CS 1110:
Introduction to Computing Using Python

Lecture 15

Recursion

[Andersen, Gries, Lee, Marschner, Van Loan, White]

Announcements: Prelim 1

- Graded and released
- **Mean**: 81 out of 104 (78%)
- Can pick up your exam in homework handback room
  - Need Cornell ID
  - Suggest printing your netid on paper
- Do not discuss exam with people taking makeups.
- **Regrade requests**: we will send email to you

Announcements: Assignment 3

- Released.
- **Due**: Thursday, March 30th, 11:59pm
- Recommendation: follow milestone deadlines.
- You MUST acknowledge help from others
  - We run software analyzers to detect similar programs
  - Have had some academic integrity violations so far
- Not a recursion assignment!

Announcement: Lab 8

- Out.
- Not a recursion lab!

Recursion

- **Recursive Definition**:

  A definition that is defined in terms of itself

A Mathematical Example: Factorial

- Non-recursive definition:

$$n! = n \times n\text{-}1 \times \ldots \times 2 \times 1$$

$$= n\,(n\text{-}1 \times \ldots \times 2 \times 1)$$

- Recursive definition:

$n! = n\,(n\text{-}1)!$ for $n \geq 0$ **Recursive case**

$0! = 1$ **Base case**

What happens if there is no base case?

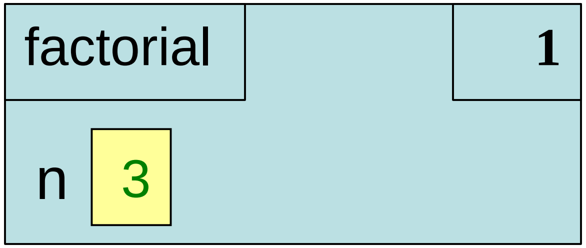
Recursion

- **Recursive Definition**:

  A definition that is defined in terms of itself
- **Recursive Function**:

  A function that calls itself (directly or indirectly)

Recursive Call Frames

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def factorial(n):
    """Returns: factorial of n.
    Pre: n ≥ 0 an int"""
    if n == 0:
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    return n*factorial(n-1)
```

Call: factorial(3)

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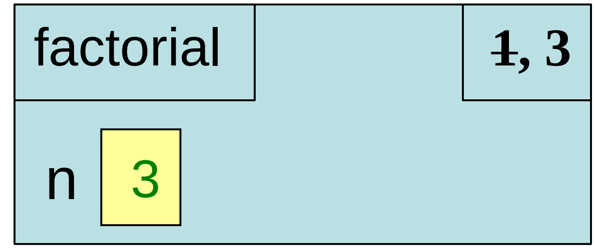

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Recursion

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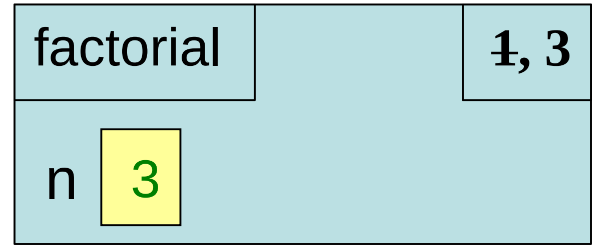

Now what?
Each call is a new frame.

Call: factorial(3)

What happens next?

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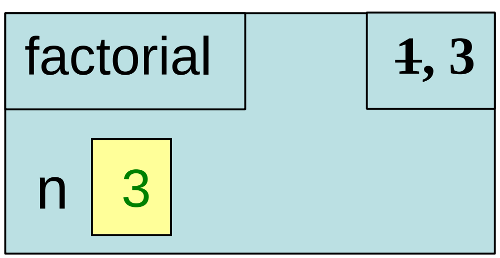

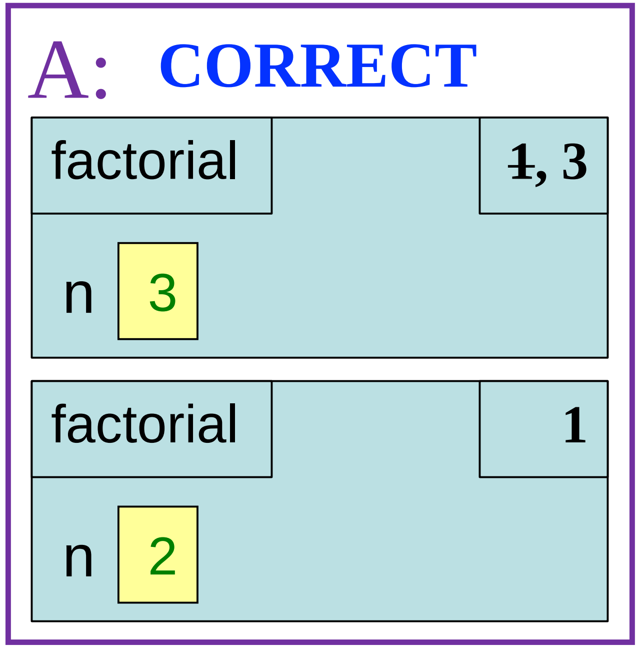

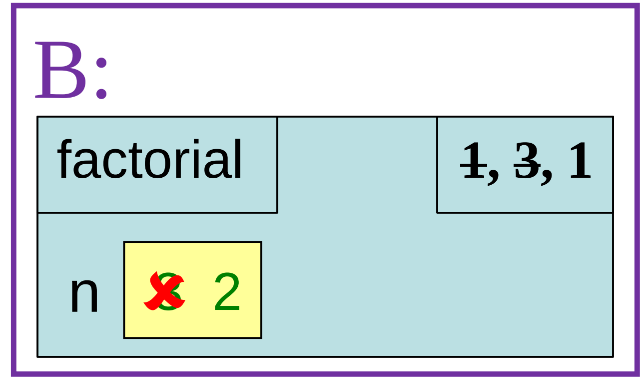

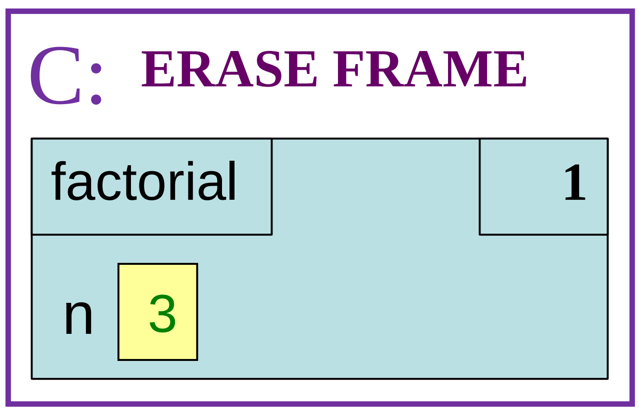

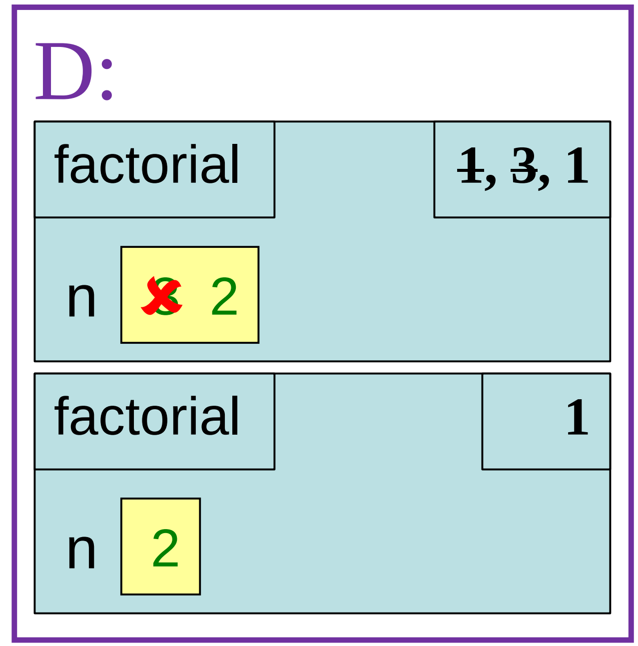

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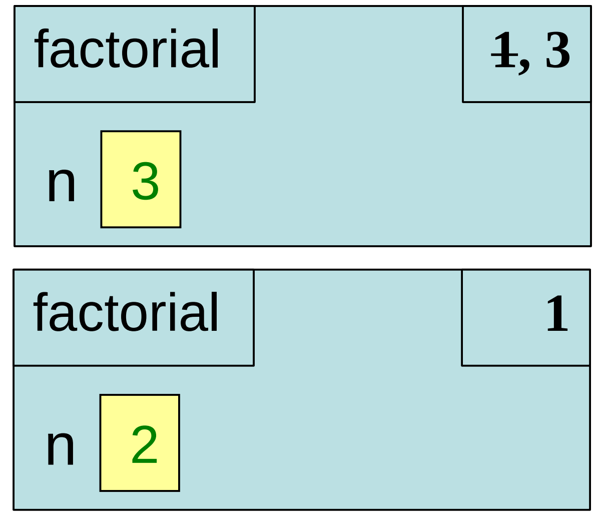

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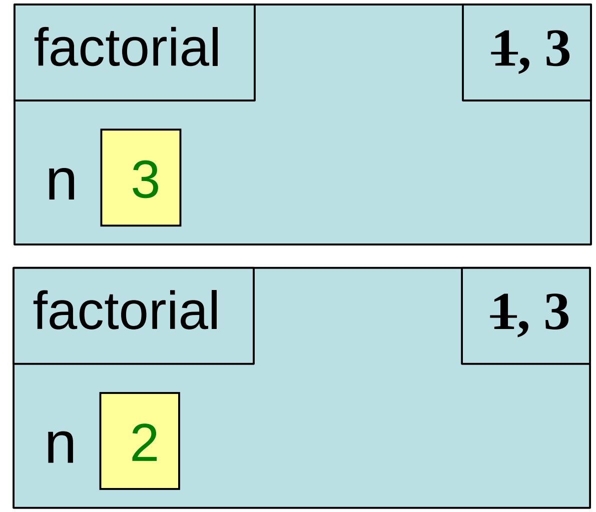

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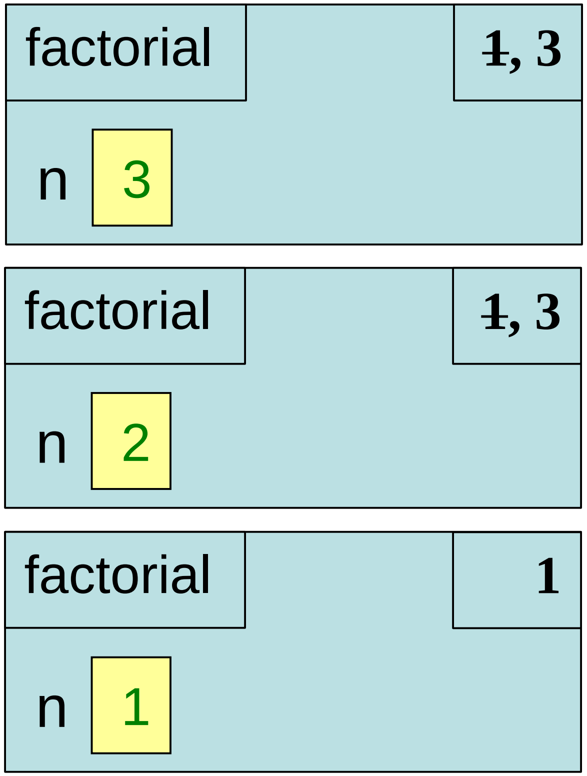

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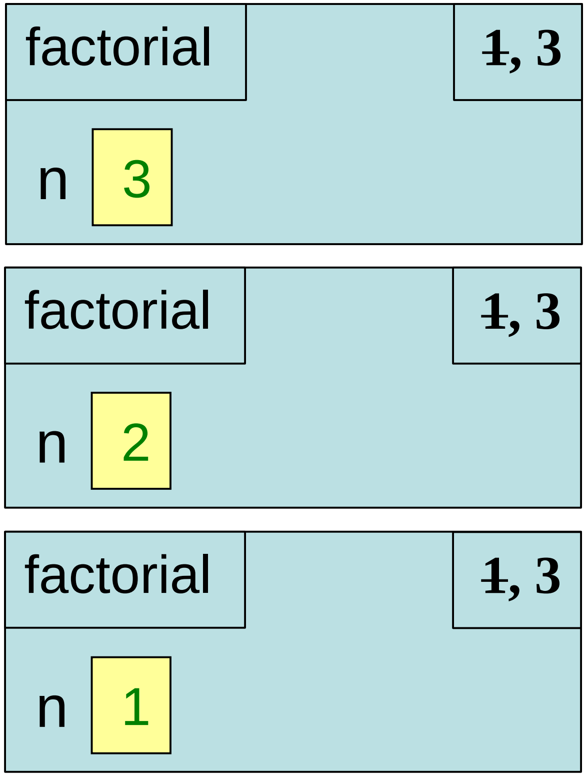

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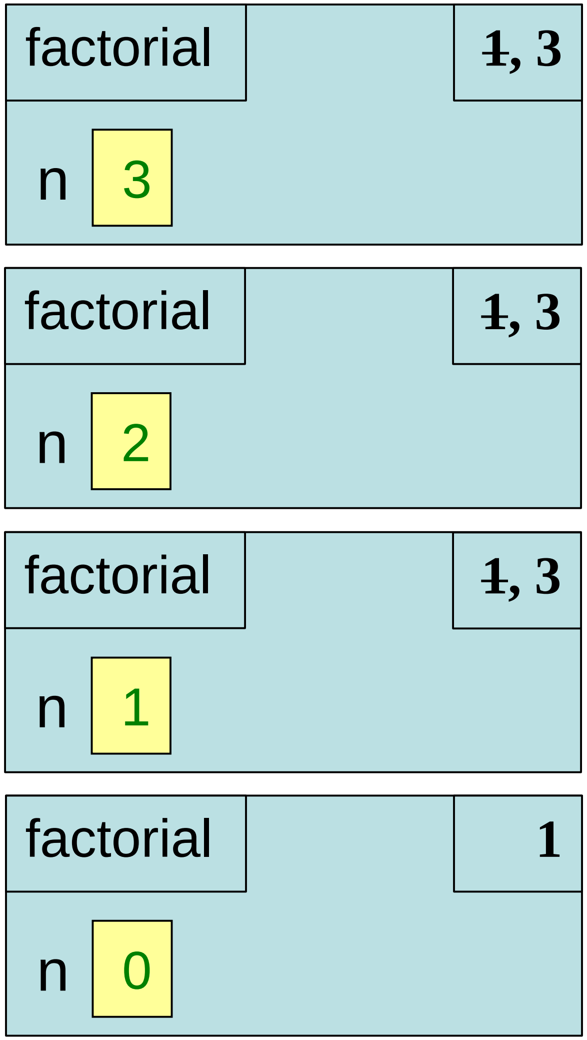

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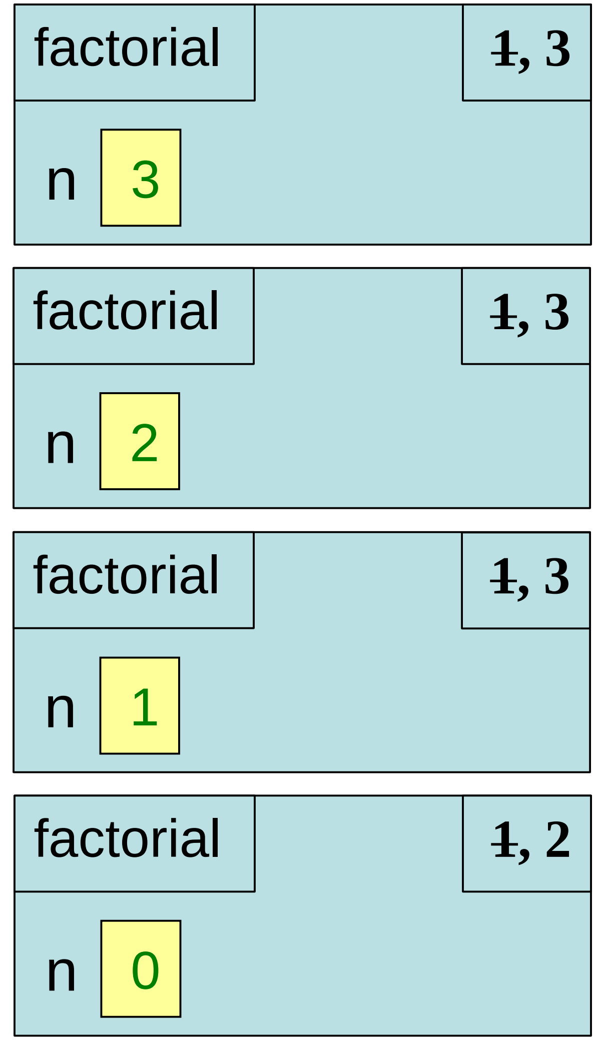

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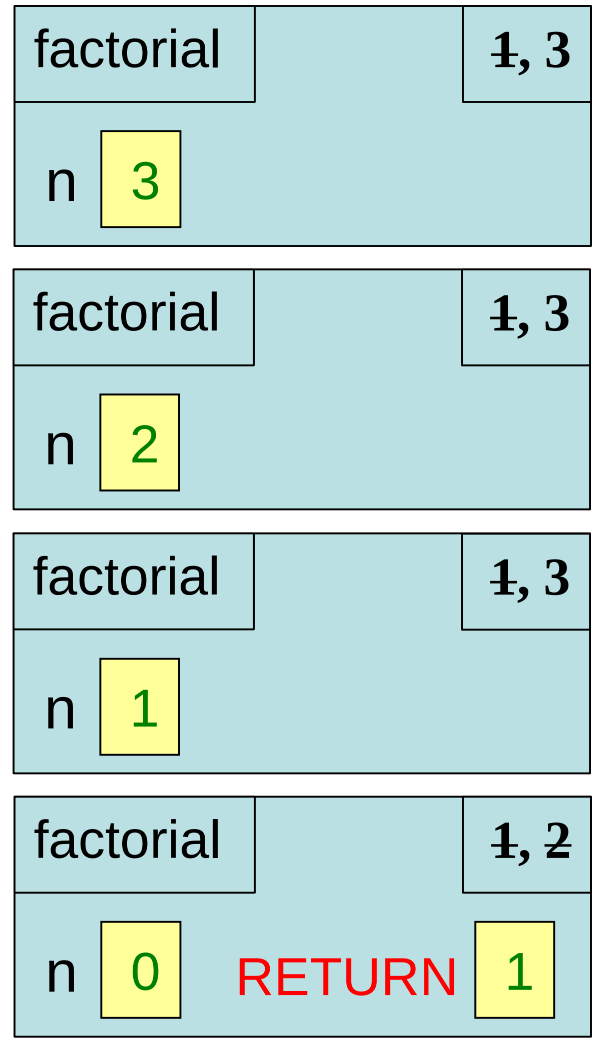

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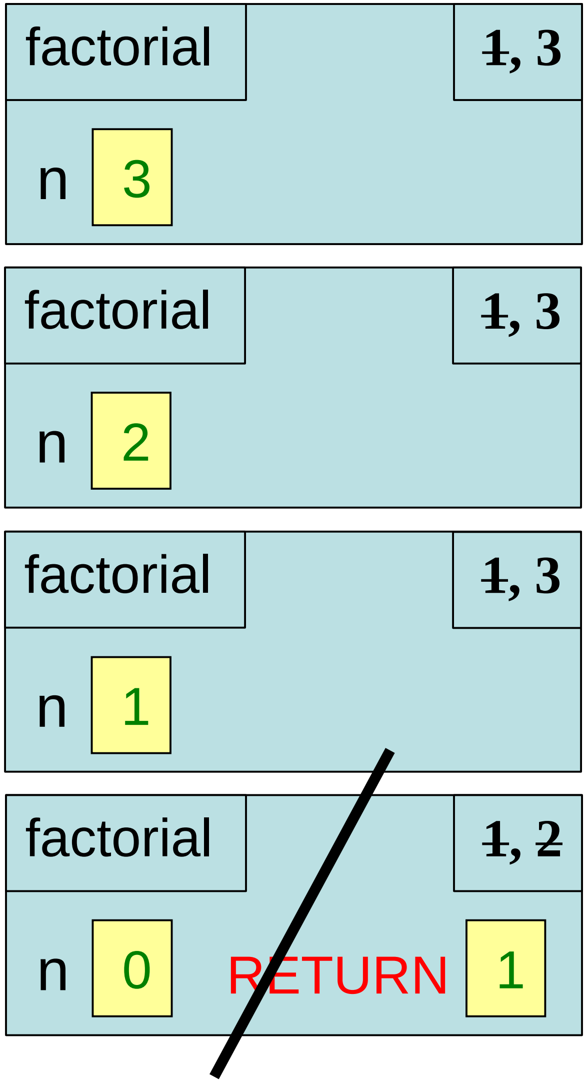

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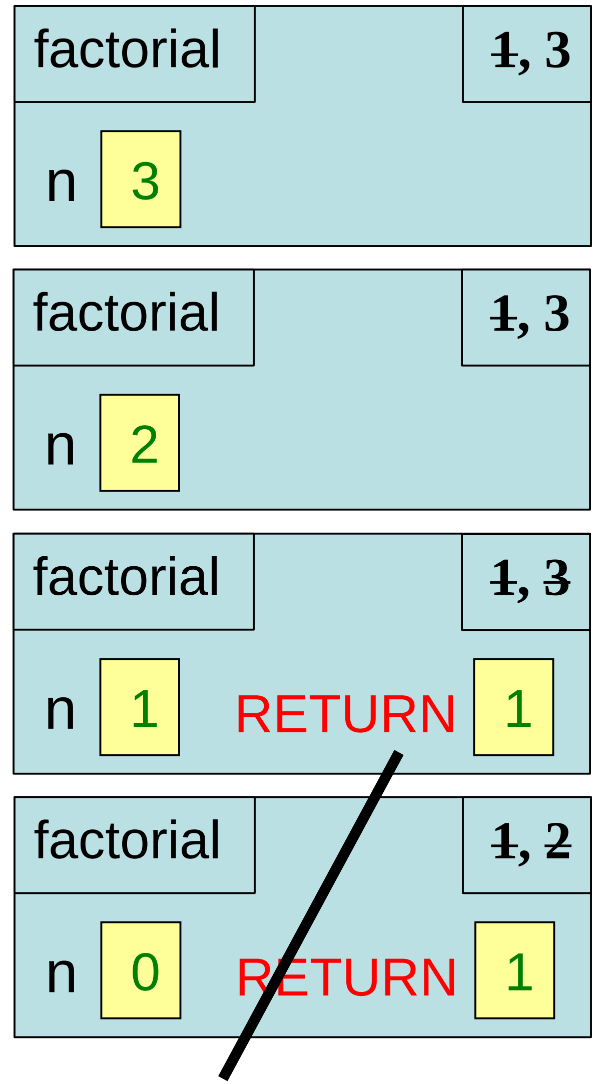

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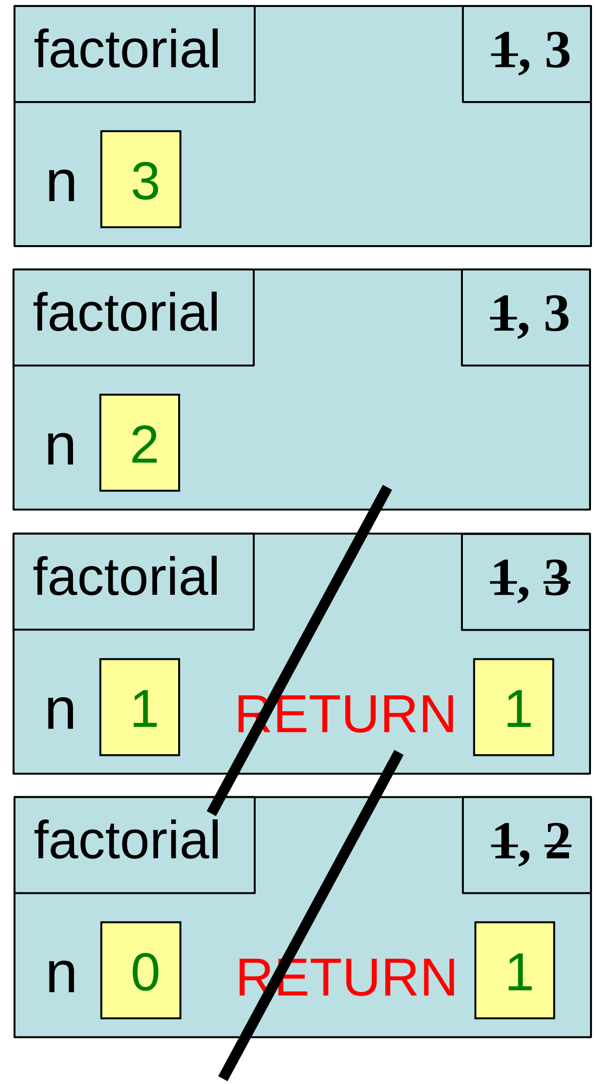

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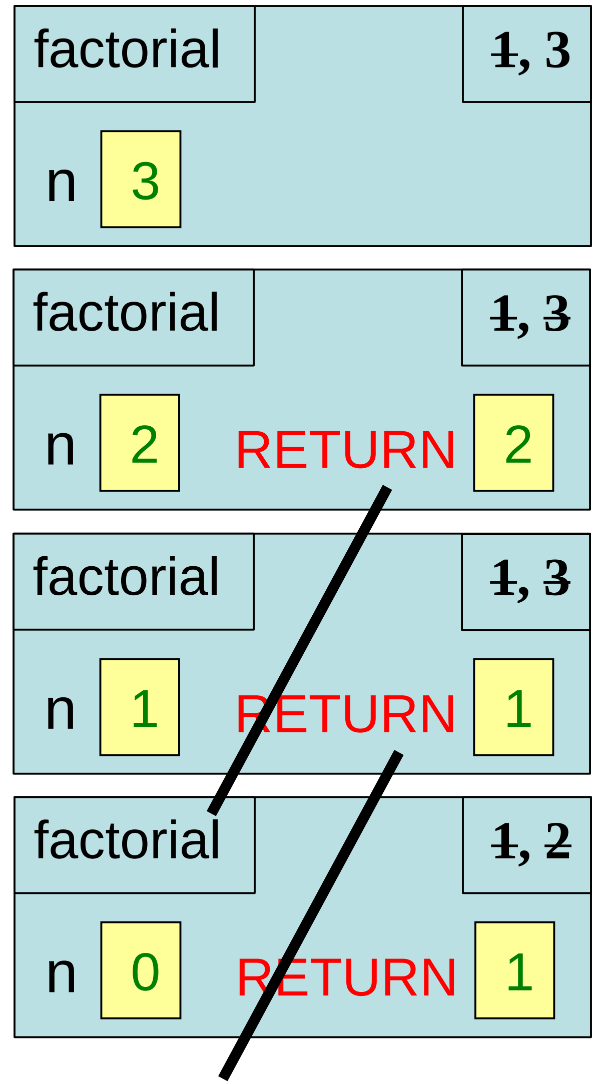

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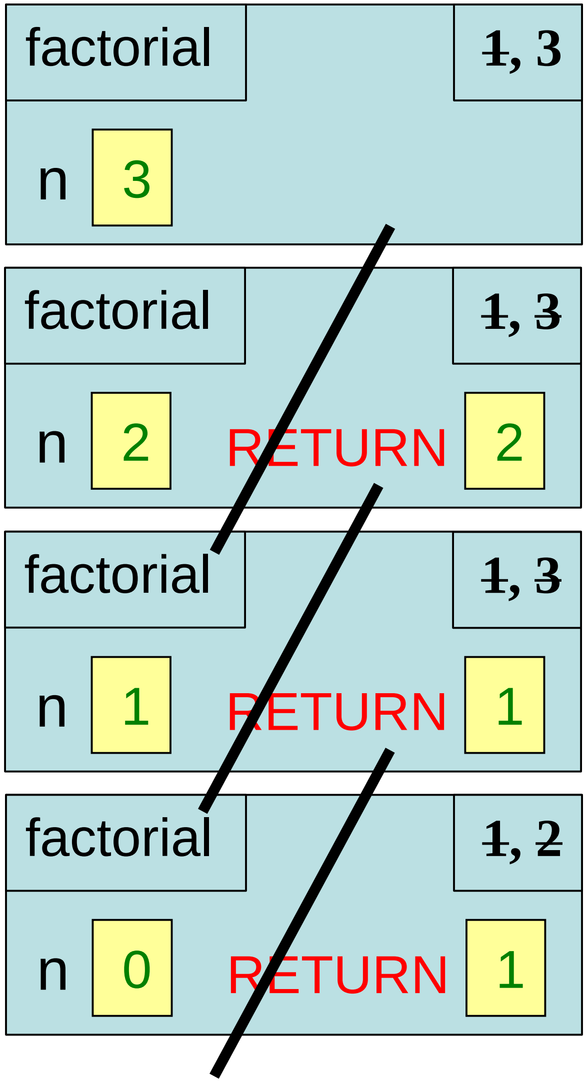

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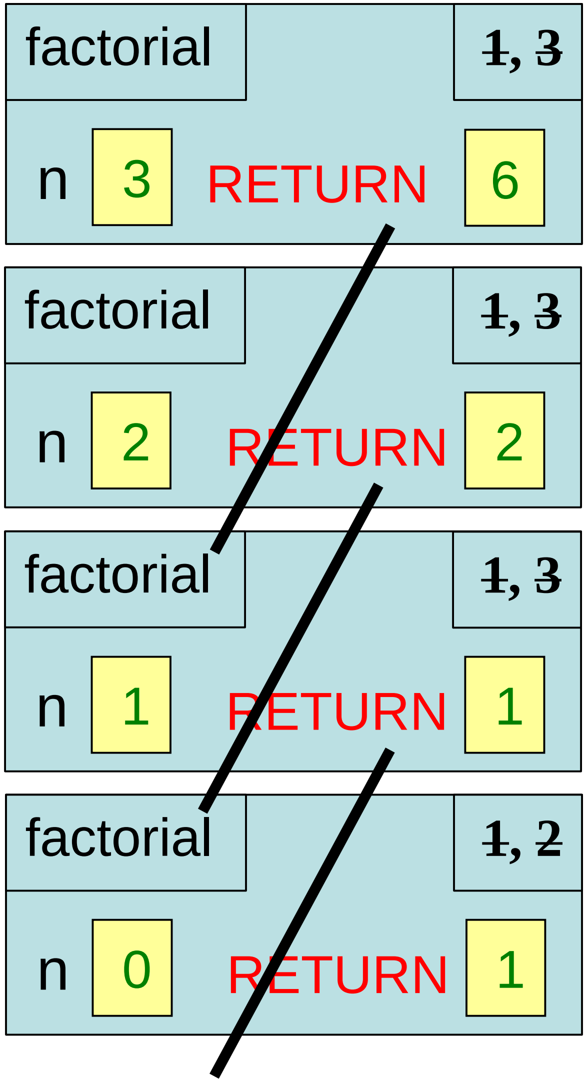

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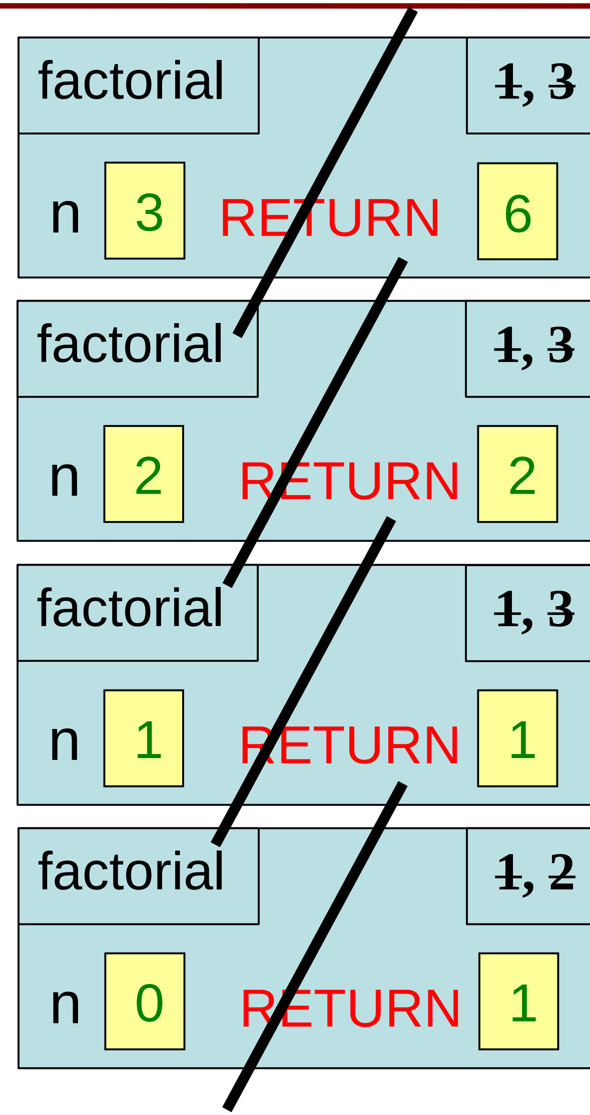

Example: Fibonnaci Sequence

- Sequence of numbers: 1, 1, 2, 3, 5, 8, 13, ...

  a_0 a_1 a_2 a_3 a_4 a_5 a_6

  - Get the next number by adding previous two
  - What is a_8?

A: $a_8 = 21$
B: $a_8 = 29$
C: $a_8 = 34$
D: None of these.

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A: $a_8 = 21$

B: $a_8 = 29$

C: $a_8 = 34$ **correct**

D: None of these.

Example: Fibonnaci Sequence

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  a_0 a_1 a_2 a_3 a_4 a_5 a_6

  - Get the next number by adding previous two
  - What is a_8?
- Recursive definition:
  - $a_n = a_{n\text{-}1} + a_{n\text{-}2}$ **Recursive Case**
  - $a_0 = 1$ **Base Case**
  - $a_1 = 1$ **(another) Base Case**

Why did we need two base cases this time?

Fibonacci as a Recursive Function

```
def fibonacci(n):
    """Returns: Fibonacci no. a_n
    Precondition: n ≥ 0 an int"""
    if n <= 1:
        return 1

    return (fibonacci(n-1)+
            fibonacci(n-2))
```

Base case(s)

Recursive case

Handles both base cases in one conditional.

Fibonacci as a Recursive Function

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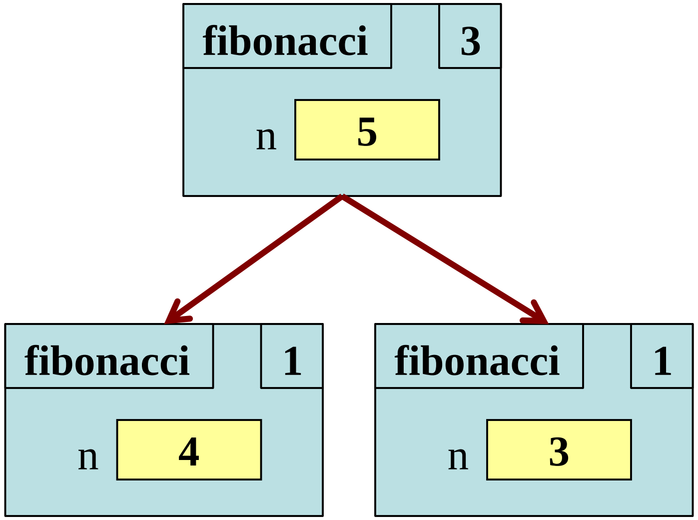


Recursion vs Iteration

- **Recursion** is ***provably equivalent*** to **iteration**
  - Iteration includes **for-loop** and **while-loop** (later)
  - Anything can do in one, can do in the other
- But some things are easier with recursion
  - And some things are easier with iteration
- Will **not** teach you when to choose recursion
- We just want you to ***understand the technique***

Recursion is best for Divide and Conquer

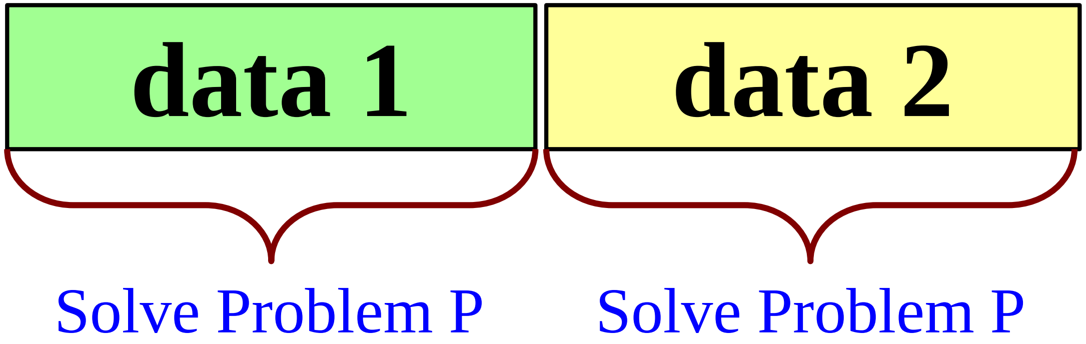
Goal: Solve problem P on a piece of data

data

Recursion is best for Divide and Conquer

Goal: Solve problem P on a piece of data

Idea: Split data into two parts and solve problem



Recursion is best for Divide and Conquer

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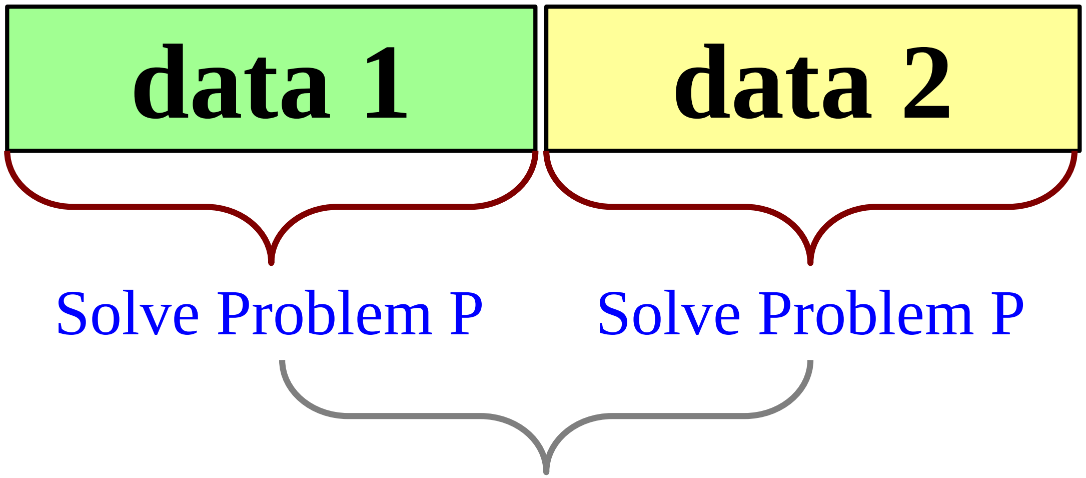
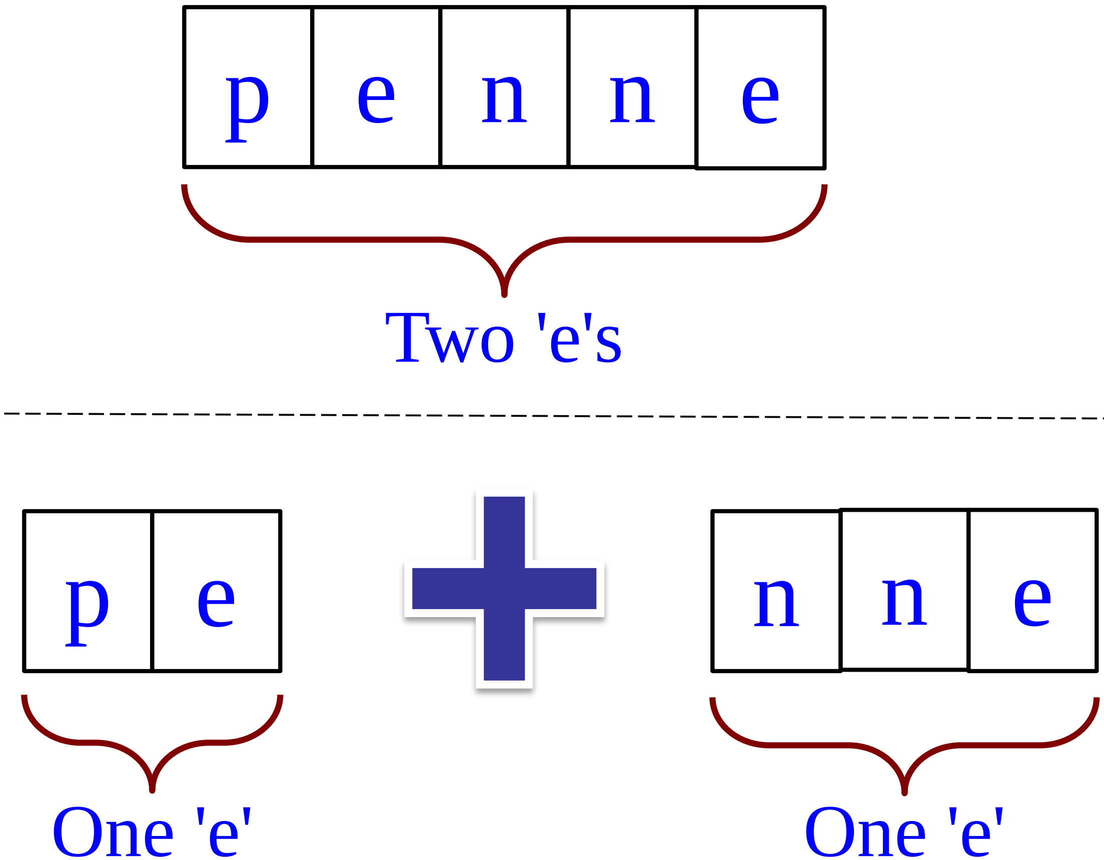


Combine Answer!

Divide and Conquer Example

Count the number of 'e's in a string:



Divide and Conquer Example

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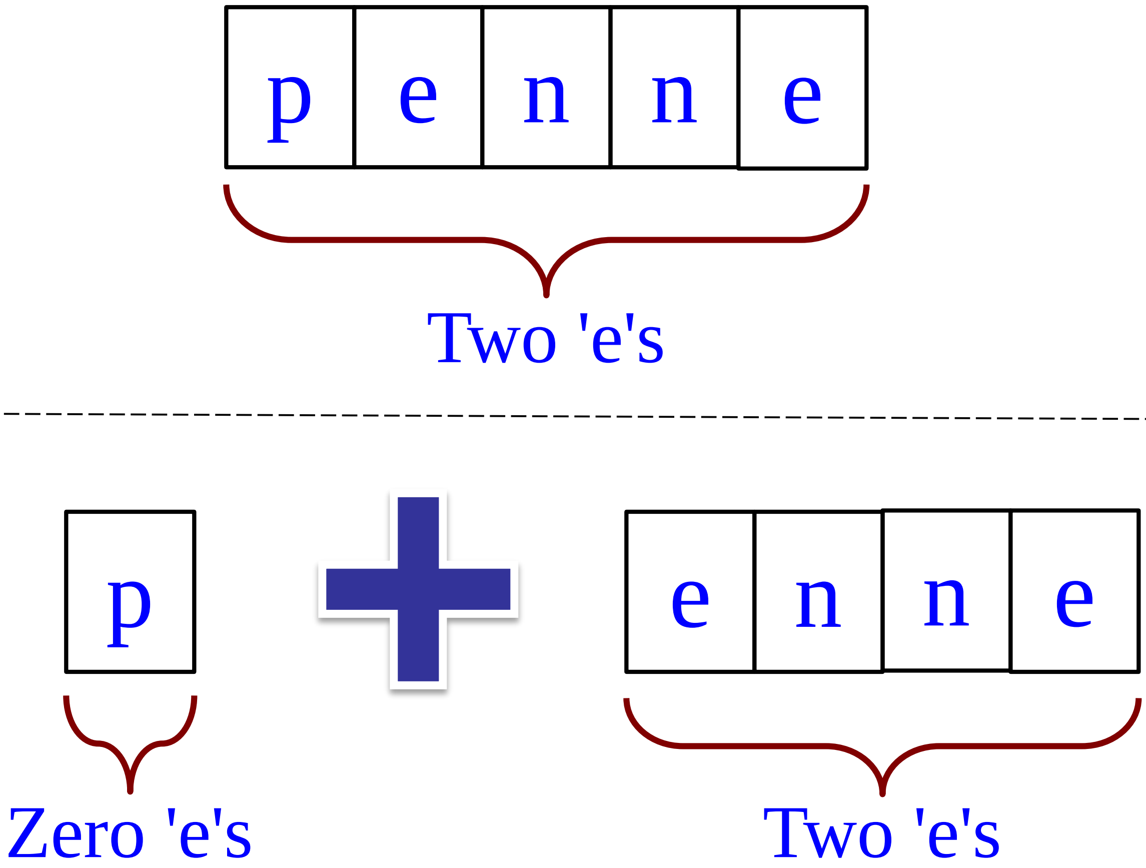


Three Steps for Divide and Conquer

1. Decide what to do on "small" data
   - Some data cannot be broken up
   - Have to compute this answer directly
2. Decide how to break up your data
   - Both "halves" should be smaller than whole
   - Often no wrong way to do this (next lecture)
3. Decide how to combine your answers
   - Assume the smaller answers are correct
   - Combining them should give bigger answer

Divide and Conquer Example

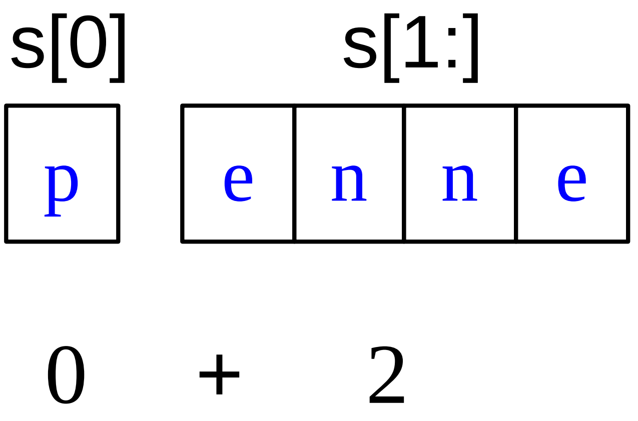
```
def num_es(s):
    """Returns: # of 'e's in s"""
    # 1. Handle small data
    if s == '':
        return 0
    elif len(s) == 1:
        return 1 if s[0] == 'e' else 0

    # 2. Break into two parts
    left = num_es(s[0])
    right = num_es(s[1:])

    # 3. Combine the result
    return left+right
```

"Short-cut" for

```
if s[0] == 'e':
    return 1
else:
    return 0
```



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p | e n n e

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p	e	n	n	e

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Divide and Conquer Example

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Base Case (step 1)

Recursive Case (steps 2 and 3)

Exercise: Remove Blanks from a String

```
def deblank(s):
    """Returns: s but with its blanks removed"""
```

1. Decide what to do on “small” data
   - If it is the **empty string**, nothing to do

```
if s == '':
    return s
```

   - If it is a **single character**, delete it if a blank

```
if s == ' ':    # There is a space here
    return ''  # Empty string
else:
    return s
```

Exercise: Remove Blanks from a String

```
def deblank(s):
    """Returns: s but with its blanks removed"""
```

2. Decide how to break it up

```
left = deblank(s[0])      # A string with no blanks
right = deblank(s[1:])   # A string with no blanks
```

3. Decide how to combine the answer

```
return left+right         # String concatenation
```

Putting it All Together

```
def deblank(s):
    """Returns: s w/o blanks"""
    if s == '':
        return s
    elif len(s) == 1:
        return '' if s[0] == ' ' else s

    left = deblank(s[0])
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    return left+right
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Handle small data

Break up the data

Combine answers

Putting it All Together

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Base Case

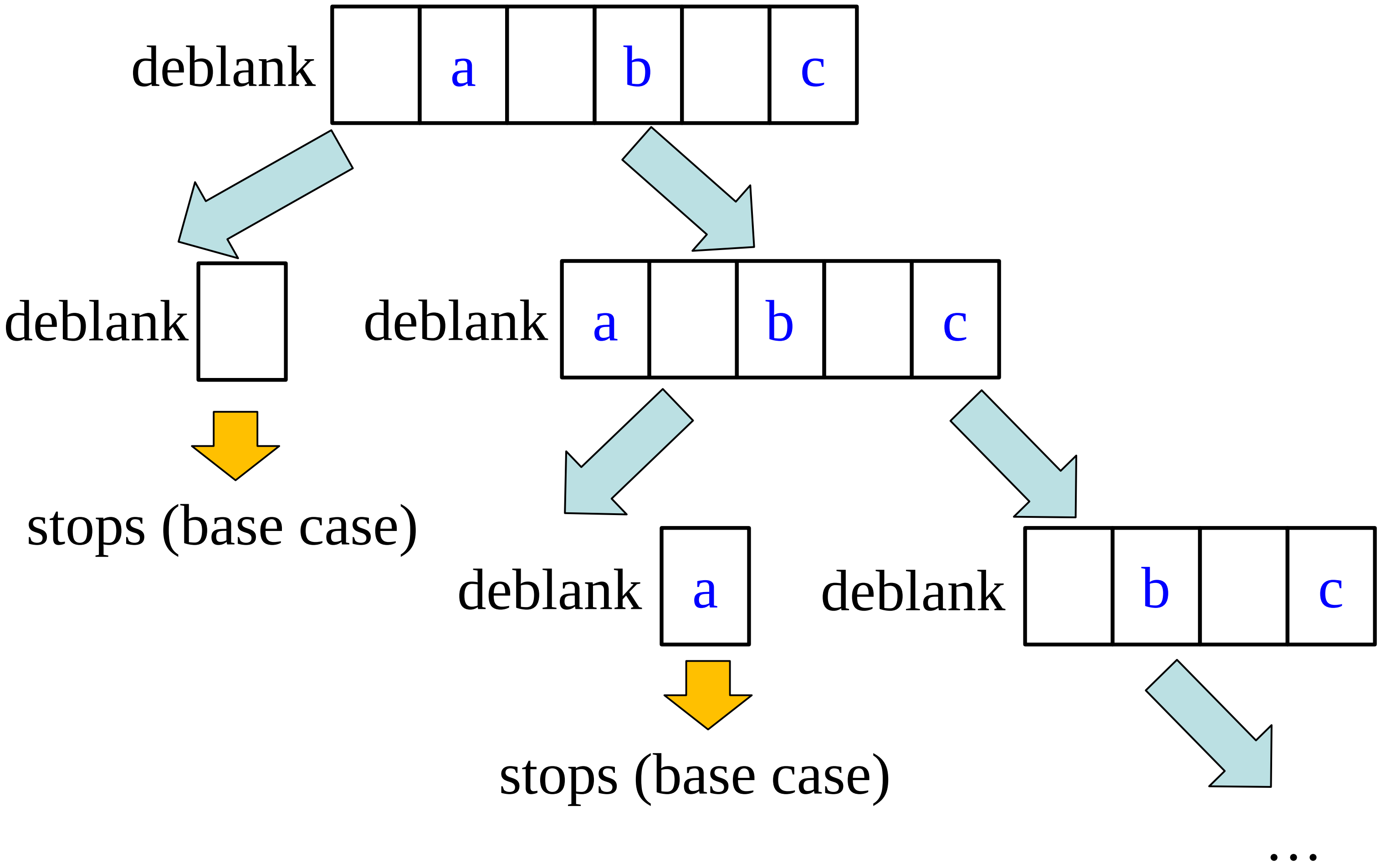
Recursive Case

Following the Recursion

deblank

	a		b		c

Following the Recursion



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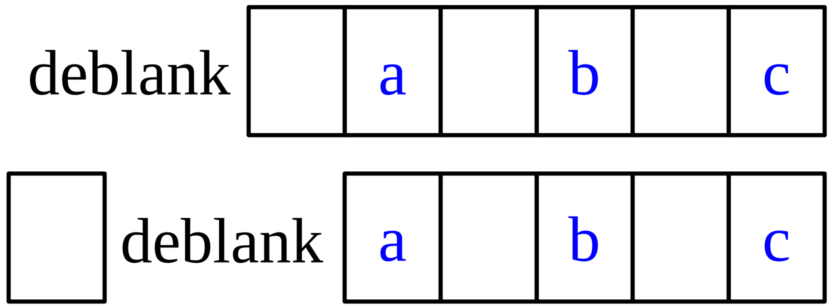

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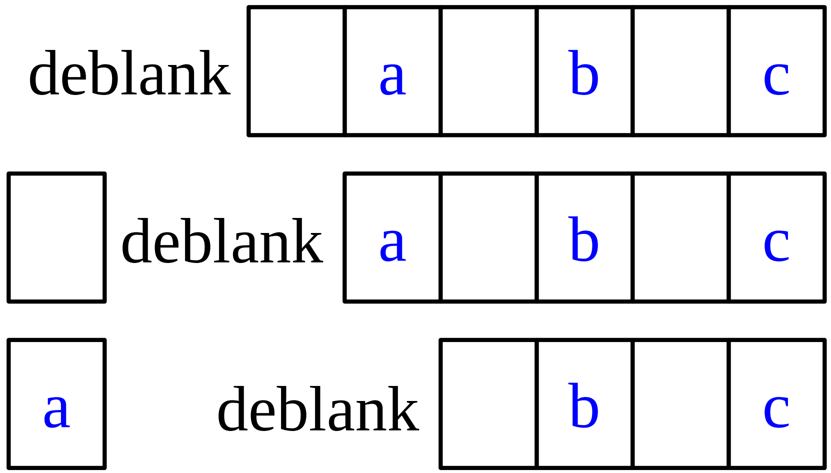

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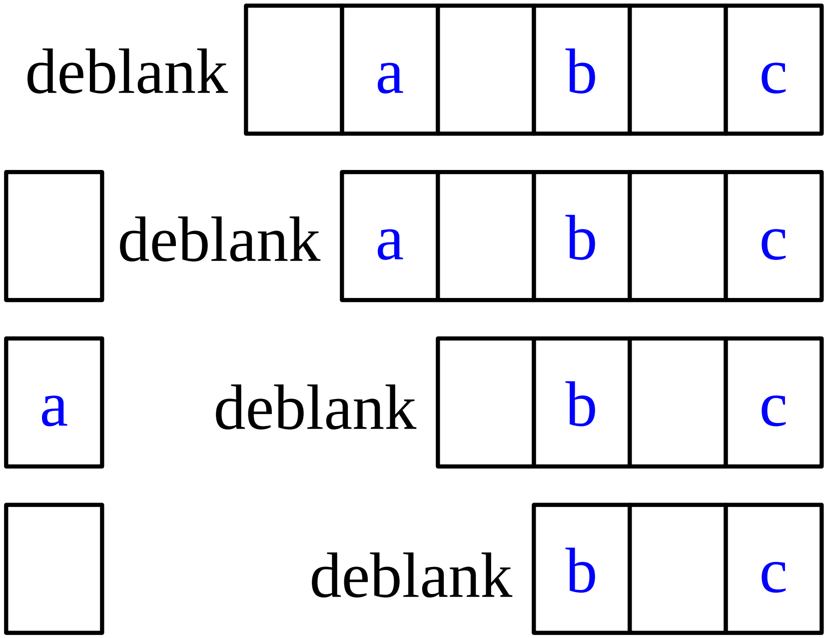

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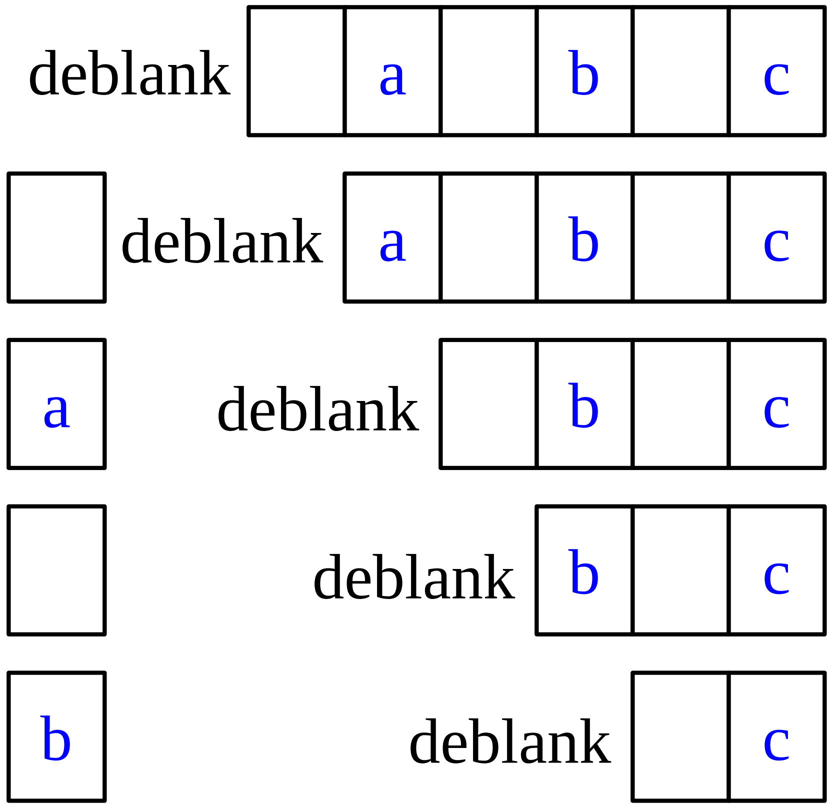

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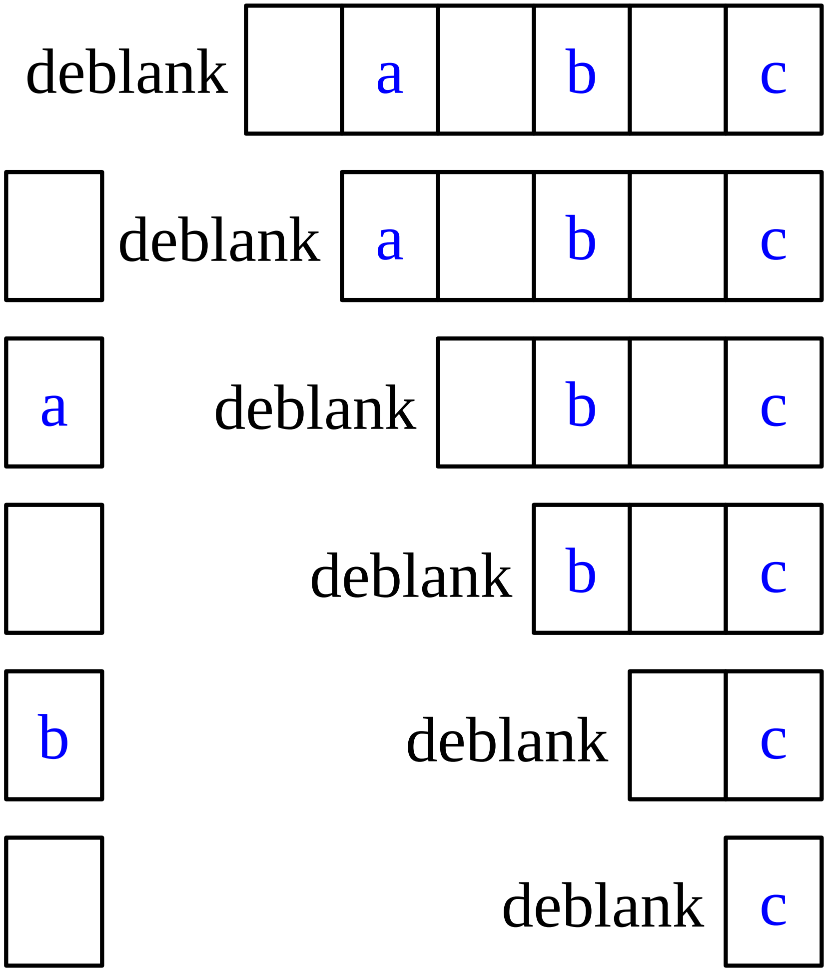

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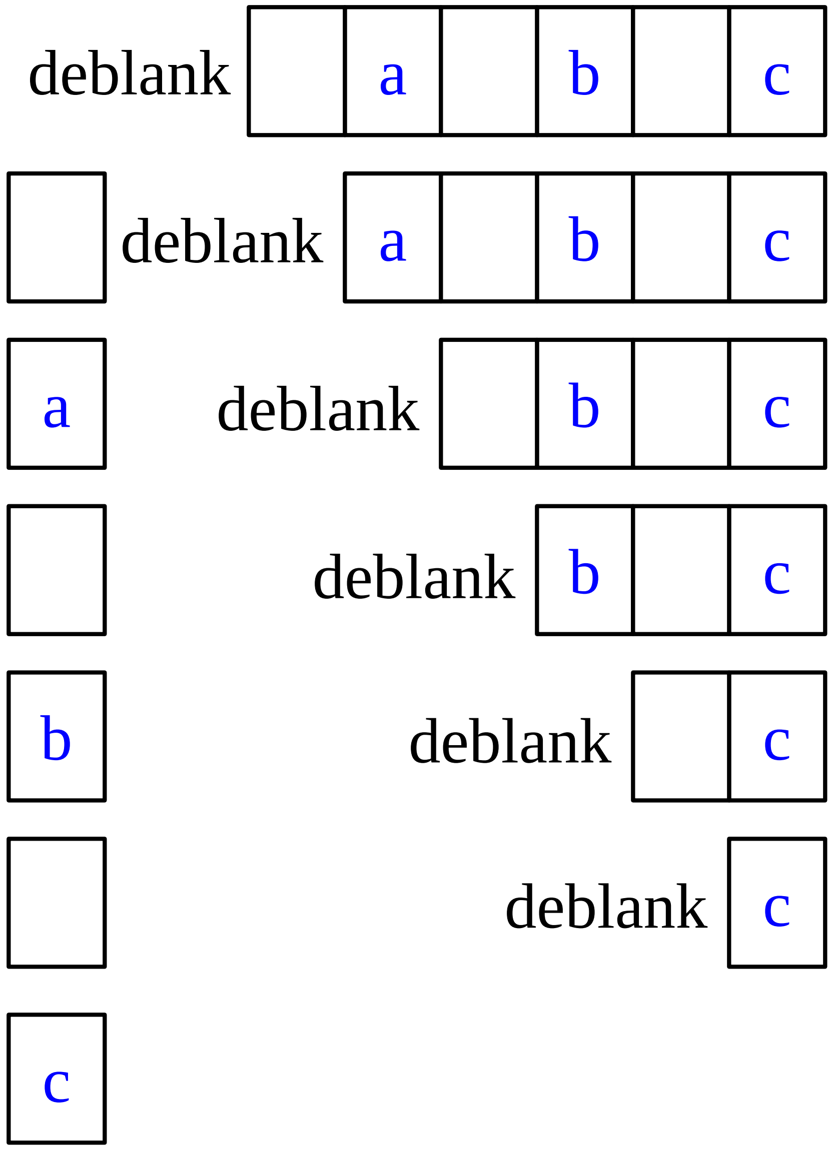

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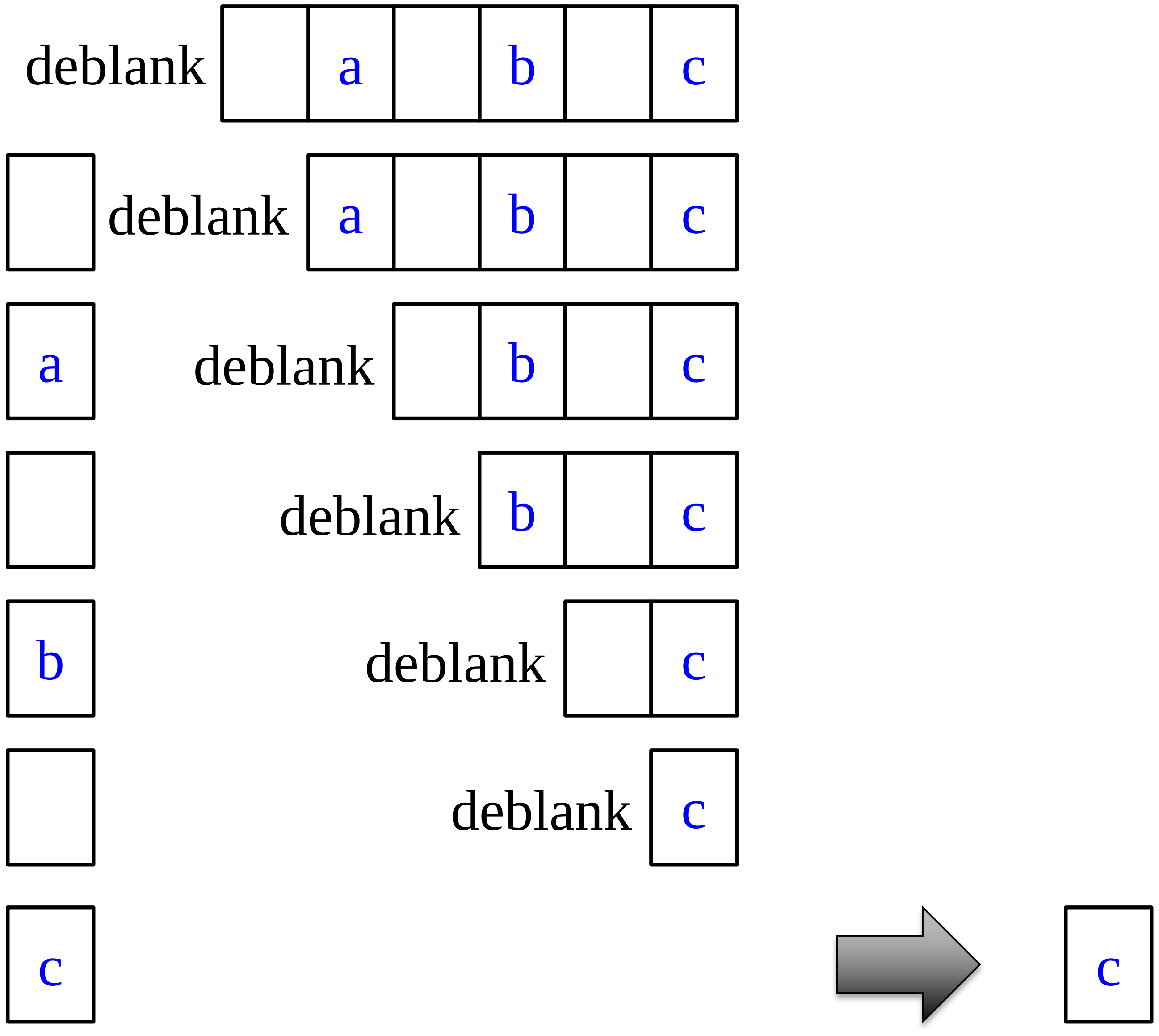
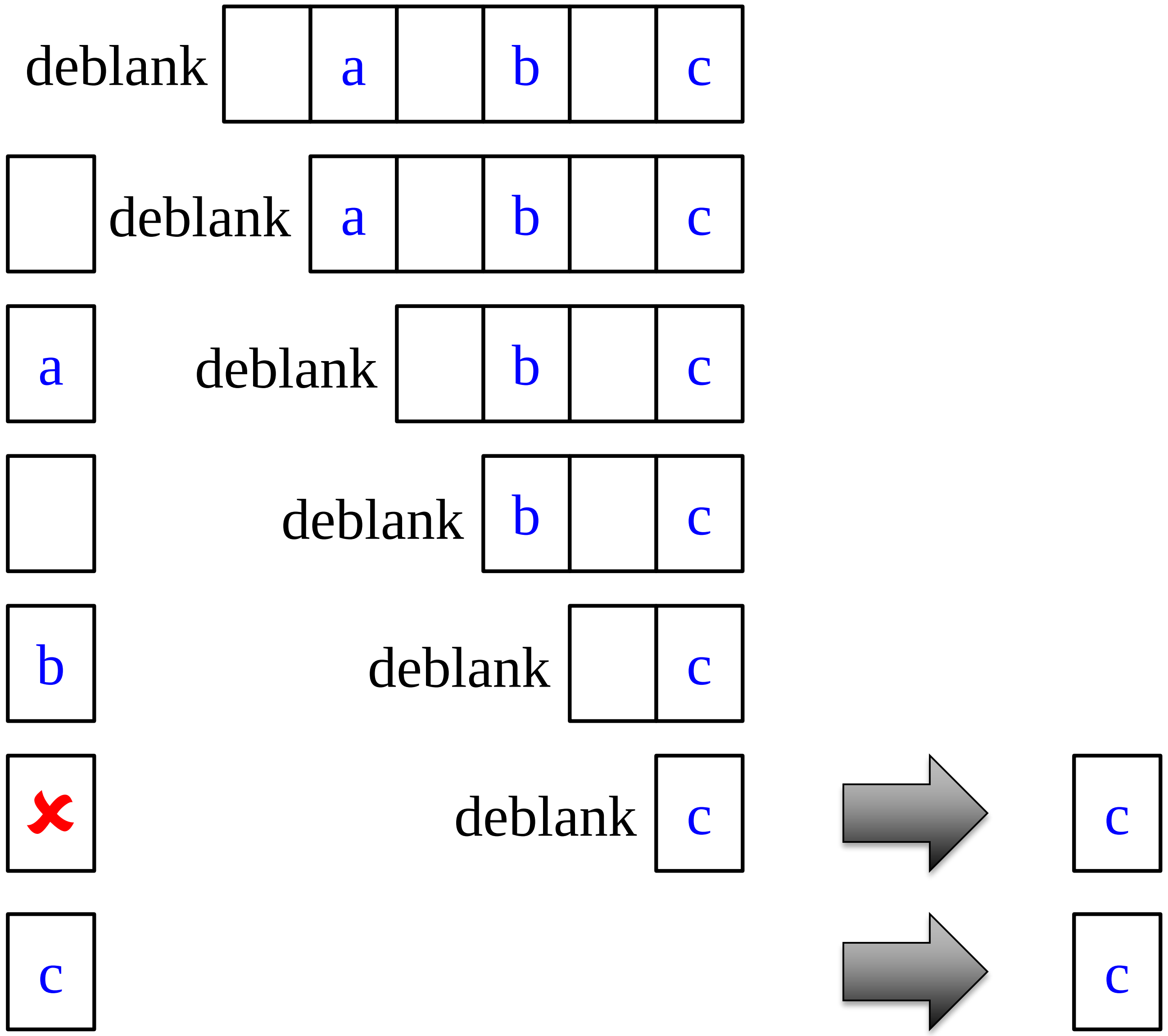
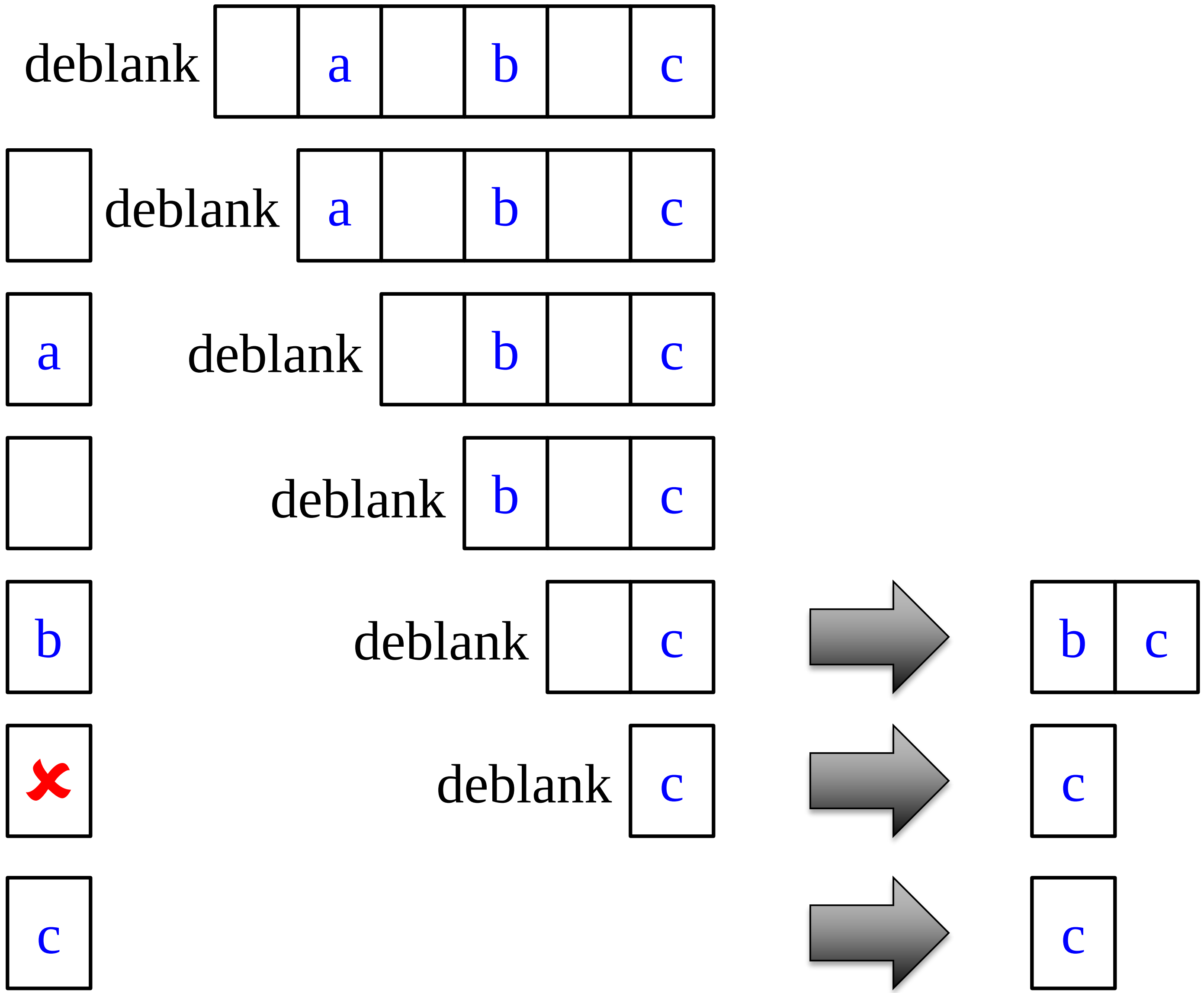


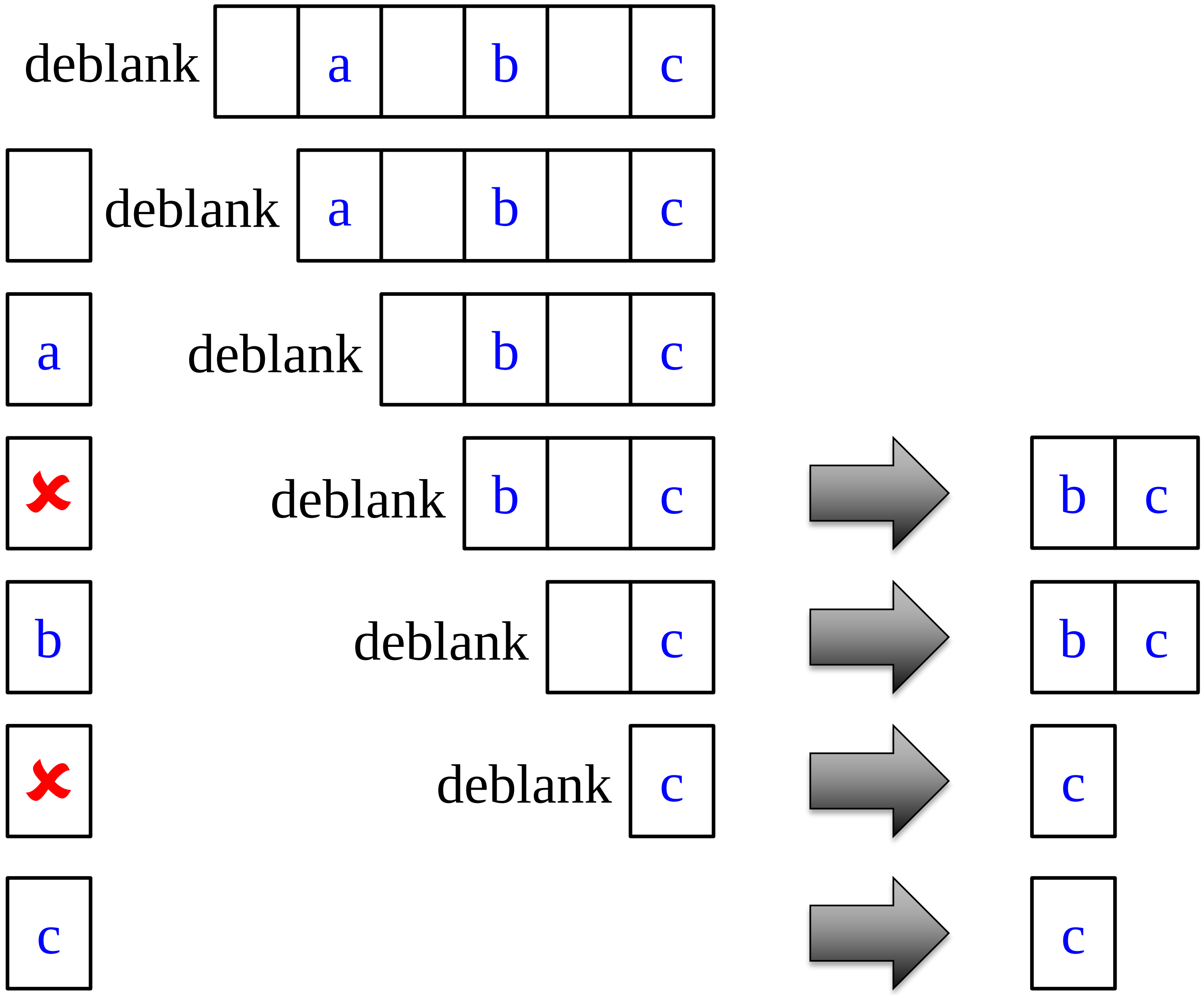
Following the Recursion



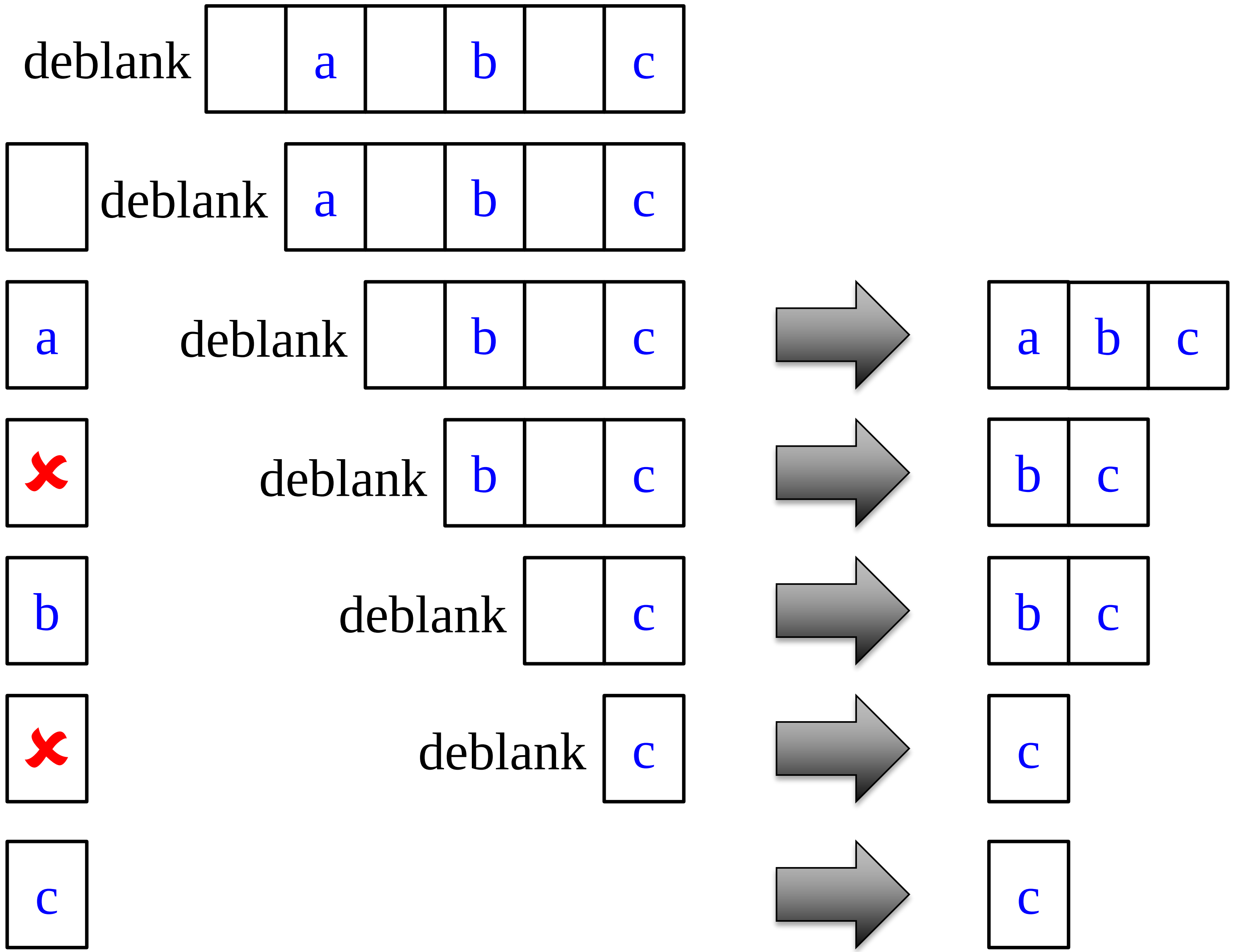
Following the Recursion



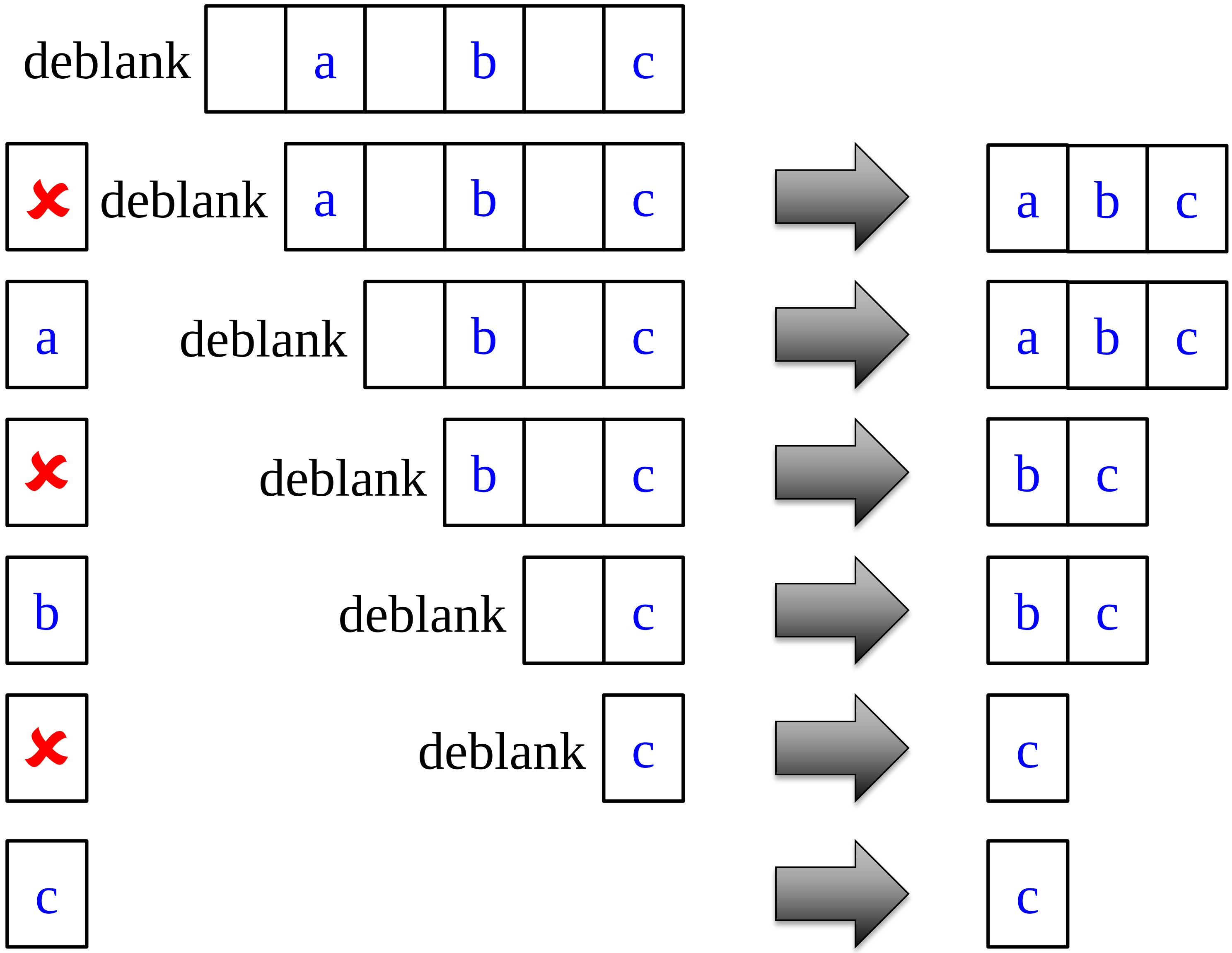
Following the Recursion



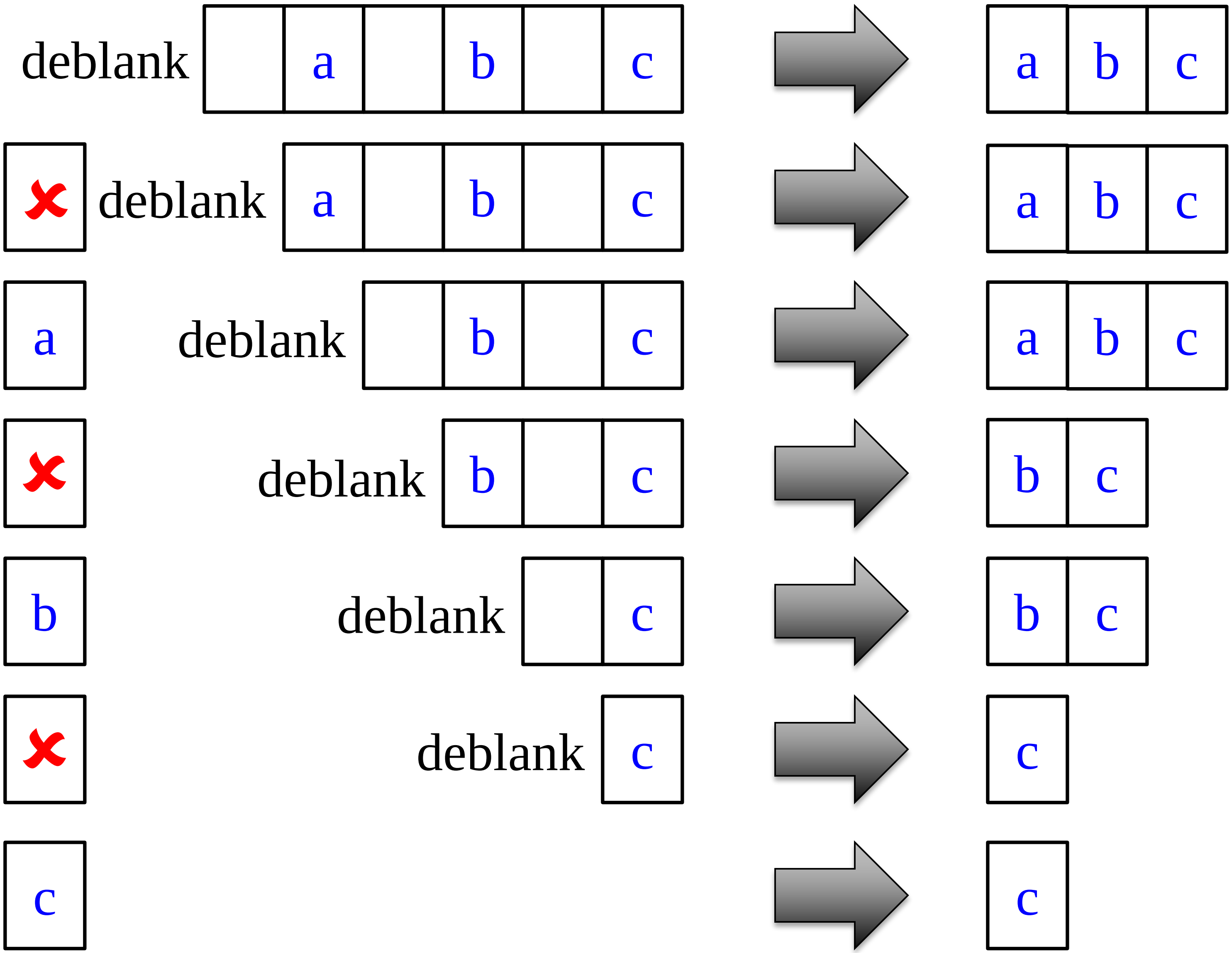
Following the Recursion



Following the Recursion

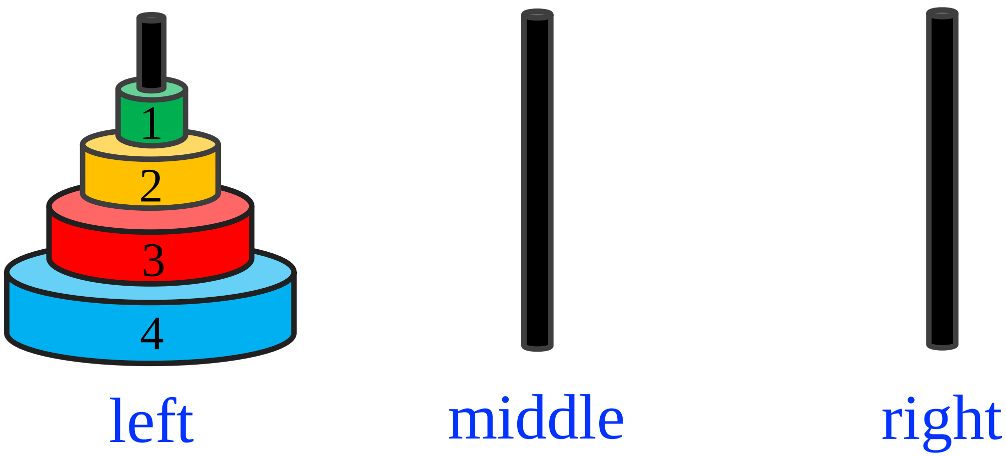


Following the Recursion



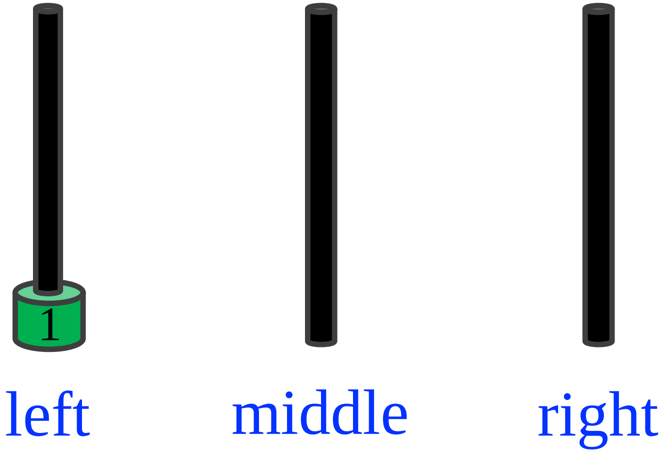
Tower of Hanoi

- Three towers: *left*, *middle*, and *right*
- *n* disks of unique sizes on *left*
- **Goal**: move all disks from *left* to *right*
- Cannot put a larger disk on top of a smaller disk



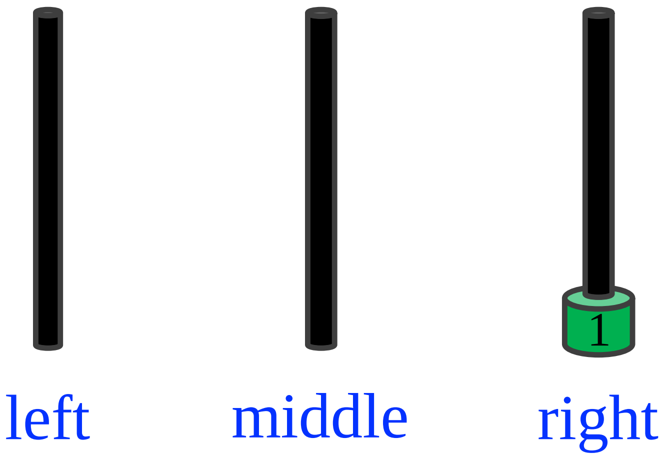
1 Disc

1. Move from *left* to *right*



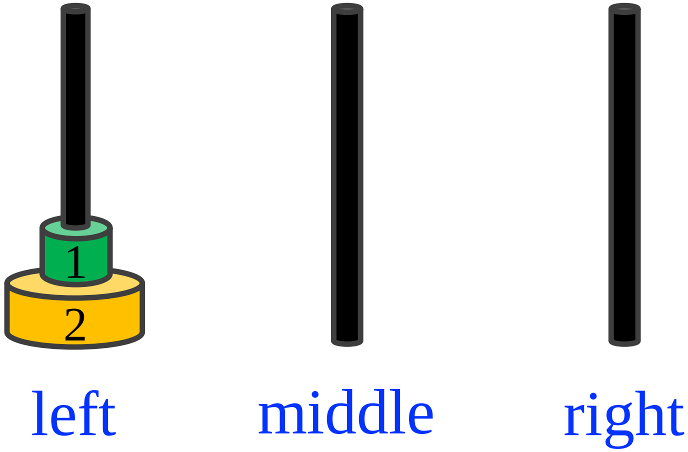
1 Disc

1. Move from *left* to *right*



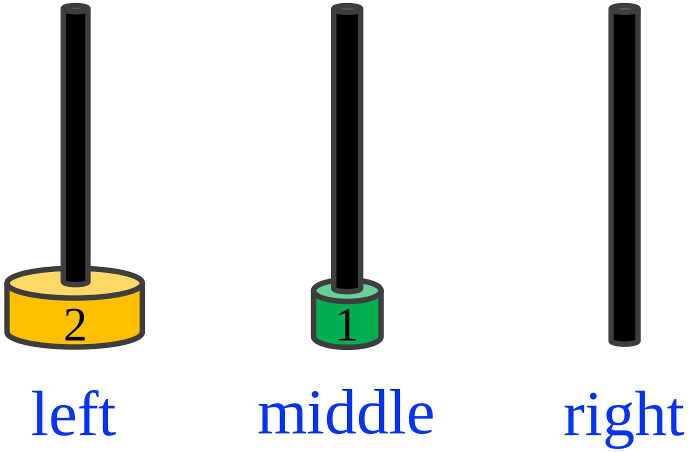
2 Discs

1. Move from *left* to *middle*

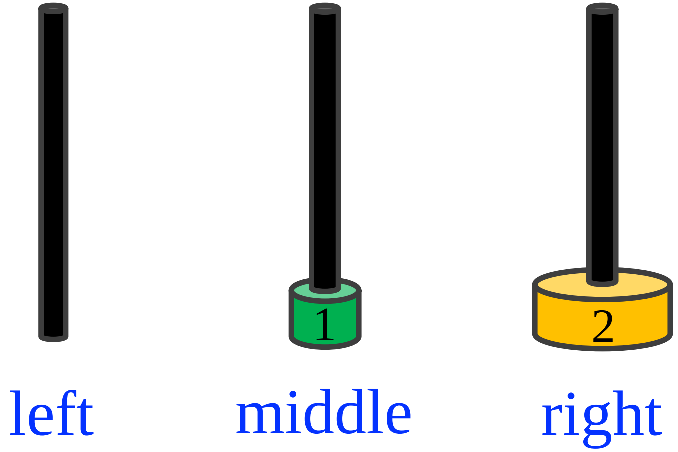


2 Discs

1. Move from *left* to *middle*
2. Move from *left* to *right*

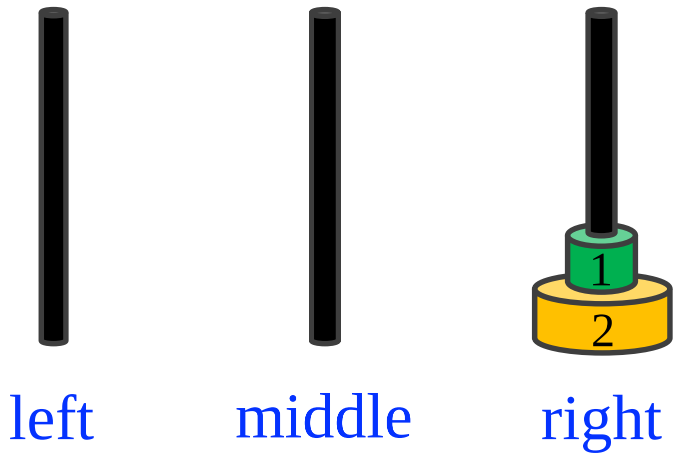


2 Discs



1. Move from *left* to *middle*
2. Move from *left* to *right*
3. Move from *middle* to *right*

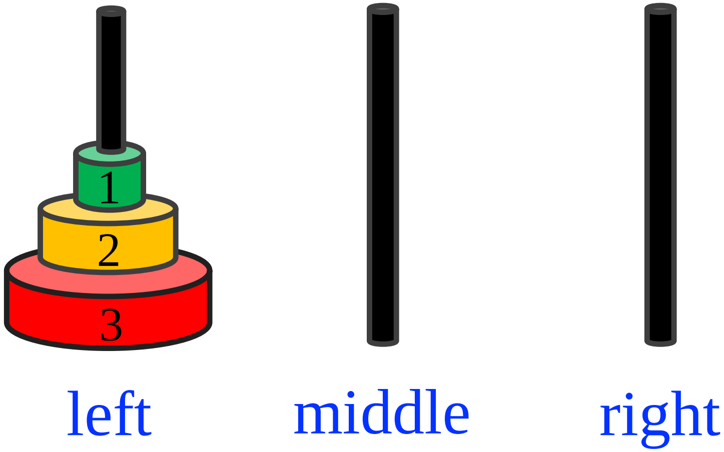
2 Discs



1. Move from *left* to *middle*
2. Move from *left* to *right*
3. Move from *middle* to *right*

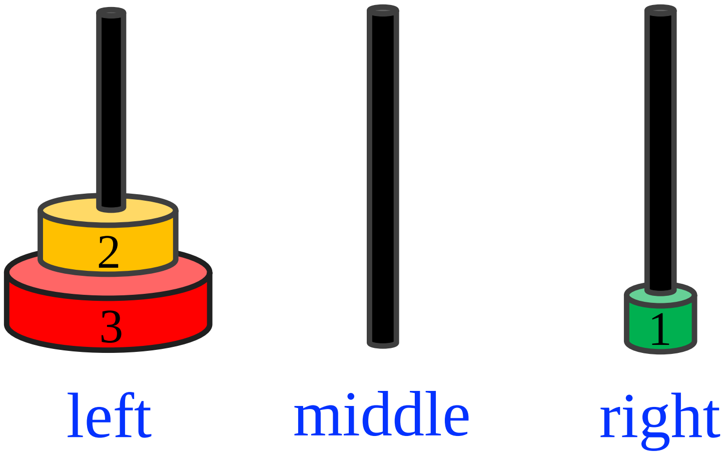
3 Discs

1. Move from *left* to *right*

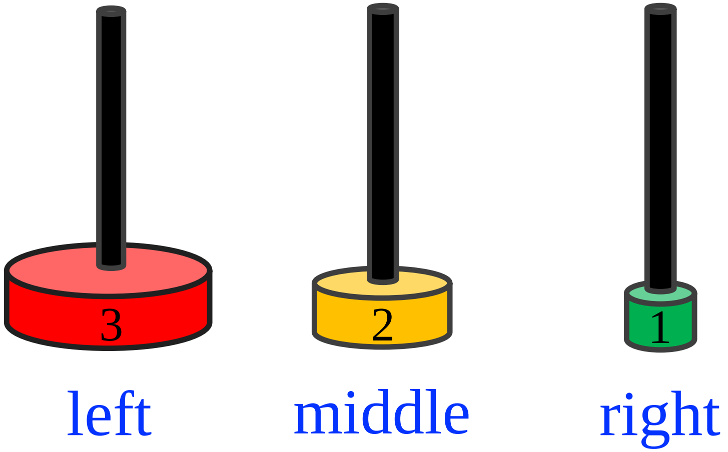


3 Discs

1. Move from *left* to *right*
2. Move from *left* to *middle*

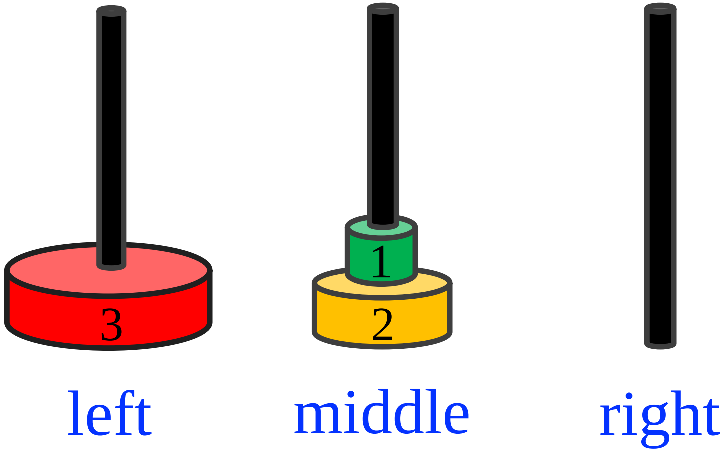


3 Discs



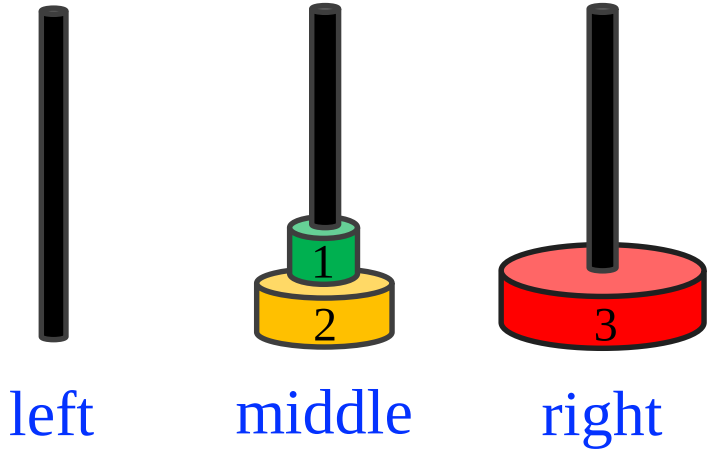
1. Move from *left* to *right*
2. Move from *left* to *middle*
3. Move from *right* to *middle*

3 Discs



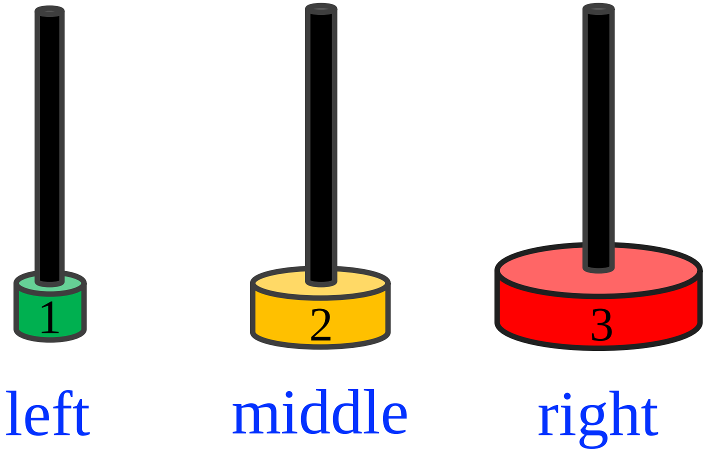
1. Move from *left* to *right*
2. Move from *left* to *middle*
3. Move from *right* to *middle*
4. Move from *left* to *right*

3 Discs



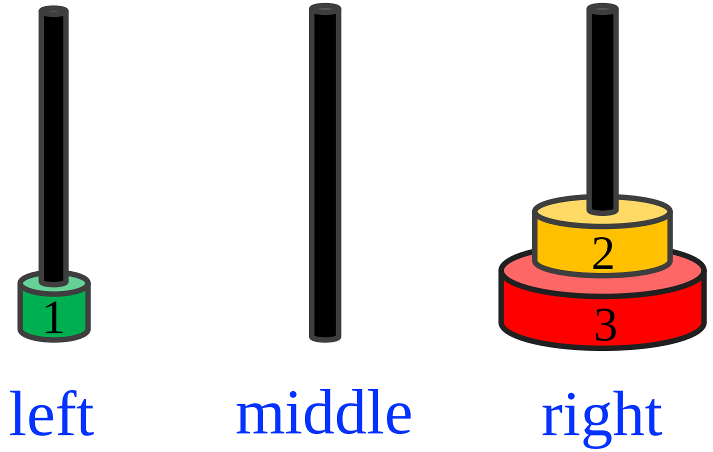
1. Move from *left* to *right*
2. Move from *left* to *middle*
3. Move from *right* to *middle*
4. Move from *left* to *right*
5. Move from *middle* to *left*

3 Discs



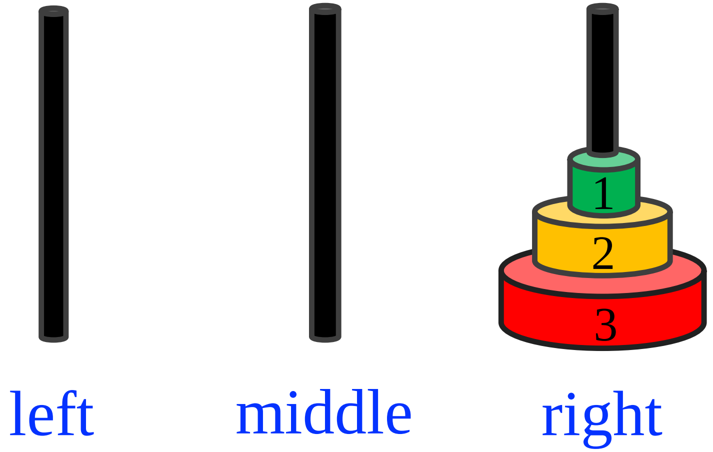
1. Move from *left* to *right*
2. Move from *left* to *middle*
3. Move from *right* to *middle*
4. Move from *left* to *right*
5. Move from *middle* to *left*
6. Move from *middle* to *right*

3 Discs



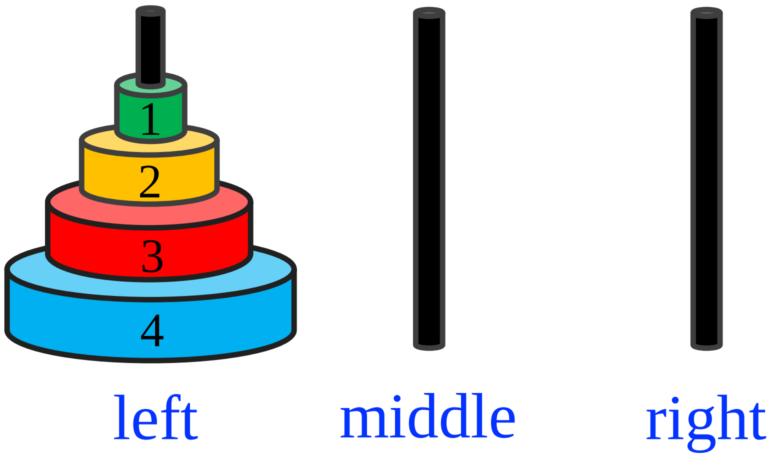
1. Move from *left* to *right*
2. Move from *left* to *middle*
3. Move from *right* to *middle*
4. Move from *left* to *right*
5. Move from *middle* to *left*
6. Move from *middle* to *right*
7. Move from *left* to *right*

3 Discs



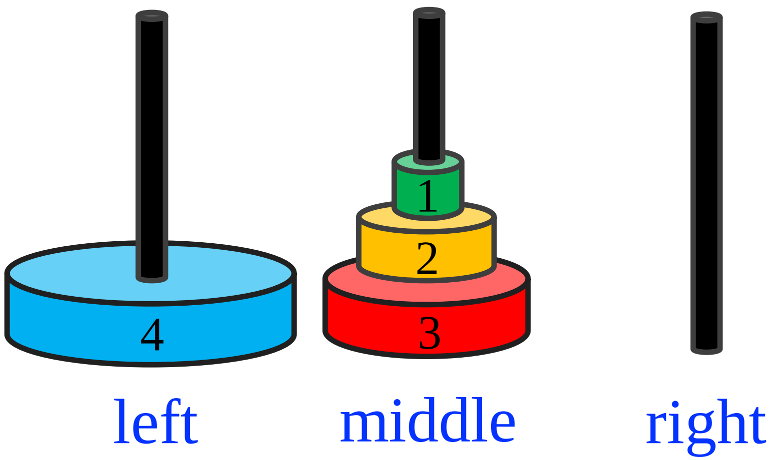
1. Move from *left* to *right*
2. Move from *left* to *middle*
3. Move from *right* to *middle*
4. Move from *left* to *right*
5. Move from *middle* to *left*
6. Move from *middle* to *right*
7. Move from *left* to *right*

4 Discs: High-level Idea

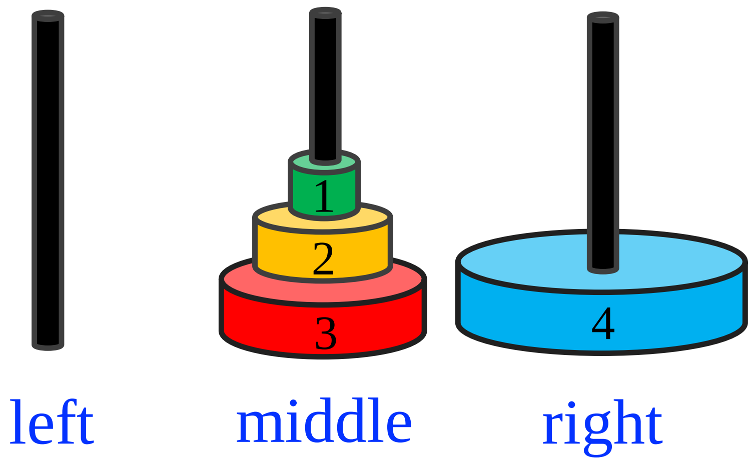


4 Discs: High-level Idea

- **Plan**: move top three disks from *left* to *middle*

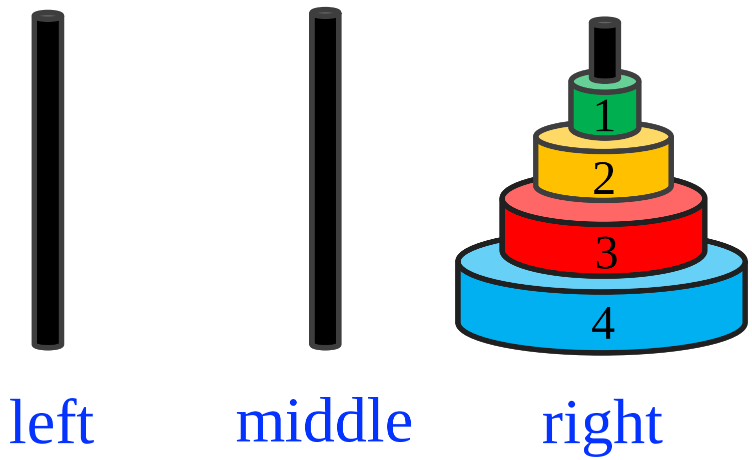


4 Discs: High-level Idea



- **Plan**: move top three disks from *left* to *middle*
- **Move**: largest disk from *left* to *right*

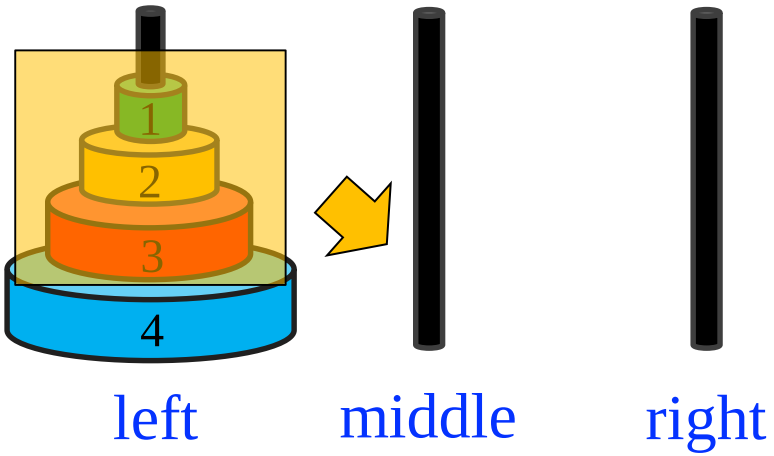
4 Discs: High-level Idea



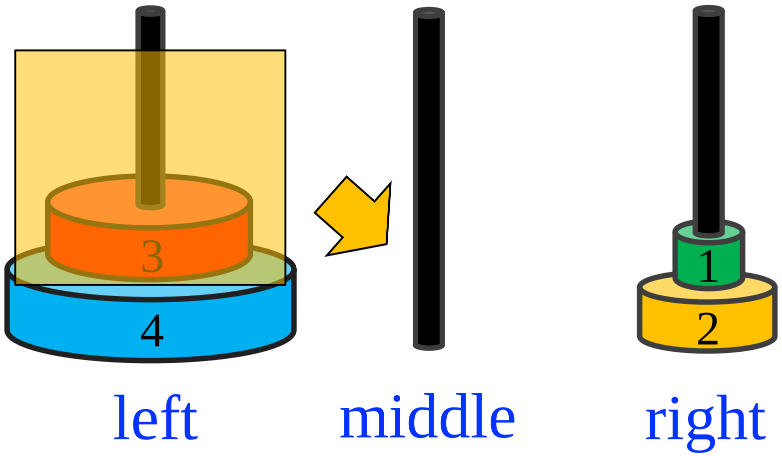
- **Plan**: move top three disks from *left* to *middle*
- **Move**: largest disk from *left* to *right*
- **Plan**: move top three disks from *middle* to *right*

4 Discs: High-level Idea

- **Plan**: move disks 1, 2, and 3 from *left* to *middle*

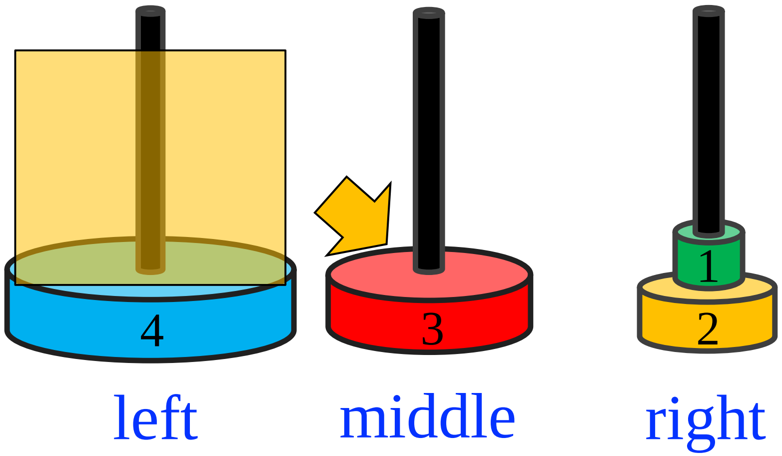


4 Discs: High-level Idea



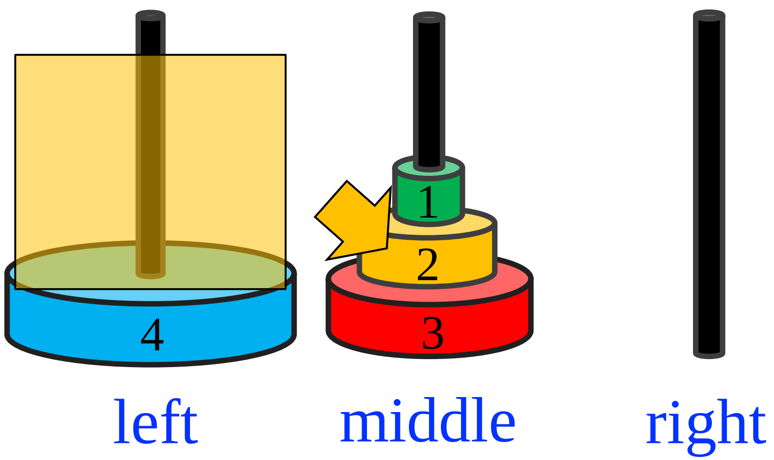
- **Plan**: move disks 1, 2, and 3 from *left* to *middle*
  - **Plan**: move disks 1 and 2 from *left* to *right*

4 Discs: High-level Idea



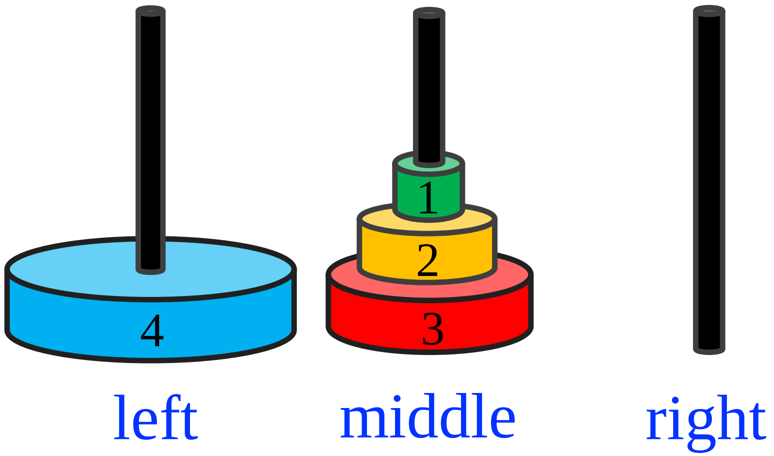
- **Plan**: move disks 1, 2, and 3 from *left* to *middle*
  - **Plan**: move disks 1 and 2 from *left* to *right*
  - **Move**: disk 3 from *left* to *right*

4 Discs: High-level Idea



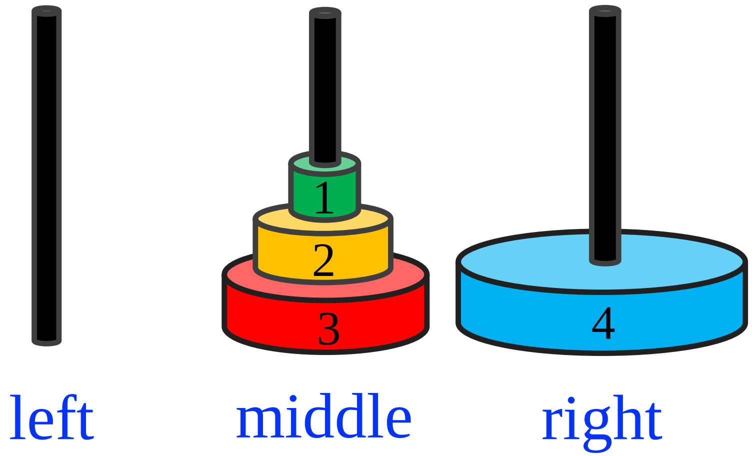
- **Plan**: move disks 1, 2, and 3 from *left* to *middle*
  - **Plan**: move disks 1 and 2 from *left* to *right*
  - **Move**: disk 3 from *left* to *right*
  - **Plan**: move disks 1 and 2 from *right* to *middle*

4 Discs: High-level Idea



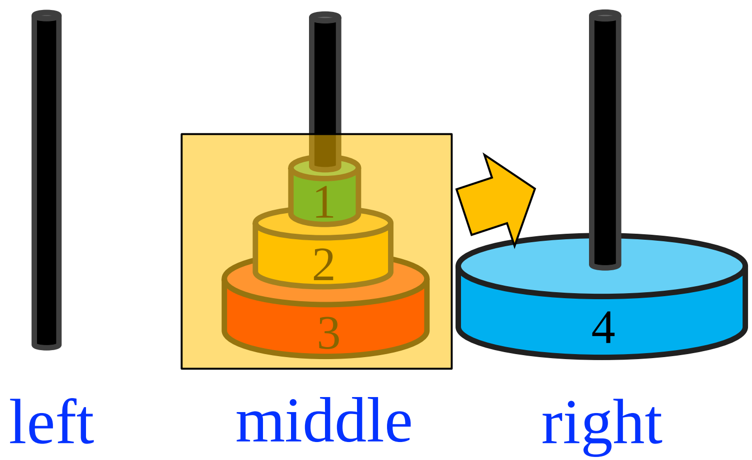
- **Plan**: move disks 1, 2, and 3 from *left* to *middle*
  - **Plan**: move disks 1 and 2 from *left* to *right*
  - **Move**: disk 3 from *left* to *right*
  - **Plan**: move disks 1 and 2 from *right* to *middle*

4 Discs: High-level Idea



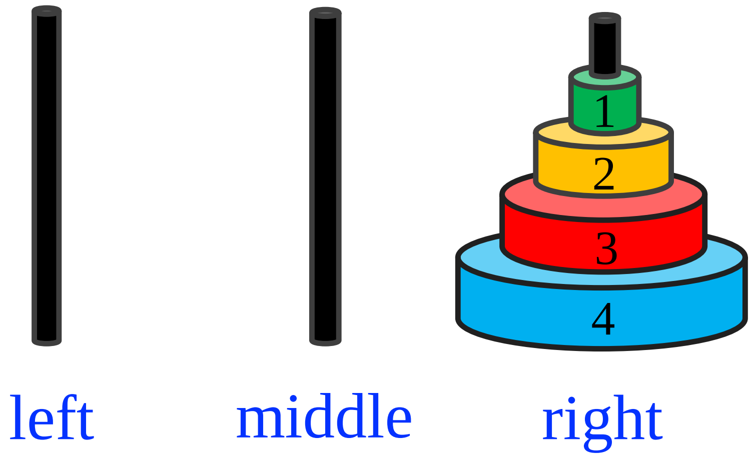
- **Plan**: move disks 1, 2, and 3 from *left* to *middle*
  - **Plan**: move disks 1 and 2 from *left* to *right*
  - **Move**: disk 3 from *left* to *right*
  - **Plan**: move disks 1 and 2 from *right* to *middle*
- **Move**: disk 4 from *left* to *right*

4 Discs: High-level Idea



- **Plan**: move disks 1, 2, and 3 from *left* to *middle*
  - **Plan**: move disks 1 and 2 from *left* to *right*
  - **Move**: disk 3 from *left* to *right*
  - **Plan**: move disks 1 and 2 from *right* to *middle*
- **Move**: disk 4 from *left* to *right*
- **Plan**: move disks 1, 2, and 3 from *middle* to *right*

4 Discs: High-level Idea



- **Plan**: move disks 1, 2, and 3 from *left* to *middle*
  - **Plan**: move disks 1 and 2 from *left* to *right*
  - **Move**: disk 3 from *left* to *right*
  - **Plan**: move disks 1 and 2 from *right* to *middle*
- **Move**: disk 4 from *left* to *right*
- **Plan**: move disks 1, 2, and 3 from *middle* to *right*

Observation: Plans within a Plan

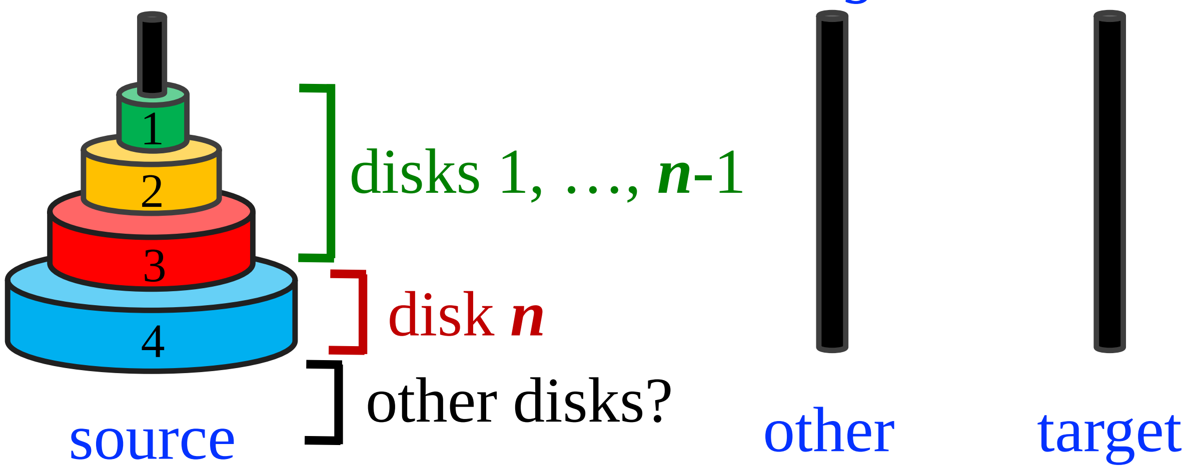
High-level plan

Low-level plan

- **Plan**: move disks 1, 2, and 3 from *left* to *middle*
  - **Plan**: move disks 1 and 2 from *left* to *right*
  - **Move**: disk 3 from *left* to *right*
  - **Plan**: move disks 1 and 2 from *right* to *middle*
- **Move**: disk 4 from *left* to *right*
- **Plan**: move disks 1, 2, and 3 from *middle* to *right*

General Pattern

To move ***n*** disks from *source* to *target:*



(*source*, *other*, and *target* can be any permutation of *left*, *middle* and *right*)

1. **Plan**: move disks 1, …, ***n***-1 from *source* to *other*
2. **Move**: disk ***n*** to from *source* to *target*
3. **Plan**: move disks 1, …, ***n***-1 from *other* to *target*