CS 1110:
Introduction to Computing Using Python

Lecture 21

Loop Invariants

[Andersen, Gries, Lee, Marschner, Van Loan, White]

Announcements

- Prelim 2 conflicts due by midnight ***tonight***
- Lab 11 is out
  - Due in 2 weeks because of Prelim 2
- Review Prelim 2 announcements from previous lecture
- A4 is due Thursday at midnight
- There will only be 5 assignments.
  - Can look at webpage for redistributed weights

Loop Invariants: Eat your Vegetables!

source: Wikipedia

Recall: The while-loop

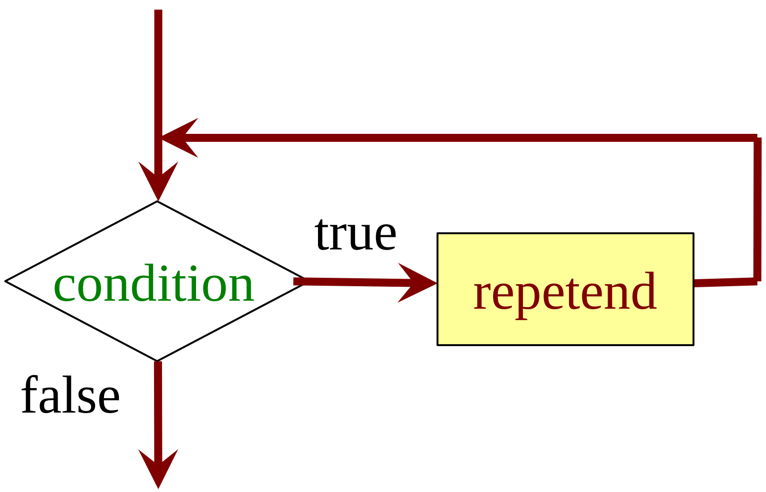
while <*condition*>:

statement 1

…

statement n

repetend or **body**



- Relationship to for-loop
  - Must explicitly ensure condition becomes false
  - *You* explicitly manage what changes per iteration

Example: Sorting

pre: b [0 … n] ?

post: b [0 … n] sorted

```
i = 0
while i < n:
    # Find minimum val in b[i..]
    # Swap min val with val at i
    i = i+1
```

0 … i−1	i … n
2 4 4 6 6	8 9 9 7 8 9
2 4 4 6 6	7 9 9 8 8 9

0 … i−1	i … n
2 4 4 6 6 7	9 9 8 8 9

Recall: Important Terminology

- **assertion**: true-false statement placed in a program to *assert* that it is true at that point
  - Can either be a **comment**, or an **assert** command
- **invariant**: assertion supposed to "always" be true
  - If temporarily invalidated, must make it true again
  - **Example**: class invariants and class methods
- **loop invariant**: assertion supposed to be true before and after each iteration of the loop
- **iteration of a loop**: one execution of its body

Preconditions & Postconditions

precondition

```
# x  = sum of 1..n-1
x = x + n
n = n + 1
# x =  sum of 1..n-1
```

postcondition

n
1 2 3 4 5 6 7 8

x contains the sum of these (6)

n
1 2 3 4 5 6 7 8

x contains the sum of these (10)

- **Precondition:** assertion placed before a segment
- **Postcondition:** assertion placed after a segment

Relationship Between Two

If precondition is true, then postcondition will be true

Solving a Problem

precondition

```
# x  = sum of 1..n

n = n + 1
# x =  sum of 1..n
```

postcondition

What statement do you put here to make the postcondition true?

A: x = x + 1

B: x = x + n

C: $\boxed{\text{x = x + n+1}}$

D: None of the above

E: I don’t know

Solving a Problem

precondition

```
# x  = sum of 1..n

n = n + 1
# x =  sum of 1..n
```

postcondition

What statement do you put here to make the postcondition true?

A: x = x + 1
B: x = x + n
C: x = x + n+1
D: None of the above
E: I don't know

Remember the new value of n

Solving a Problem

precondition

```
# x  = sum of 1..n

n = n + 1
# x =  sum of 1..n
```

postcondition

n
1 2 3 4 5 6 7 8
n+1
x contains the sum of these (10)

n
1 2 3 4 5 6 7 8
x contains the sum of these (15)

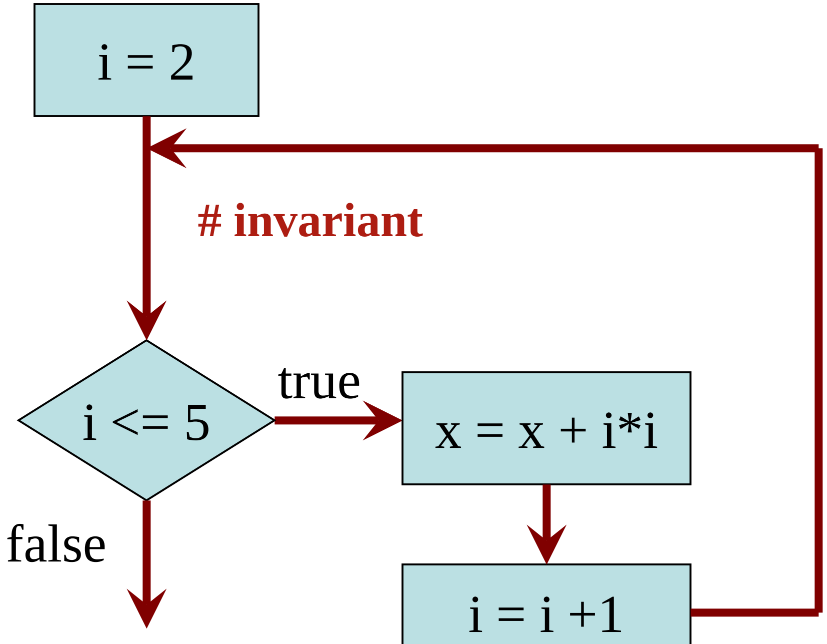
A: x = x + 1
B: x = x + n
C: x = x + n+1
D: None of the above
E: I don't know

Remember the new value of n

Invariants: Assertions That Do Not Change

- **Loop Invariant:** an assertion that is true before and after each iteration (execution of repetend)

```
x = 0; i = 2
while i <= 5:
    x = x + i*i
    i = i +1
# x = sum of squares of 2..5
```



Invariant:

x = sum of squares of 2..i-1

in terms of the range of integers that have been processed so far

The loop processes the range 2..5

Invariants: Assertions That Do Not Change

- **Loop Invariant**: an assertion that is true before and after each iteration (execution of repetend)
- Should help you ***understand the loop***
- There are good invariants and bad invariants
- Bad:
  - 2 != 1

True, but *doesn't help you understand the loop*

- Good:
  - s[0…k] is sorted

Seems useful in order to conclude that s is sorted.

Key Difference

```
x = 0; i = 2
# Inv: x = sum of squares of 2..i-1
while i <= 5:
    x = x + i*i
    i = i +1
# Post: x = sum of squares of 2..5
```

Invariant:
True when loop terminates

Loop termination condition:
False when loop terminates

Invariants: Assertions That Do Not Change

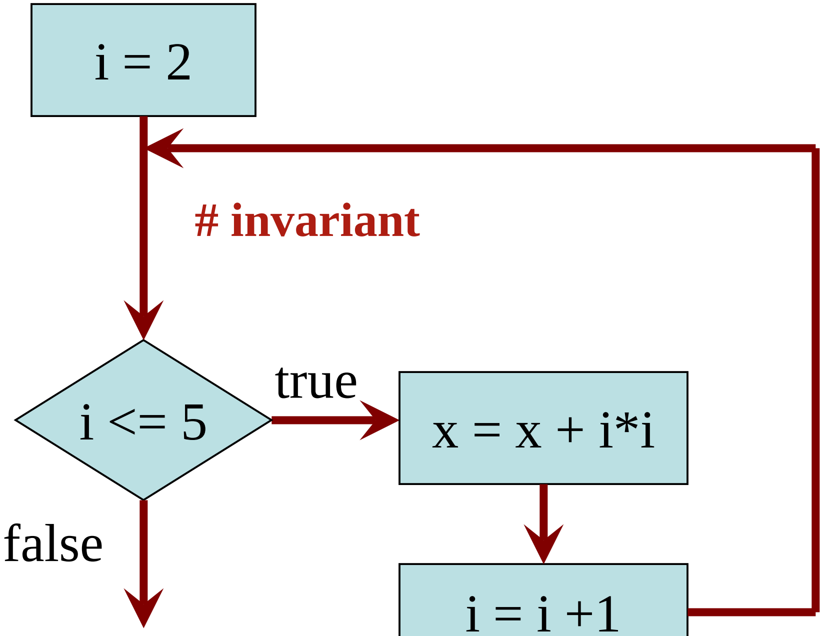
```
x = 0; i = 2
# Inv: x = sum of squares of 2..i-1
while i <= 5:
    x = x + i*i
    i = i +1
# Post: x = sum of squares of 2..5
```

Integers that have been processed:

Range 2..i-1:

x | 0

i | ?



The loop processes the range 2..5

Invariants: Assertions That Do Not Change

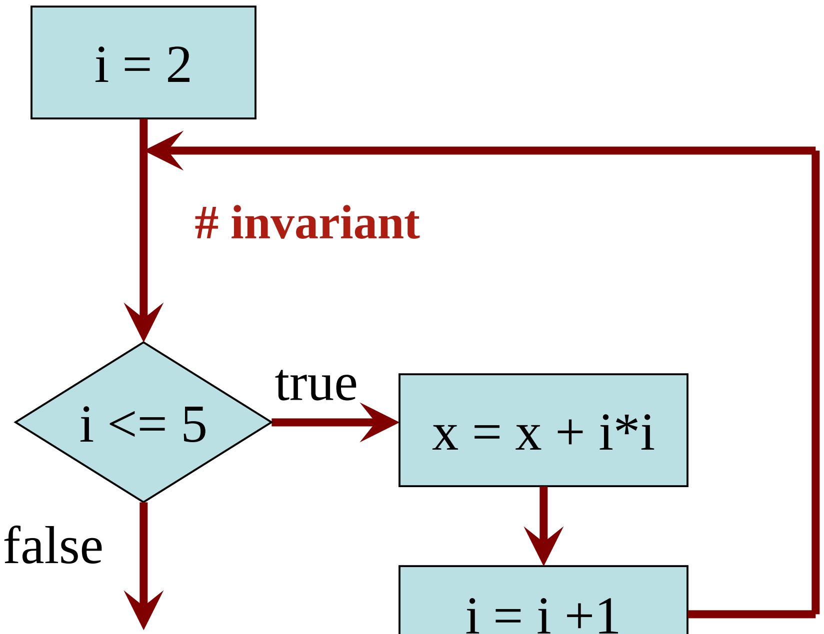
```
x = 0; i = 2
# Inv: x = sum of squares of 2..i-1
while i <= 5:
    x = x + i*i
    i = i +1
# Post: x = sum of squares of 2..5
```

x: 0

i: ~~?~~ 2

Integers that have been processed:

Range 2..i-1: 2..1 (empty)



The loop processes the range 2..5

Invariants: Assertions That Do Not Change

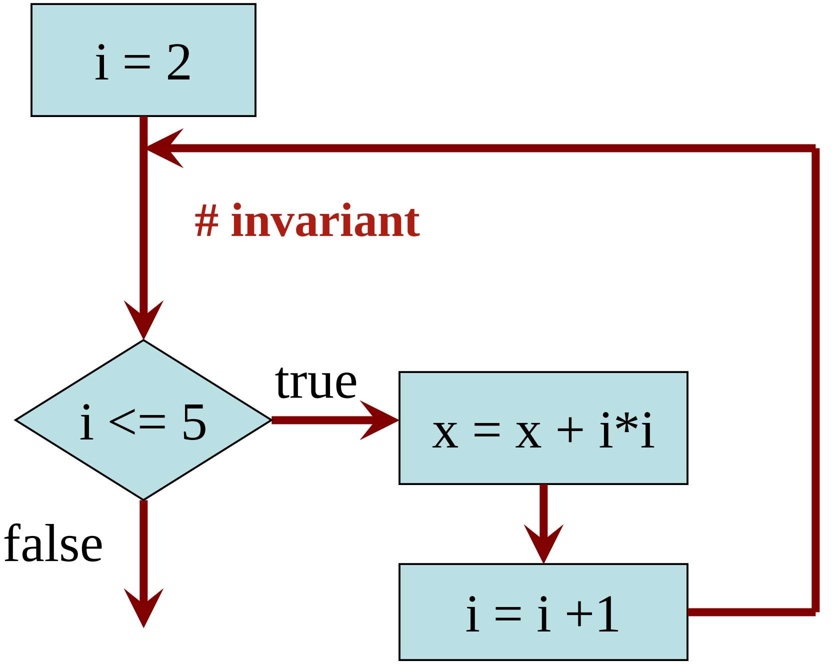
```
x = 0; i = 2
# Inv: x = sum of squares of 2..i-1
while i <= 5:
    x = x + i*i
    i = i +1
# Post: x = sum of squares of 2..5
```

x ~~0~~ 4

i ~~?~~ ~~2~~ 3

Integers that have
been processed: 2

Range 2..i-1: 2..2



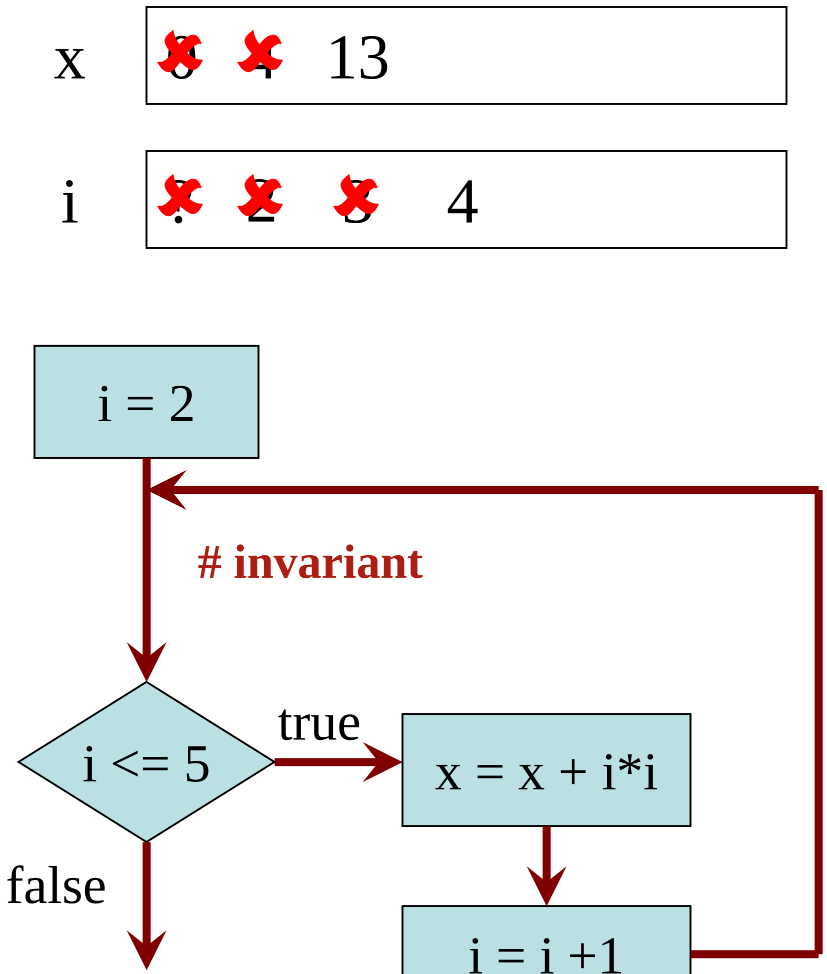
The loop processes the range 2..5

Invariants: Assertions That Do Not Change

```
x = 0; i = 2
# Inv: x = sum of squares of 2..i-1
while i <= 5:
    x = x + i*i
    i = i +1
# Post: x = sum of squares of 2..5
```

Integers that have been processed: 2, 3

Range 2..i-1: 2..3



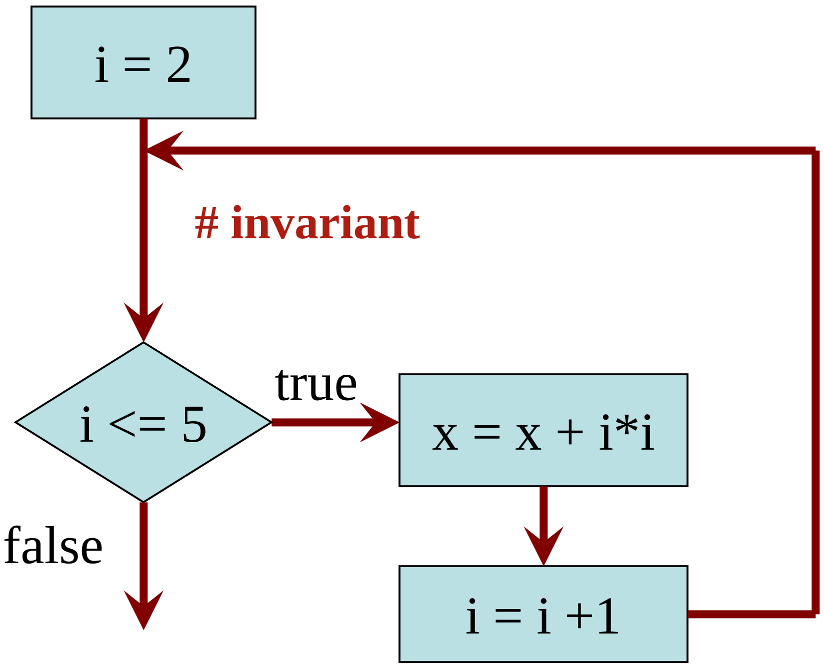
The loop processes the range 2..5

Invariants: Assertions That Do Not Change

```
x = 0; i = 2
# Inv: x = sum of squares of 2..i-1
while i <= 5:
    x = x + i*i
    i = i +1
# Post: x = sum of squares of 2..5
```

x	~~0~~ ~~4~~ ~~13~~ 29
i	~~?~~ ~~2~~ ~~3~~ ~~4~~ 5

Integers that have been processed:	2, 3, 4
Range 2..i-1:	2..4



The loop processes the range 2..5

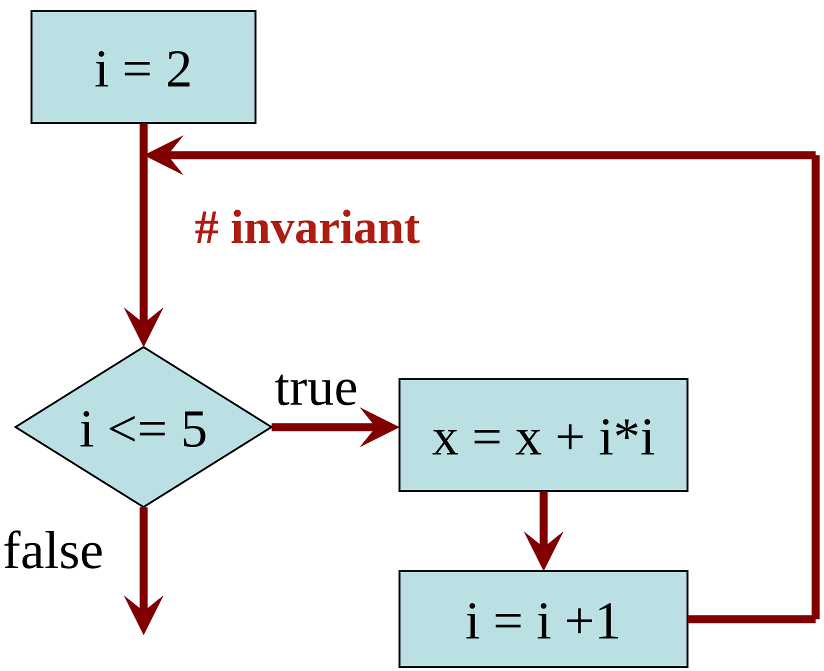
Invariants: Assertions That Do Not Change

```
x = 0; i = 2
# Inv: x = sum of squares of 2..i-1
while i <= 5:
    x = x + i*i
    i = i +1
# Post: x = sum of squares of 2..5
```

x: ~~0~~ ~~4~~ ~~13~~ ~~29~~ 54

i: ~~?~~ ~~2~~ ~~3~~ ~~4~~ ~~5~~ 6

Integers that have been processed:	2, 3, 4, 5
Range 2..i-1:	2..5



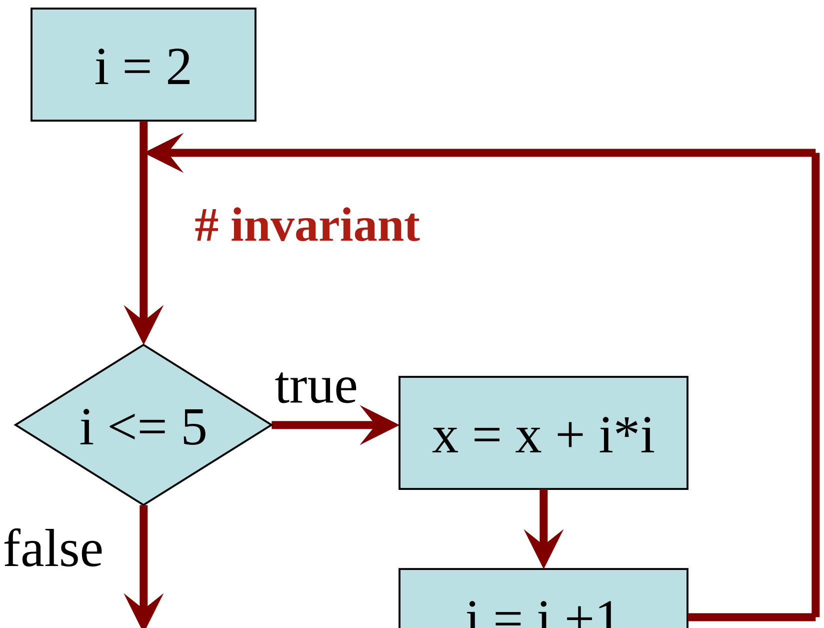
The loop processes the range 2..5

Invariants: Assertions That Do Not Change

```
x = 0; i = 2
# Inv: x = sum of squares of 2..i-1
while i <= 5:
    x = x + i*i
    i = i +1
# Post: x = sum of squares of 2..5
```

x: ~~0~~ ~~4~~ ~~13~~ ~~29~~ 54

i: ~~?~~ ~~2~~ ~~3~~ ~~4~~ ~~5~~ 6



Integers that have been processed:	2, 3, 4, 5
Range 2..i-1:	2..5

Invariant was always true just before test of loop condition. So it's true when loop terminates

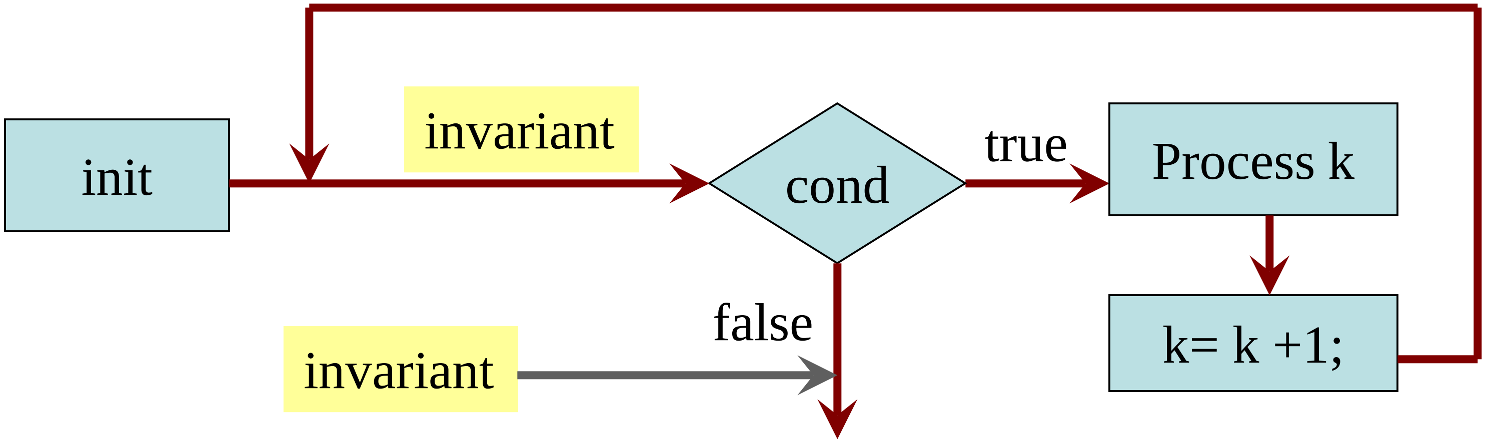
The loop processes the range 2..5

Designing Integer while-loops

Command to do something

```
# Process integers in a..b
# inv: integers in a..k-1 have been processed
k = a
while  k <= b:
    process integer k
    k = k + 1
# post: integers in a..b have been processed
```

Equivalent postcondition



Designing Integer while-loops

1. Recognize that a range of integers b..c has to be processed
2. Write the command and equivalent postcondition
3. Write the basic part of the while-loop
4. Write loop invariant
5. Figure out any initialization
6. Implement the repetend (process k)

Designing Integer while-loops

1. Recognize that a range of integers b..c has to be processed
2. Write the command and equivalent postcondition
3. Write the basic part of the while-loop
4. Write loop invariant
5. Figure out any initialization
6. Implement the repetend (process k)

```
# Process b..c




# Postcondition: range b..c has been processed
```

Designing Integer while-loops

1. Recognize that a range of integers b..c has to be processed
2. Write the command and equivalent postcondition
3. Write the basic part of the while-loop
4. Write loop invariant
5. Figure out any initialization
6. Implement the repetend (process k)

```
# Process b..c



while  k <= c:

    k = k + 1
# Postcondition: range b..c has been processed
```

Designing Integer while-loops

1. Recognize that a range of integers b..c has to be processed
2. Write the command and equivalent postcondition
3. Write the basic part of the while-loop
4. Write loop invariant
5. Figure out any initialization
6. Implement the repetend (process k)

```
# Process b..c


# Invariant: range b..k-1 has been processed
while  k <= c:

    k = k + 1
# Postcondition: range b..c has been processed
```

Designing Integer while-loops

1. Recognize that a range of integers b..c has to be processed
2. Write the command and equivalent postcondition
3. Write the basic part of the while-loop
4. Write loop invariant
5. Figure out any initialization
6. Implement the repetend (process k)

```
# Process b..c
Initialize variables (if necessary) to make invariant true
# Invariant: range b..k-1 has been processed
while  k <= c:
    # Process k
    k = k + 1
# Postcondition: range b..c has been processed
```

Finding an Invariant

Command to do something

```
# Make b True if n is prime, False otherwise
```

```
# b is True if no int in 2..n-1 divides n, False otherwise
```

Equivalent postcondition

What is the invariant?

Finding an Invariant

Command to do something

```
# Make b True if n is prime, False otherwise

while k < n:
    # Process k;

    k = k +1
# b is True if no int in 2..n-1 divides n, False otherwise
```

Equivalent postcondition

What is the invariant?

Finding an Invariant

Command to do something

```
# Make b True if n is prime, False otherwise



# invariant: b is True if no int in 2..k-1 divides n, False otherwise
while k < n:
    # Process k;



    k = k +1
# b is True if no int in 2..n-1 divides n, False otherwise
```

Equivalent postcondition

What is the invariant?

1 2 3 … k-1 k k+1 … n

Finding an Invariant

Command to do something

```
# Make b True if n is prime, False otherwise
b = True
k = 2
# invariant: b is True if no int in 2..k-1 divides n, False otherwise
while k < n:
    # Process k;



    k = k +1
# b is True if no int in 2..n-1 divides n, False otherwise
```

Equivalent postcondition

What is the invariant?

1 2 3 … k-1 k k+1 … n

Finding an Invariant

Command to do something

```
# Make b True if n is prime, False otherwise
b = True
k = 2
# invariant: b is True if no int in 2..k-1 divides n, False otherwise
while k < n:
    # Process k;
    if n % k == 0:
        b = False
    k = k +1
# b is True if no int in 2..n-1 divides n, False otherwise
```

Equivalent postcondition

What is the invariant?

1 2 3 … k-1 k k+1 … n

Finding an Invariant

```
# set x to # adjacent equal pairs in s
```

Command to do something

for s = 'ebeee', x = 2

```
while k < len(s):
    # Process k

    k = k + 1
# x = # adjacent equal pairs in s[0..len(s)-1]
```

Equivalent postcondition

k: next integer to process.
Which have been processed?

A: 0..k

B: 1..k

C: 0..k–1

D: 1..k–1

E: I don’t know

Finding an Invariant

```
# set x to # adjacent equal pairs in s
```

Command to do something

for s = 'ebeee', x = 2

```
while k < len(s):
    # Process k

    k = k + 1
# x = # adjacent equal pairs in s[0..len(s)-1]
```

Equivalent postcondition

k: next integer to process.
Which have been processed?

A: 0..k
B: 1..k
C: $\boxed{0..k–1}$
D: 1..k–1
E: I don’t know

What is the invariant?

A: x = no. adj. equal pairs in s[1..k]
B: x = no. adj. equal pairs in s[0..k]
C: x = no. adj. equal pairs in s[1..k–1]
D: x = no. adj. equal pairs in s[0..k–1]
E: I don’t know

Finding an Invariant

```
# set x to # adjacent equal pairs in s


# inv: x = # adjacent equal pairs in s[0..k-1]
while k < len(s):
    # Process k

    k = k + 1
# x = # adjacent equal pairs in s[0..len(s)-1]
```

Command to do something

for s = 'ebeee', x = 2

Equivalent postcondition

k: next integer to process.

What ***indices*** have been considered?

A: 0..k

B: 1..k

C: $\boxed{\text{0..k–1}}$

D: 1..k–1

E: I don't know

What is the invariant?

A: x = no. adj. equal pairs in s[1..k]

B: x = no. adj. equal pairs in s[0..k]

C: x = no. adj. equal pairs in s[1..k–1]

D: $\boxed{\text{x = no. adj. equal pairs in s[0..k–1]}}$

E: I don't know

Finding an Invariant

```
# set x to # adjacent equal pairs in s
x = 0

# inv: x = # adjacent equal pairs in s[0..k-1]
while k < len(s):
    # Process k

    k = k + 1
# x = # adjacent equal pairs in s[0..len(s)-1]
```

Command to do something

for s = 'ebeee', x = 2

Equivalent postcondition

k: next integer to process.
What is initialization for k?

A: k = 0

B: k = 1

C: k = –1

D: I don't know

Finding an Invariant

```
# set x to # adjacent equal pairs in s
x = 0
k = 1
# inv: x = # adjacent equal pairs in s[0..k-1]
while k < len(s):
    # Process k

    k = k + 1
# x = # adjacent equal pairs in s[0..len(s)-1]
```

Command to do something

for s = 'ebeee', x = 2

Equivalent postcondition

k: next integer to process.
What is initialization for k?

A: k = 0
B: $\boxed{k = 1}$
C: k = –1
D: I don’t know

Which do we compare to “process” k?

A: s[k] and s[k+1]
B: s[k-1] and s[k]
C: s[k-1] and s[k+1]
D: s[k] and s[n]
E: I don’t know

Finding an Invariant

```
# set x to # adjacent equal pairs in s
x = 0
k = 1
# inv: x = # adjacent equal pairs in s[0..k-1]
while k < len(s):
    # Process k
    x = x + 1 if (s[k-1] == s[k]) else 0
    k = k + 1
# x = # adjacent equal pairs in s[0..len(s)-1]
```

Command to do something

for s = 'ebeee', x = 2

Equivalent postcondition

k: next integer to process.
What is initialization for k?

A: k = 0
B: k = 1
C: k = –1
D: I don't know

Which do we compare to "process" k?

A: s[k] and s[k+1]
B: s[k-1] and s[k]
C: s[k-1] and s[k+1]
D: s[k] and s[n]
E: I don't know

Reason carefully about initialization

```
# s is a list of ints; len(s) >= 1
# Set c to largest element in s
c = ??
k = ??
# inv:
while k < len(s):
    # Process k
    k = k+1
#  c = largest int in s[0..len(s)–1]
```

Command to do something

Equivalent postcondition

1. What is the invariant?

Reason carefully about initialization

```
# s is a list of ints; len(s) >= 1
# Set c to largest element in s
c = ??
k = ??
# inv: c is largest element in s[0..k–1]
while k < len(s):
    # Process k
    k = k+1
#  c = largest int in s[0..len(s)–1]
```

Command to do something

Equivalent postcondition

1. What is the invariant?

Reason carefully about initialization

```
# s is a list of ints; len(s) >= 1
# Set c to largest element in s
c = ??
k = ??
# inv: c is largest element in s[0..k–1]
while k < len(s):
    # Process k
    k = k+1
#  c = largest int in s[0..len(s)–1]
```

Command to do something

Equivalent postcondition

1. What is the invariant?
2. How do we initialize c and k?

A: k = 0; c = s[0]

B: k = 1; c = s[0]

C: k = 1; c = s[1]

D: k = 0; c = s[1]

E: None of the above

Reason carefully about initialization

```
# s is a list of ints; len(s) >= 1
# Set c to largest element in s
c = ??
k = ??
# inv: c is largest element in s[0..k–1]
while k < len(s):
    # Process k
    k = k+1
#  c = largest int in s[0..len(s)–1]
```

Command to do something

Equivalent postcondition

1. What is the invariant?
2. How do we initialize c and k?

A: k = 0; c = s[0]

B: $\boxed{\text{k = 1; c = s[0]}}$

C: k = 1; c = s[1]

D: k = 0; c = s[1]

E: None of the above

An empty set of characters or integers has no maximum. Therefore, be sure that 0..k–1 is not empty. You must start with k = 1.

What is the Invariant?

pre: b (0 … n)

?

post: b (0 … n)

sorted

inv: b (0 … i … n)

sorted, ≤ b[i..]	≥ b[0..i-1]

First segment always contains smaller values

```
i = 0
while i < n:
    # Find minimum val in b[i..]
    # Swap min val with val at i
    i = i+1
```

(i, n)

2 4 4 6 6	8 9 9 7 8 9

(i, n)

2 4 4 6 6	7 9 9 8 8 9

(i, n)

2 4 4 6 6 7	9 9 8 8 9