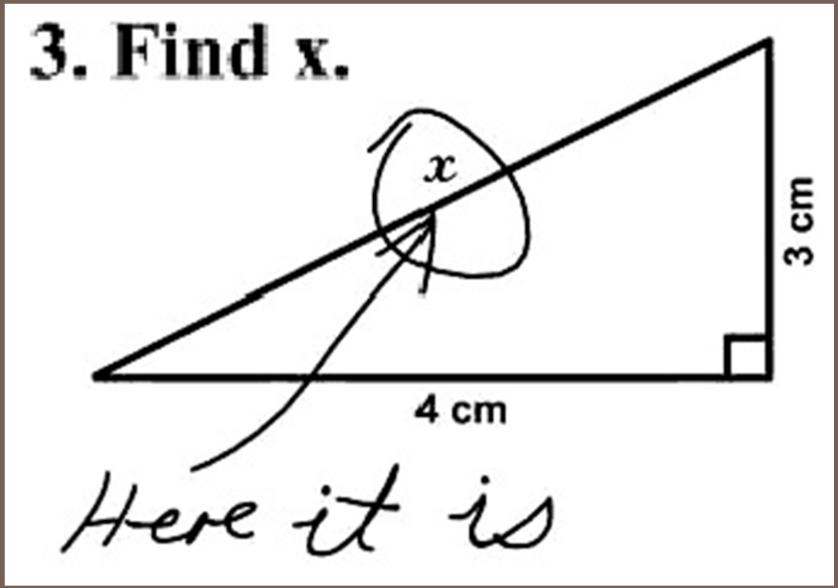



# SEARCHING, SORTING, AND ASYMPTOTIC COMPLEXITY

Lecture 13

CS2110 – Fall 2013

# Readings, Homework



- Textbook: Chapter 4
- Homework:
  - Recall our discussion of linked lists from two weeks ago.
  - What is the worst case complexity for appending N items on a linked list? For testing to see if the list contains X? What would be the best case complexity for these operations?
  - If we were going to talk about O() complexity for a list, which of these makes more sense: worst, average or best-case complexity? Why?

# What Makes a Good Algorithm?



- Suppose you have two possible algorithms or data structures that basically do the same thing; which is *better*?
- Well… what do we mean by *better*?
  - Faster?
  - Less space?
  - Easier to code?
  - Easier to maintain?
  - Required for homework?
- How do we measure time and space for an algorithm?

# Sample Problem: Searching



- Determine if *sorted* array a contains integer v
- First solution: Linear Search (check each element)

```
/** return true iff v is in a */
static boolean find(int[] a, int v) {
  for (int i = 0; i < a.length; i++) {
    if (a[i] == v) return true;
  }
  return false;
}
```

```
static boolean find(int[] a, int v) {
  for (int x : a) {
    if (x == v) return true;
  }
  return false;
}
```

# Sample Problem: Searching



## Second solution: *Binary Search*

Still returning true iff v is in a

Keep true: all occurrences of v are in b[low..high]

```
static boolean find (int[] a, int v) {
    int low= 0;
    int high= a.length - 1;
    while (low <= high) {
        int mid = (low + high)/2;
        if (a[mid] == v) return true;
        if (a[mid] < v)
            low= mid + 1;
        else  high= mid - 1;
    }
    return false;
}
```

# Linear Search vs Binary Search



Which one is better?

- Linear: easier to program
- Binary: faster… isn't it?

How do we measure speed?

- Experiment?
- Proof?
- What inputs do we use?

- Simplifying assumption #1: Use *size* of input rather than input itself
  - For sample search problem, input size is n+1 where n is array size

- Simplifying assumption #2: Count number of "*basic steps*" rather than computing exact times

# One Basic Step = One Time Unit



**Basic step:**

- Input/output of scalar value
- Access value of scalar variable, array element, or object field
- assign to variable, array element, or object field
- do one arithmetic or logical operation
- method invocation (not counting arg evaluation and execution of method body)

- For conditional: number of basic steps on branch that is executed
- For loop: (number of basic steps in loop body) * (number of iterations)
- For method: number of basic steps in method body (include steps needed to prepare stack-frame)

# Runtime vs Number of Basic Steps



## Is this cheating?

- The runtime is not the same as number of basic steps
- Time per basic step varies depending on computer, compiler, details of code…

## Well … yes, in a way

- But the number of basic steps is *proportional* to the actual runtime

Which is better?

- n or $n^2$ time?
- 100 n or $n^2$ time?
- 10,000 n or $n^2$ time?

As n gets large, multiplicative constants become less important

Simplifying assumption #3: Ignore multiplicative constants

# Using Big-O to Hide Constants

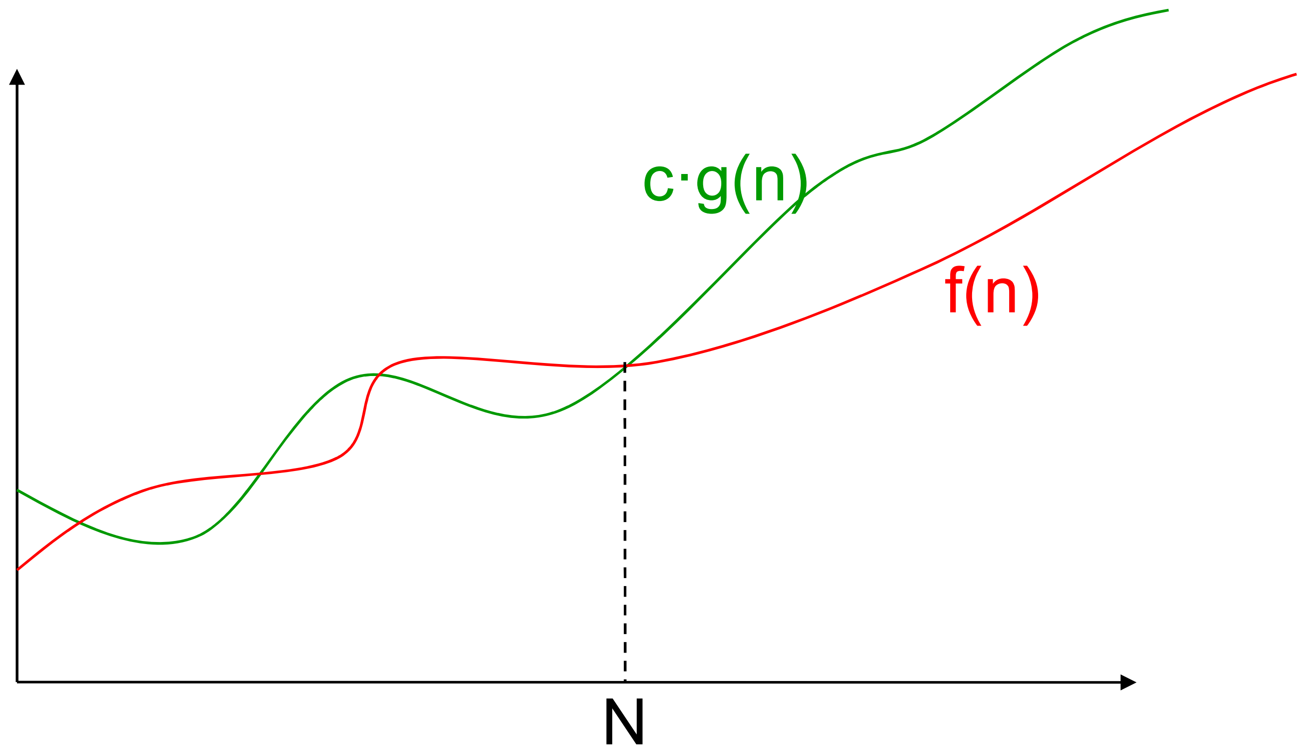


- We say f(n) is *order of* g(n) if f(n) is bounded by a constant times g(n)
- Notation: f(n) is O(g(n))
- Roughly, f(n) is O(g(n)) means that f(n) grows like g(n) or slower, to within a constant factor
- "Constant" means fixed and independent of n

- Example: $(n^2 + n)$ is $O(n^2)$
- We know $n \leq n^2$ for $n \geq 1$
- So $n^2 + n \leq 2\, n^2$ for $n \geq 1$
- So by definition, $n^2 + n$ is $O(n^2)$ for c=2 and N=1

Formal definition: f(n) is O(g(n)) if there exist constants c and N such that for all $n \geq N$, $f(n) \leq c \cdot g(n)$

# A Graphical View




To prove that f(n) is O(g(n)):

- Find N and c such that $f(n) \leq c\, g(n)$ for all $n > N$
- Pair (c, N) is a *witness pair* for proving that f(n) is O(g(n))

# Big-O Examples



Claim: $100\ n + \log n$ is $O(n)$

We know $\log n \leq n$ for $n \geq 1$

So $100\ n + \log n \leq 101\ n$ for $n \geq 1$

So by definition,

$100\ n + \log n$ is $O(n)$ for $c = 101$ and $N = 1$

Claim: $\log_B n$ is $O(\log_A n)$

since $\log_B n$ is $(\log_B A)(\log_A n)$

Question: Which grows faster: n or log n?

# Big-O Examples



Let $f(n) = 3n^2 + 6n - 7$

- $f(n)$ is $O(n^2)$
- $f(n)$ is $O(n^3)$
- $f(n)$ is $O(n^4)$
- …

$g(n) = 4\ n \log n + 34\ n - 89$

- $g(n)$ is $O(n \log n)$
- $g(n)$ is $O(n^2)$

$h(n) = 20 \cdot 2^n + 40n$

$h(n)$ is $O(2^n)$

$a(n) = 34$

- $a(n)$ is $O(1)$

Only the *leading* term (the term that grows most rapidly) matters

# Problem-Size Examples



- Consisider a computing device that can execute 1000 operations per second; how large a problem can we solve?

| | 1 second | 1 minute | 1 hour |
|---|---|---|---|
| n | 1000 | 60,000 | 3,600,000 |
| n log n | 140 | 4893 | 200,000 |
| $n^2$ | 31 | 244 | 1897 |
| $3n^2$ | 18 | 144 | 1096 |
| $n^3$ | 10 | 39 | 153 |
| $2^n$ | 9 | 15 | 21 |

# Commonly Seen Time Bounds



| | | |
|---|---|---|
| $O(1)$ | constant | excellent |
| $O(\log n)$ | logarithmic | excellent |
| $O(n)$ | linear | good |
| $O(n \log n)$ | n log n | pretty good |
| $O(n^2)$ | quadratic | OK |
| $O(n^3)$ | cubic | maybe OK |
| $O(2^n)$ | exponential | too slow |

# Worst-Case/Expected-Case Bounds



We can't possibly determine time bounds for all imaginable inputs of size n

Simplifying assumption #4:
Determine number of steps for either

- worst-case or
- expected-case

- Worst-case
  - Determine how much time is needed for the *worst possible* input of size n
- Expected-case
  - Determine how much time is needed *on average* for all inputs of size n

# Simplifying Assumptions



Use the size of the input rather than the input itself – n

Count the number of "basic steps" rather than computing exact time

Ignore multiplicative constants and small inputs (order-of, big-O)

Determine number of steps for either

- worst-case
- expected-case

These assumptions allow us to analyze algorithms effectively

# Worst-Case Analysis of Searching



## Linear Search

```
/** return true iff v is in a */
static bool find (int[] a, int v) {
    for (int x : a) {
        if (x == v) return true;
    }
   return false;
}
```

worst-case time: O(n)

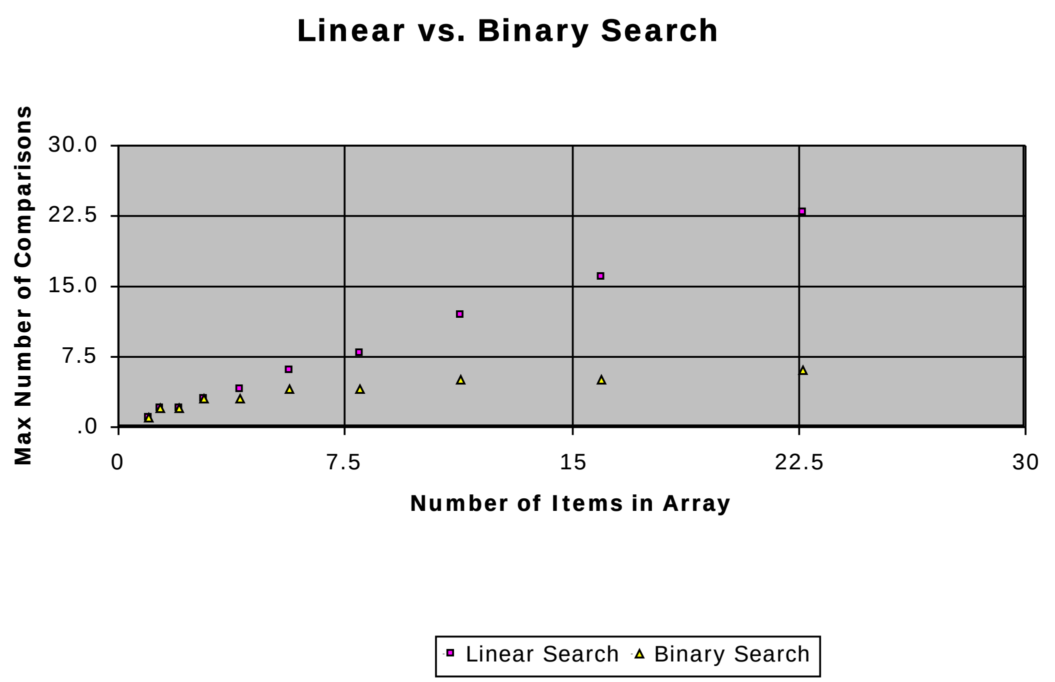
## Binary Search

```
static bool find (int[] a, int v) {
   int low= 0;
   int high= a.length – 1;
   while (low <= high) {
      int mid = (low + high)/2;
      if (a[mid] == v) return true;
      if (a[mid] < v)
            low= mid + 1;
      else   high= mid - 1;
     }
    return false;
 }
```

worst-case time: O(log n)

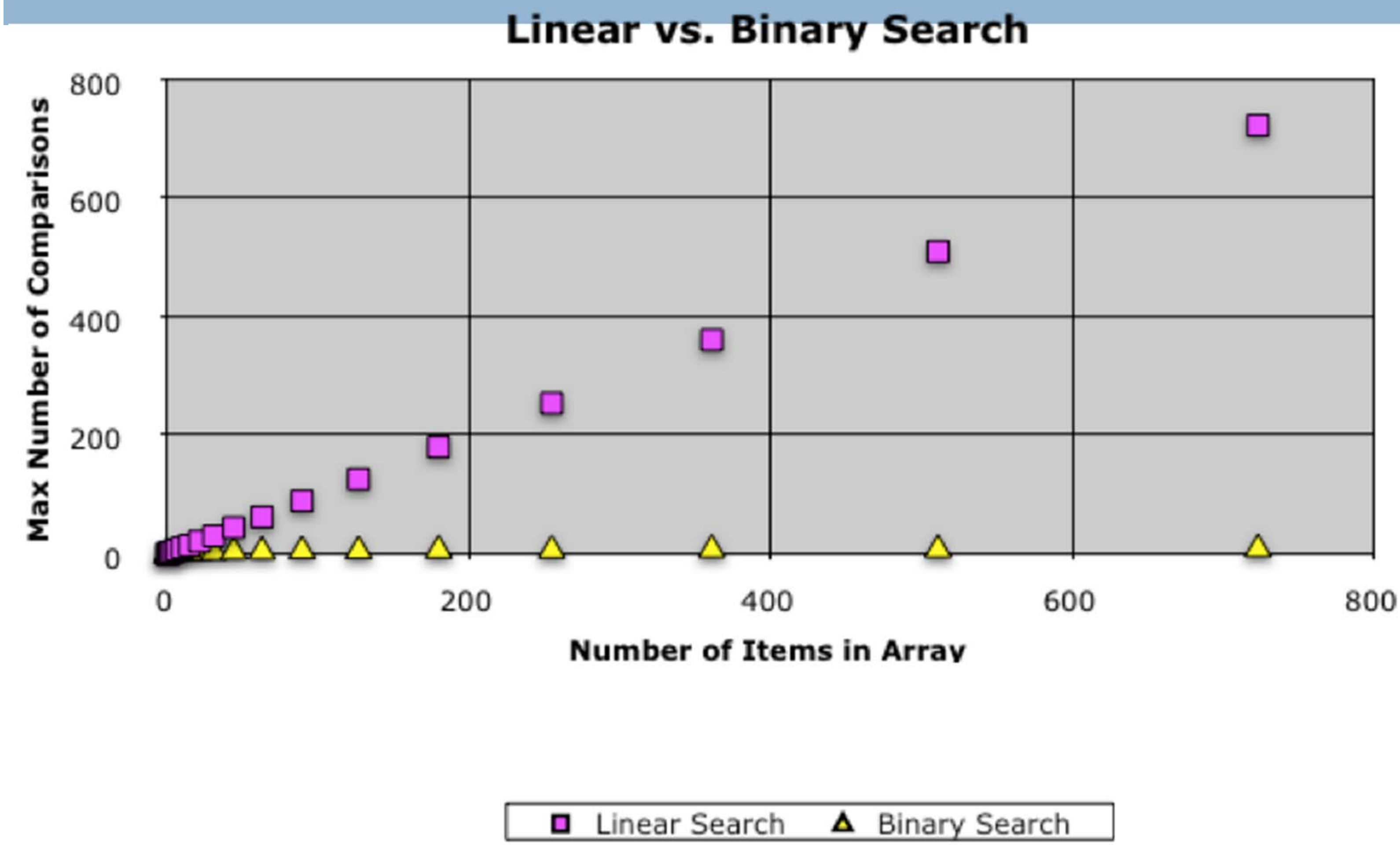
# Comparison of Algorithms





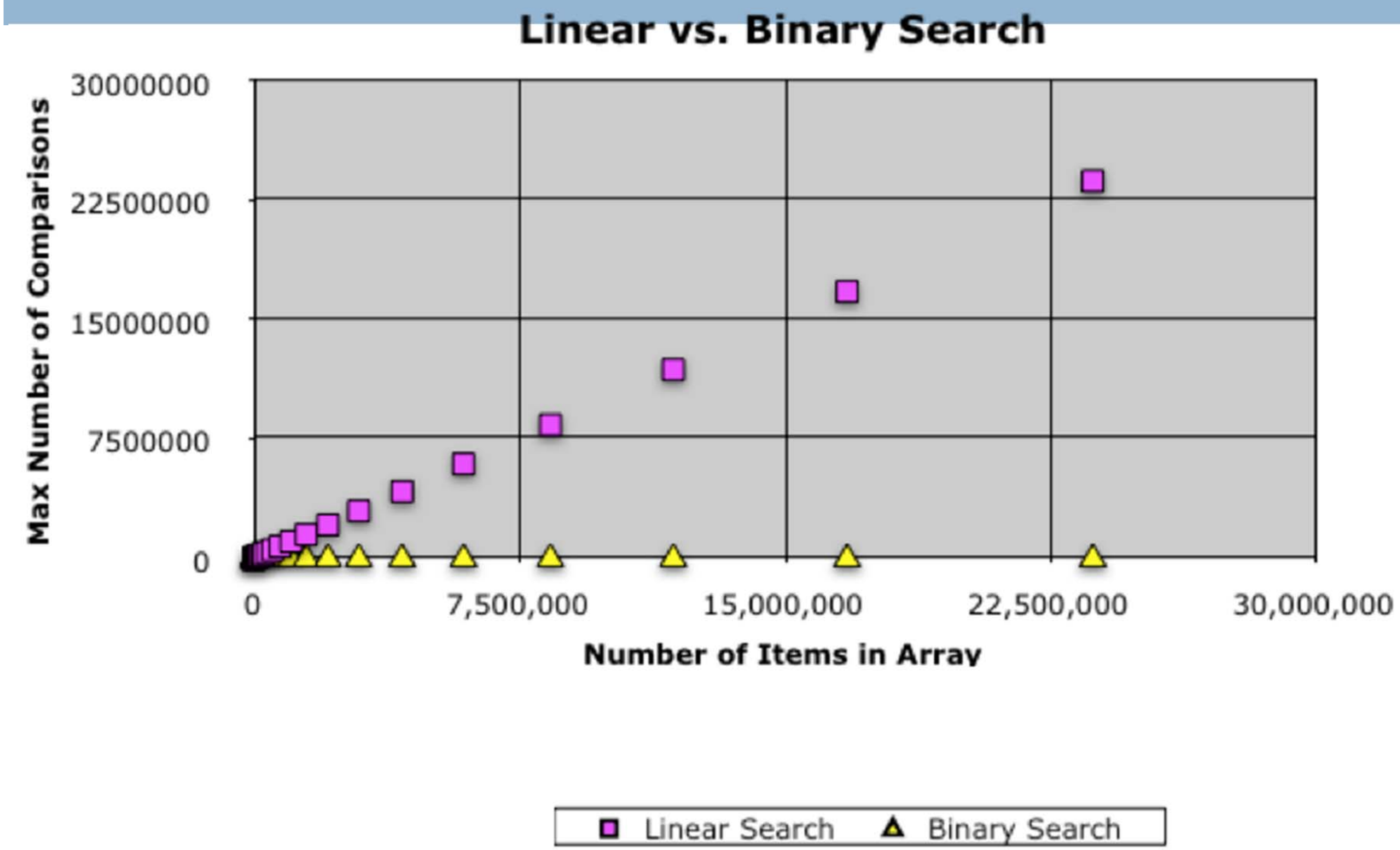
# Comparison of Algorithms





# Comparison of Algorithms





# Analysis of Matrix Multiplication



Multiply n-by-n matrices A and B:

Convention, matrix problems measured in terms of n, the number of rows, columns

- Input size is really $2n^2$, not n
- Worst-case time: $O(n^3)$
- Expected-case time: $O(n^3)$

```
for (i = 0; i < n; i++)
  for (j = 0; j < n; j++) {
    c[i][j] = 0;
    for (k = 0; k < n; k++)
      c[i][j] += a[i][k]*b[k][j];
  }
```

# Remarks



Once you get the hang of this, you can quickly zero in on what is relevant for determining asymptotic complexity

- Example: you can usually ignore everything that is not in the innermost loop. Why?

Main difficulty:

- Determining runtime for recursive programs

# Why Bother with Runtime Analysis?



Computers so fast that we can do whatever we want using simple algorithms and data structures, right?

Not really – data-structure/algorithm improvements can be a *very big* win

Scenario:

- A runs in $n^2$ msec
- A' runs in $n^2/10$ msec
- B runs in 10 n log n msec

Problem of size $n=10^3$

- A: $10^3$ sec ≈ 17 minutes
- A': $10^2$ sec ≈ 1.7 minutes
- B: $10^2$ sec ≈ 1.7 minutes

Problem of size $n=10^6$

- A: $10^9$ sec ≈ 30 years
- A': $10^8$ sec ≈ 3 years
- B: $2 \cdot 10^5$ sec ≈ 2 days

1 day = 86,400 sec ≈ $10^5$ sec

1,000 days ≈ 3 years

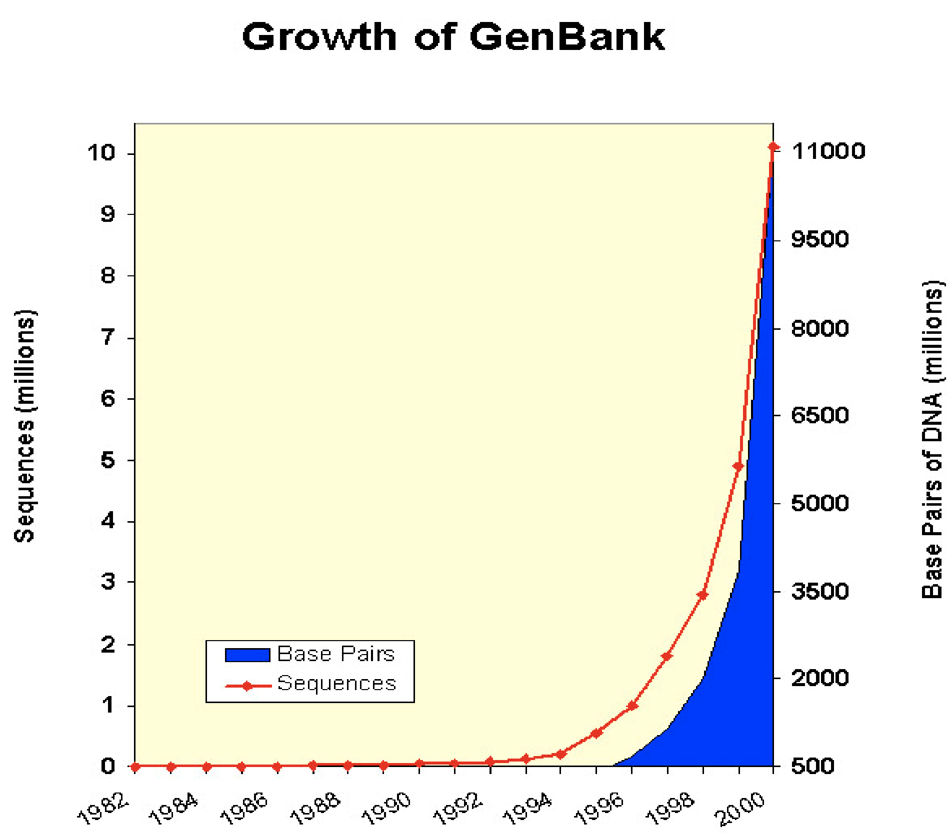
# Algorithms for the Human Genome



Human genome
= 3.5 billion nucleotides
~ 1 Gb

@1 base-pair instruction/$\mu$sec

- $n^2 \rightarrow$ 388445 years
- $n \log n \rightarrow$ 30.824 hours
- $n \rightarrow$ 1 hour



# Limitations of Runtime Analysis



Big-O can hide a very large constant

- Example: selection
- Example: small problems

The specific problem you want to solve may not be the worst case

- Example: Simplex method for linear programming

Your program may not be run often enough to make analysis worthwhile

- Example: one-shot vs. every day
- You may be analyzing and improving the wrong part of the program

- Very common situation
- Should use profiling tools

# Summary



- Asymptotic complexity
  - Used to measure of time (or space) required by an algorithm
  - Measure of the *algorithm*, not the *problem*
- Searching a sorted array
  - Linear search: O(n) worst-case time
  - Binary search: O(log n) worst-case time
- Matrix operations:
  - Note: n = number-of-rows = number-of-columns
  - Matrix-vector product: $O(n^2)$ worst-case time
  - Matrix-matrix multiplication: $O(n^3)$ worst-case time
- More later with sorting and graph algorithms