RECURSION

Lecture 6

CS2110 – Fall 2009

Recursion

- Arises in two forms in computer science

- We'll explore both
  - Recursion as a *mathematical* tool for defining a function in terms of its own value in a simpler case

  - Recursion as a *programming* tool. You've seen this previously but we'll take it to mind-bending extremes (by the end of the class it will seem easy!)

Recursion as a math technique

- Broadly, recursion is a powerful technique for specifying functions, sets, and programs

- Example recursively-defined functions and programs
  - factorial
  - combinations
  - exponentiation (raising to an integer power)

- Example recursively-defined sets
  - grammars
  - expressions
  - data structures (lists, trees, ...)

The Factorial Function (n!)

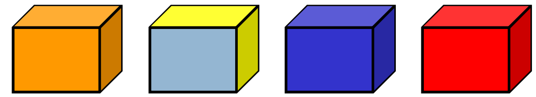
- Define n! = n·(n–1)·(n–2)···3·2·1 *read: "n factorial"*
  - E.g., 3! = 3·2·1 = 6
- By convention, 0! = 1
- The function int → int that gives n! on input n is called the factorial function

The Factorial Function (n!)

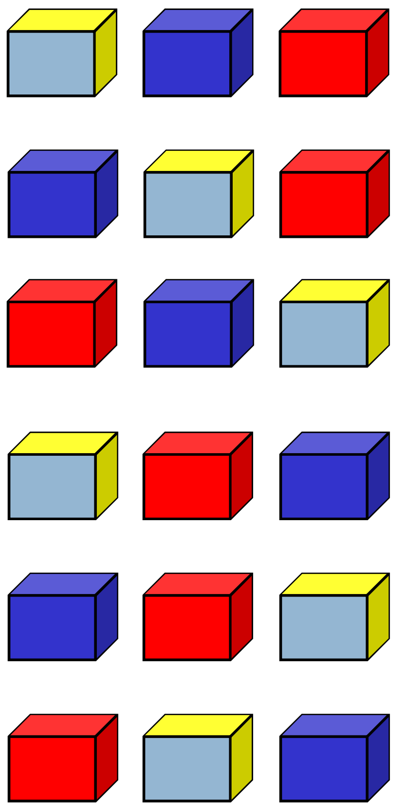
- n! is the number of permutations of n distinct objects
  - There is just one permutation of one object. 1! = 1
  - There are two permutations of two objects: 2! = 2

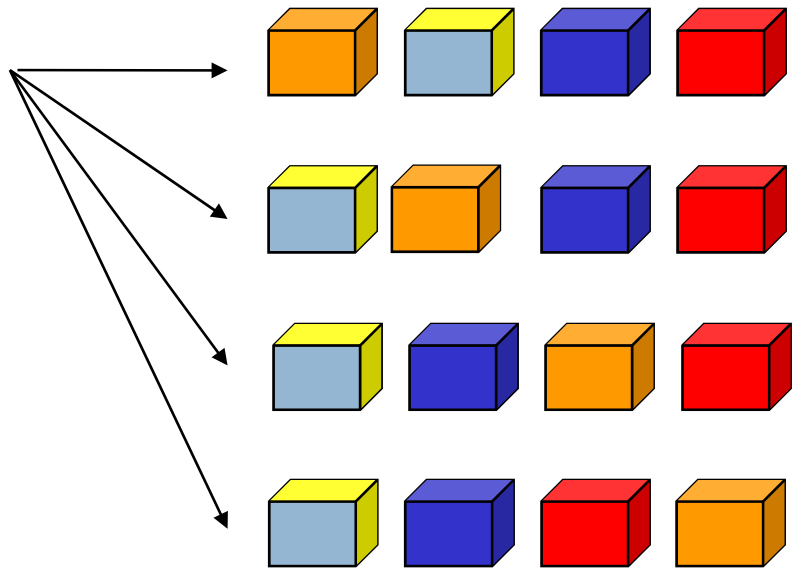
    1 2   2 1
  - There are six permutations of three objects: 3! = 6

    1 2 3   1 3 2   2 1 3   2 3 1   3 1 2   3 2 1
- If $n > 0$, $n! = n \cdot (n - 1)!$

Permutations of 

Permutations of
non-orange blocks





Each permutation of the three non-orange blocks gives four permutations when the orange block is included

- Total number = 4·3! = 4·6 = 24: 4!

Observation

- One way to think about the task of permuting the four colored blocks was to start by computing all permutations of three blocks, then finding all ways to add a fourth block
  - And this "explains" why the number of permutations turns out to be 4!
  - Can generalize to prove that the number of permutations of n blocks is n!

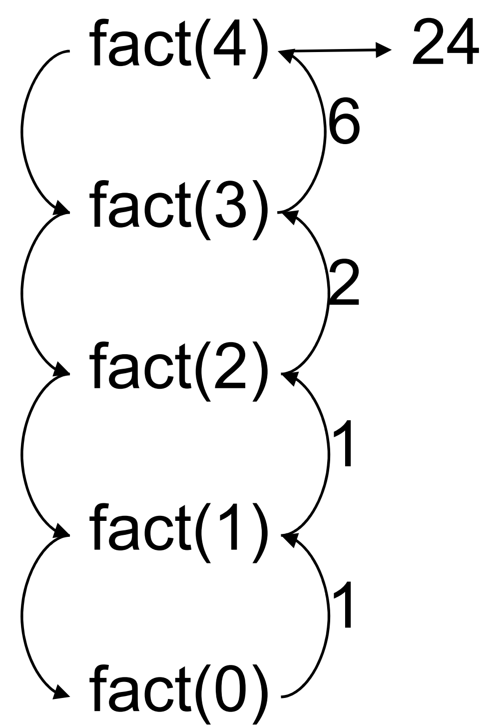
A Recursive Program

$0! = 1$

$n! = n \cdot (n-1)!, \quad n > 0$

```
static int fact(int n) {
    if (n == 0)
         return 1;
    else
         return n*fact(n-1);
}
```

Execution of fact(4)



General Approach to Writing Recursive Functions

1. Try to find a parameter, say n, such that the solution for n can be obtained by combining solutions to the *same problem using smaller values of n* (e.g., (n-1) in our factorial example)

2. Find *base case(s)* – small values of n for which you can just write down the solution (e.g., 0! = 1)

3. Verify that, for any valid value of n, applying the reduction of step 1 repeatedly will ultimately hit one of the base cases

A cautionary note

- Keep in mind that each instance of your recursive function has its own local variables
- Also, remember that “higher” instances are waiting while “lower” instances run

- Not such a good idea to touch global variables from within recursive functions
  - Legal… but a common source of errors
  - Must have a really clear mental picture of how recursion is performed to get this right!

The Fibonacci Function

- Mathematical definition:

  fib(0) = 0
  fib(1) = 1
  fib(n) = fib(n – 1) + fib(n – 2), n ≥ 2

  two base cases!

- Fibonacci sequence: 0, 1, 1, 2, 3, 5, 8, 13, …

```
static int fib(int n) {
   if (n == 0) return 0;
   else if (n == 1) return 1;
   else return fib(n-1) + fib(n-2);
}
```

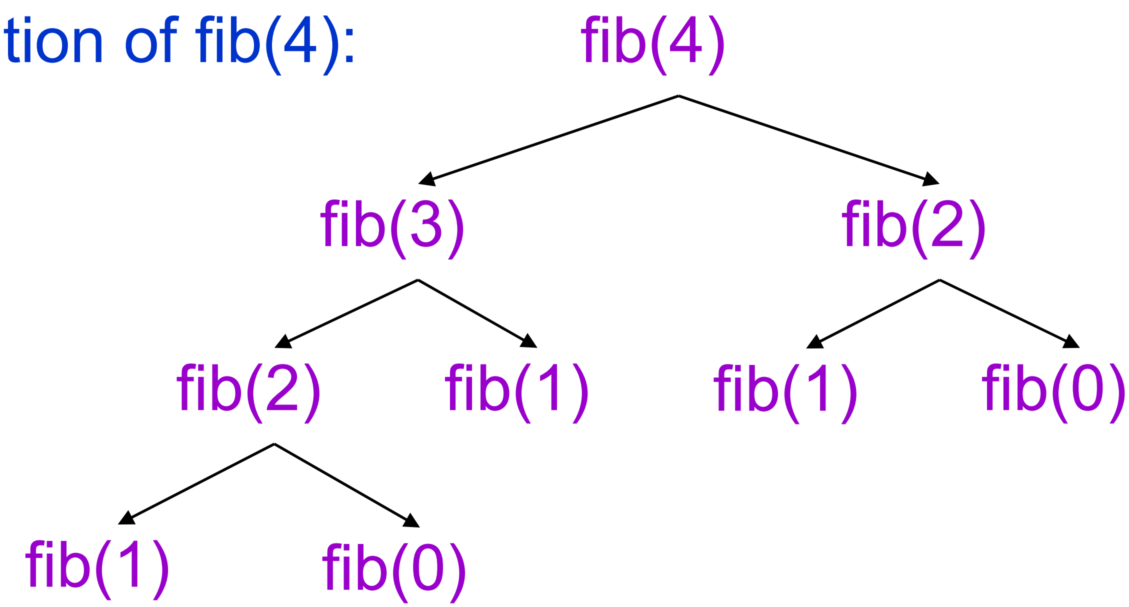
Fibonacci (Leonardo Pisano) 1170–1240?

Statue in Pisa, Italy
Giovanni Paganucci
1863

Recursive Execution

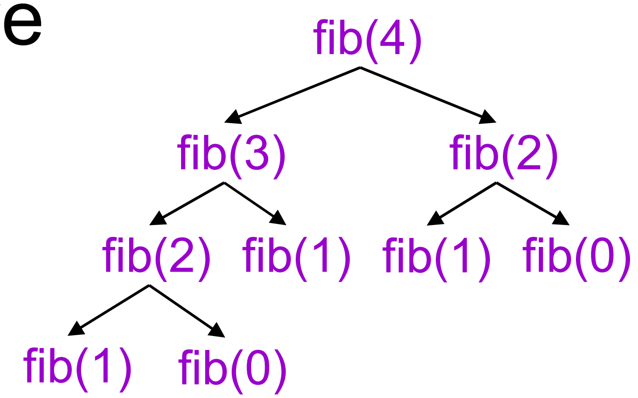

```
static int fib(int n) {
   if (n == 0) return 0;
   else if (n == 1) return 1;
   else return fib(n-1) + fib(n-2);
}
```

Execution of fib(4):



One thing to notice

- This way of computing the Fibonacci function is elegant, but inefficient
- It "recomputes" answers again and again!
- To improve speed, need to save known answers in a table!
- Called a *cache*



Adding caching to our solution

- Before:

```
static int fib(int n
   if (n == 0)
       return 0;
   else if (n == 1)
       return 1;
   else
       return fib(n-
}
```

- After

```
ArrayList<boolean> known = new ArrayList<boolean>;
ArrayList<int> cached = new ArrayList<cached>;
static int fib(int n) {
   int v;
   if(known[n])
         return cached[n];
   if (n == 0)
       v = 0;
   else if (n == 1)
       v = 1;
   else
       v = fib(n-1) + fib(n-2);
   known[n] = true;
   cached[n] = v;
   return v;
}
```

Notice the development process

- We started with the idea of recursion
- Created a very simple recursive procedure
- Noticed it will be slow, because it wastefully recomputes the same thing again and again
- So made it a bit more complex but gained a lot of speed in doing so

- This is a common software engineering pattern

Combinations
(a.k.a. Binomial Coefficients)

- How many ways can you choose r items from a set of n distinct elements? $\binom{n}{r}$ "n choose r"

  $\binom{5}{2}$ = number of 2-element subsets of {A,B,C,D,E}

  2-element subsets containing A: $\binom{4}{1}$
  {A,B}, {A,C}, {A,D}, {A,E}

  2-element subsets not containing A: {B,C},{B,D},{B,E},{C,D},{C,E},{D,E} $\binom{4}{2}$

- Therefore, $\binom{5}{2} = \binom{4}{1} + \binom{4}{2}$
- … in perfect form to write a recursive function!

Combinations

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}, \quad n > r > 0$$

$$\binom{n}{n} = 1$$

$$\binom{n}{0} = 1$$

Can also show that $\binom{n}{r} = \frac{n!}{r!(n-r)!}$

$$\binom{0}{0}$$
$$\binom{1}{0} \quad \binom{1}{1}$$
$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$
$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$
$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

Pascal's triangle =

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

Binomial Coefficients

Combinations are also called *binomial coefficients* because they appear as coefficients in the expansion of the binomial power **$(x+y)^n$** :

$$(x + y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n} y^n$$

$$= \sum_{i=0}^{n} \binom{n}{i} x^{n-i}y^i$$

Combinations Have Two Base Cases

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}, \quad n > r > 0$$

$$\binom{n}{n} = 1$$

$$\binom{n}{0} = 1$$

Two base cases

- Coming up with right base cases can be tricky!
- General idea:
  - Determine argument values for which recursive case does not apply
  - Introduce a base case for each one of these

Recursive Program for Combinations

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}, \quad n > r > 0$$

$$\binom{n}{n} = 1$$

$$\binom{n}{0} = 1$$

```
static int combs(int n, int r) {   //assume n>=r>=0
   if (r == 0 || r == n) return 1; //base cases
   else return combs(n-1,r) + combs(n-1,r-1);
}
```

Exercise for the reader (you!)

- Modify our recursive program so that it caches results
- Same idea as for our caching version of the fibonacci series

- Question to ponder: When is it worthwhile to adding caching to a recursive function?
  - *Certainly not always…*
  - *Must think about tradeoffs: space to maintain the cached results vs speedup obtained by having them*

Positive Integer Powers

- $a^n = a \cdot a \cdot a \cdots a$ (n times)

- Alternate description:
  - $a^0 = 1$
  - $a^{n+1} = a \cdot a^n$

```
static int power(int a, int n) {
   if (n == 0) return 1;
   else return a*power(a,n-1);
}
```

A Smarter Version

- Power computation:
  - $a^0 = 1$
  - If n is nonzero and even, $a^n = (a^{n/2})^2$
  - If n is odd, $a^n = a \cdot (a^{n/2})^2$
    - Java note: If x and y are integers, “x/y” returns the integer part of the quotient
- Example:

  $a^5 = a \cdot (a^{5/2})^2 = a \cdot (a^2)^2 = a \cdot ((a^{2/2})^2)^2 = a \cdot (a^2)^2$

  Note: this requires 3 multiplications rather than 5!

- What if n were larger?
  - Savings would be more significant
- This is much faster than the straightforward computation
  - Straightforward computation: n multiplications
  - Smarter computation: log(n) multiplications

Smarter Version in Java

- n = 0: $a^0 = 1$
- n nonzero and even: $a^n = (a^{n/2})^2$
- n nonzero and odd: $a^n = a \cdot (a^{n/2})^2$

parameters

local variable

```
static int power(int a, int n) {
    if (n == 0) return 1;
    int halfPower = power(a,n/2);
    if (n%2 == 0) return halfPower*halfPower;
    return halfPower*halfPower*a;
}
```

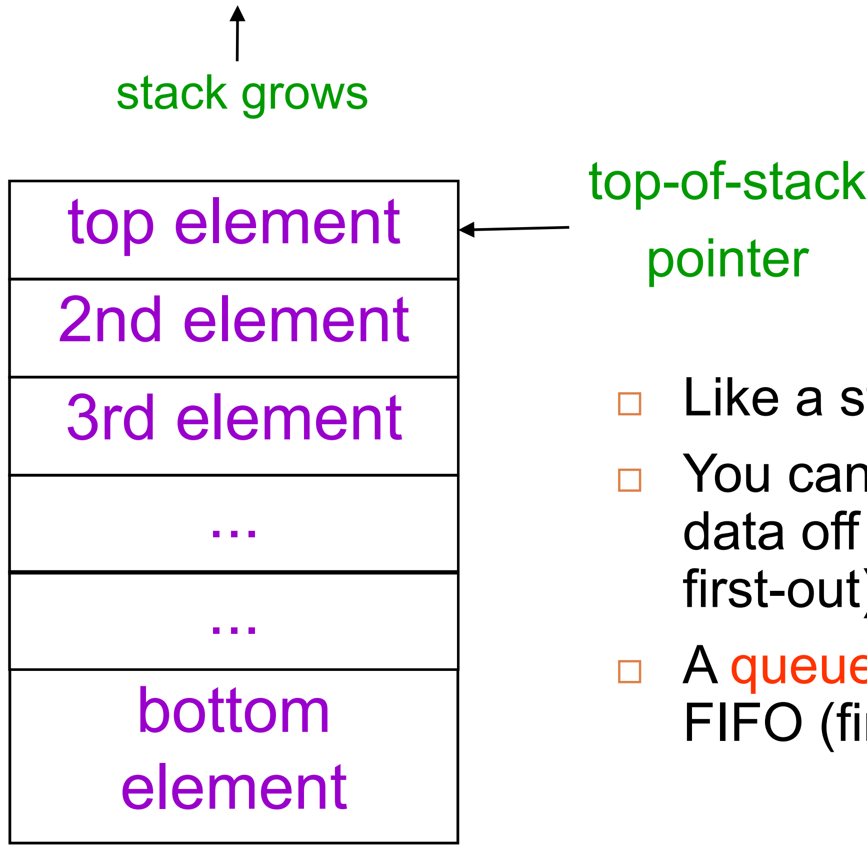
- The method has two parameters and a local variable
- Why aren't these overwritten on recursive calls?

Implementation of Recursive Methods

- Key idea:
  - Use a stack to remember parameters and local variables across recursive calls
  - Each method invocation gets its own stack frame

- A stack frame contains storage for
  - Local variables of method
  - Parameters of method
  - Return info (return address and return value)
  - Perhaps other bookkeeping info

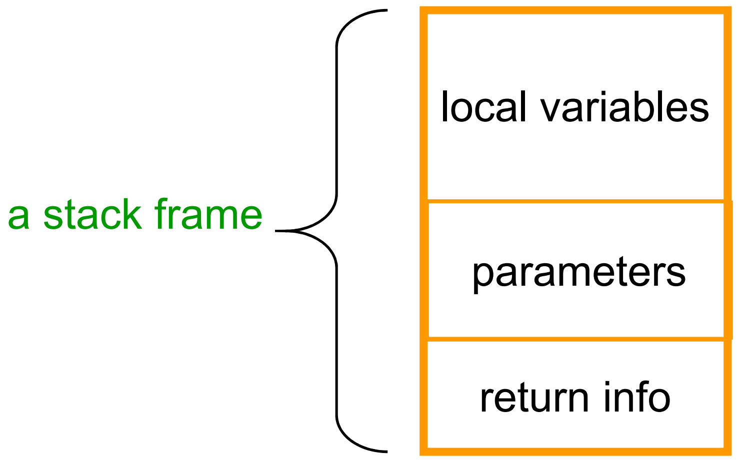
Stacks



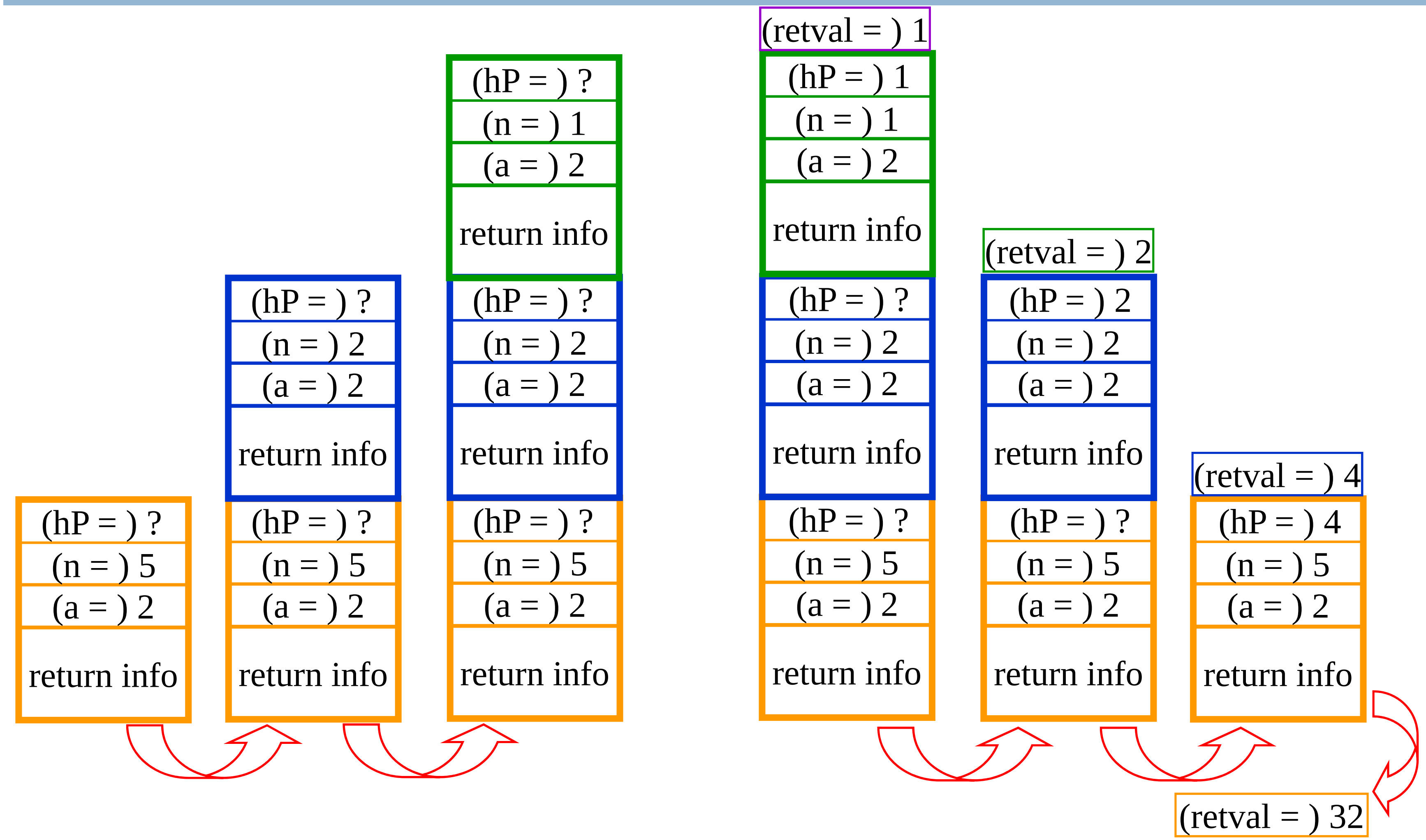
- Like a stack of dinner plates
- You can push data on top or pop data off the top in a LIFO (last-in-first-out) fashion
- A queue is similar, except it is FIFO (first-in-first-out)

Stack Frame

- A new stack frame is pushed with each recursive call
- The stack frame is popped when the method returns
  - Leaving a return value (if there is one) on top of the stack



Example: power(2, 5)



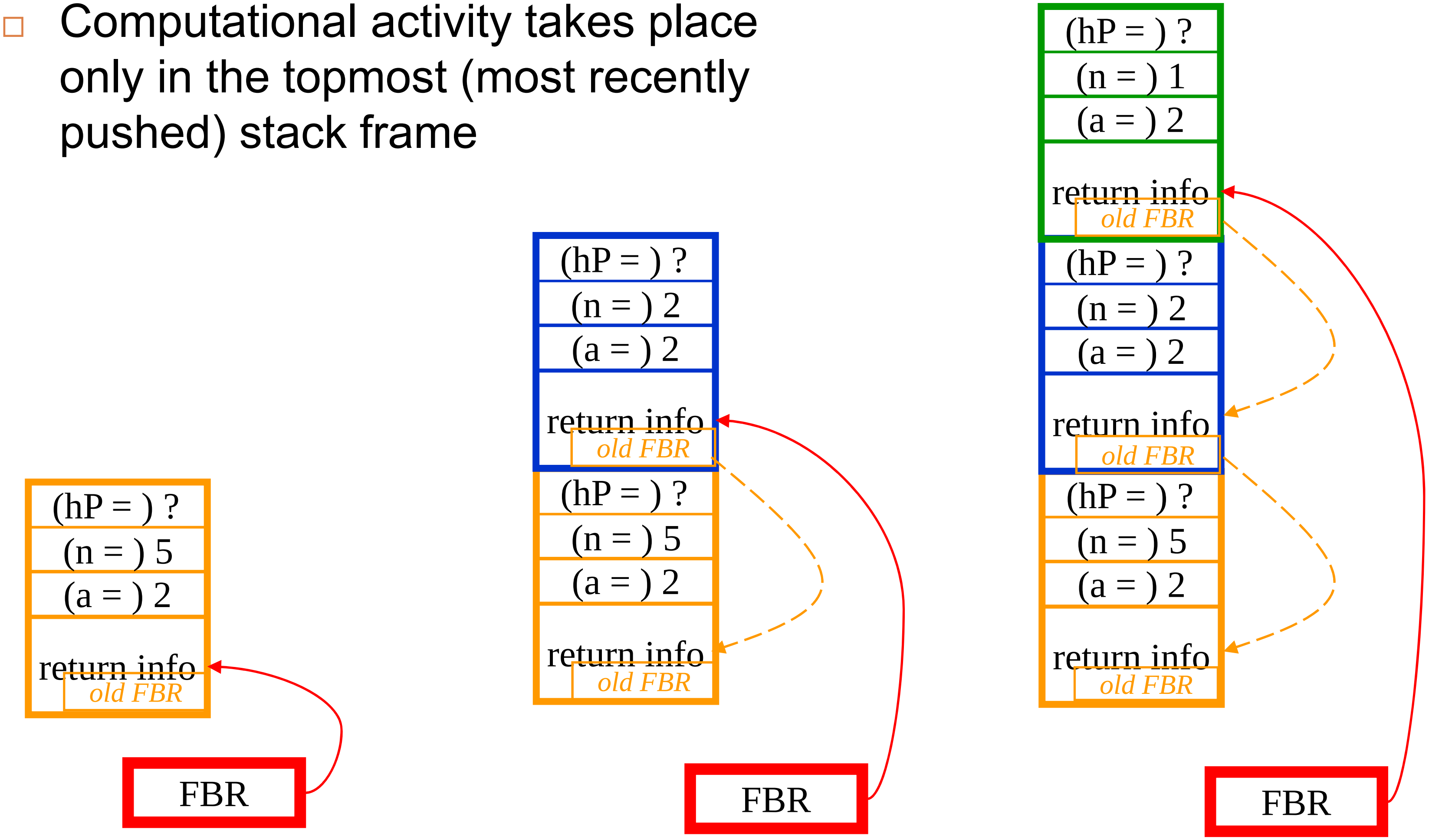
How Do We Keep Track?

- At any point in execution, many invocations of *power* may be in existence
  - Many stack frames (all for *power*) may be in Stack
  - Thus there may be several different versions of the variables *a* and *n*
- How does processor know which location is relevant at a given point in the computation?

- Answer:
  Frame Base Register
  - When a method is invoked, a frame is created for that method invocation, and FBR is set to point to that frame
  - When the invocation returns, FBR is restored to what it was before the invocation
- How does machine know what value to restore in the FBR?
  - This is part of the return info in the stack frame

FBR

- Computational activity takes place only in the topmost (most recently pushed) stack frame



Conclusion

- Recursion is a convenient and powerful way to define functions

- Problems that seem insurmountable can often be solved in a "divide-and-conquer" fashion:
  - Reduce a big problem to smaller problems of the same kind, solve the smaller problems
  - Recombine the solutions to smaller problems to form solution for big problem

- Important application (next lecture): parsing