17

- Intuition: A dag has a vertex with indegree 0. Why?
- This idea leads to an algorithm:

  A digraph is a dag if and only if one can iteratively delete indegree-0 vertices until the graph disappears

Is this a dag?

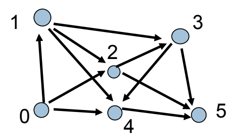
18

- Intuition: A dag has a vertex with indegree 0. Why?
- This idea leads to an algorithm:

  A digraph is a dag if and only if one can iteratively delete indegree-0 vertices until the graph disappears

Topological Sort

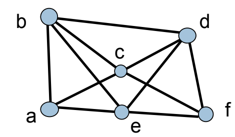
- We just computed a topological sort of the dag
  This is a numbering of the vertices such that all edges go from lower- to higher-numbered vertices



- Useful in job scheduling with precedence constraints

Graph Coloring

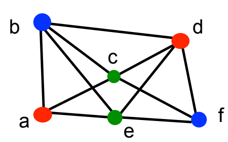
Coloring of an undirected graph: an assignment of a color to each node such that no two adjacent vertices get the same color



How many colors are needed to color this graph?

Graph Coloring

A coloring of an undirected graph: an assignment of a color to each node such that no two adjacent vertices get the same color

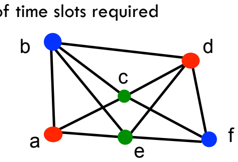


How many colors are needed to color this graph?

3

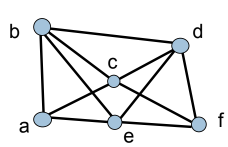
An Application of Coloring

- Vertices are jobs
- Edge (u,v) is present if jobs u and v each require access to the same shared resource, so they cannot execute simultaneously
- Colors are time slots to schedule the jobs
- Minimum number of colors needed to color the graph = minimum number of time slots required



Planarity

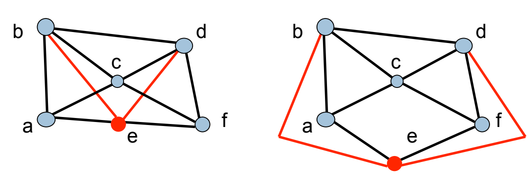
A graph is planar if it can be embedded in the plane with no edges crossing



Is this graph planar?

Planarity

A graph is planar if it can be embedded in the plane with no edges crossing

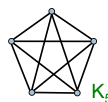


Is this graph planar? YES

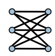
Detecting Planarity

25

Kuratowski's Theorem



K_5



$K_{3,3}$

A graph is planar if and only if it does not contain a copy of K_5 or $K_{3,3}$ (possibly with other nodes along the edges shown)

Detecting Planarity

26

Early 1970's John Hopcroft spent time at Stanford, talked to grad student Bob Tarjan (now at Princeton). Together, they developed a linear-time algorithm to test a graph for planarity. Significant achievement.

Won Turing Award

The Four-Color Theorem

27

Every planar graph is 4-colorable
(Appel & Haken, 1976)

Interesting history. "Proved" in about 1876 and published, but ten years later, a mistake was found. It took 90 more years for a proof to be found.

Countries are nodes; edge between them if they have a common boundary. You need 5 colors to color a map —water has to be blue!

The Four-Color Theorem

28

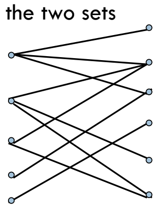
Every planar graph is 4-colorable
(Appel & Haken, 1976)

Proof rests on a lot of computation! A program checks thousands of "configurations", and if none are colorable, theorem holds.

Program written in assembly language. Recursive, contorted, to make it efficient. Gries found an error in it but a "safe kind": it might say a configuration was colorable when it wasn't.

Bipartite Graphs

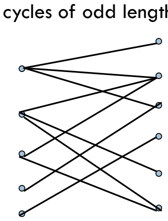
A directed or undirected graph is bipartite if the vertices can be partitioned into two sets such that all edges go between the two sets



Bipartite Graphs

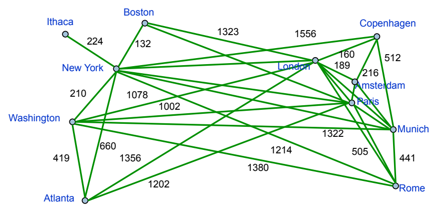
30

The following are equivalent
- G is bipartite
- G is 2-colorable
- G has no cycles of odd length



Traveling Salesperson

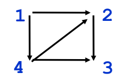
31



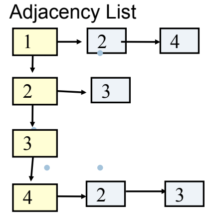
Find a path of minimum distance that visits every city

Representations of Graphs

32



Adjacency List



Adjacency Matrix

	1	2	3	4
1	0	1	0	1
2	0	0	1	0
3	0	0	0	0
4	0	1	1	0

Adjacency Matrix or Adjacency List?

33

n: number of vertices
m: number of edges
d(u): outdegree of u

Adjacency Matrix

Uses space O(n^2)

Can iterate over all edges in time O(n^2)

Can answer "Is there an edge from u to v?" in O(1) time

Better for dense graphs (lots of edges)

- Adjacency List
  - Uses space O(m+n)
  - Can iterate over all edges in time O(m+n)
  - Can answer "Is there an edge from u to v?" in O(d(u)) time
  - Better for sparse graphs (fewer edges)

Graph Algorithms

34

- Search
  - depth-first search
  - breadth-first search
- Shortest paths
  - Dijkstra's algorithm
- Minimum spanning trees
  - Prim's algorithm
  - Kruskal's algorithm

Depth-First Search

35

- Follow edges depth-first starting from an arbitrary vertex r, using a stack to remember where you came from
- When you encounter a vertex previously visited, or there are no outgoing edges, retreat and try another path
- Eventually visit all vertices reachable from r
- If there are still unvisited vertices, repeat
- O(m) time

Difficult to understand!
Let's write a recursive procedure

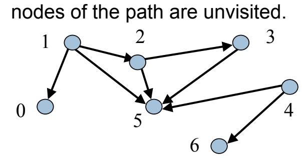
Depth-First Search

36

boolean[] visited;

node u is visited means: visited[u] is true
To visit u means to: set visited[u] to true

Node u is REACHABLE from node v if there is a path (u, …, v) in which all nodes of the path are unvisited.



Suppose all nodes are unvisited.

The nodes that are REACHABLE from node 1 are 1, 0, 2, 3, 5

The nodes that are REACHABLE from 4 are 4, 5, 6.

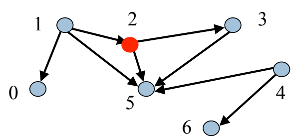
Depth-First Search

37

boolean[] visited;

To “visit” a node u: set visited[u] to true.

Node u is REACHABLE from node v if there is a path (u, …, v) in which all nodes of the path are unvisited.



Suppose 2 is already visited, others unvisited.

The nodes that are REACHABLE from node 1 are 1, 0, 5

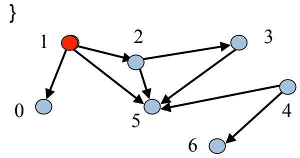
The nodes that are REACHABLE from 4 are 4, 5, 6.

Depth-First Search

38

```
/** Node u is unvisited. Visit all nodes
    that are REACHABLE from u. */
public static void dfs(int u) {
    visited[u]= true;

}
```



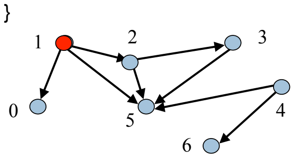
Let u be 1
The nodes that are REACHABLE from node 1 are 1, 0, 2, 3, 5

Depth-First Search

39

```
/** Node u is unvisited. Visit all nodes
    that are REACHABLE from u. */
public static void dfs(int u) {
    visited[u]= true;

    for each edge (u, v)
        if v is unvisited then dfs(v);
}
```



Let u be 1
The nodes to be visited are 0, 2, 3, 5

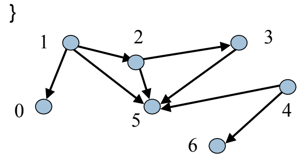
Have to do dfs on all unvisited neighbors of u

Depth-First Search

40

```
/** Node u is unvisited. Visit all nodes
    that are REACHABLE from u. */
public static void dfs(int u) {
    visited[u]= true;

    for each edge (u, v)
        if v is unvisited then dfs(v);
}
```



Let u be 1
The nodes to be visited are 0, 2, 3, 5

Suppose the for each loop visits neighbors in numerical order. Then dfs(1) visits the nodes in this order:
1, 0, 2, 3, 5

Depth-First Search

41

```
/** Node u is unvisited. Visit all nodes
    that are REACHABLE from u. */
public static void dfs(int u) {
    visited[u]= true;
    for each edge (u, v)
        if v is unvisited then dfs(v);
}
```

That’s all there is to the basic dfs. You may have to change it to fit a particular situation.

Example: There may be a different way (other than array visited) to know whether a node has been visited

Example: Instead of using recursion, use a loop and maintain the stack yourself.

Depth-First Search

42

```
/** Node u is unvisited. Visit all nodes
    that are REACHABLE from u. */
public static void dfs(int u) {
    visited[u]= true;
    for each edge (u, v)
        if v is unvisited then dfs(v);
}
```

That’s all there is to the basic dfs. You may have to change it to fit a particular situation.

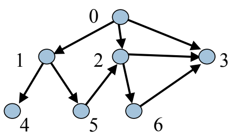
Example: In Bfly, there is no need for a parameter, because the current position of the Bfly takes the place of u. But then the specification must be changed, and probably the Bfly should be back at its original tile after each iteration of the loop. Make sure that is in the specification!

Breadth-First Search (BFS)

43

BFS visits all neighbors first before visiting their neighbors. It goes level by level.

Use a queue instead of a stack
- stack: last-in, first-out (LIFO)
- queue: first-in, first-out (FIFO)



dfs(0) visits in this order:
0, 1, 4, 5, 2, 3, 6

bfs(0) visits in this order:
0,1, 2, 3, 4, 5, 6

Breadth-first not good for the Bfly: too much flying back and forth

Summary

44

- We have seen an introduction to graphs and will return to this topic on Thursday
  - Definitions
  - Testing for a dag
  - Depth-first and breadth-first search