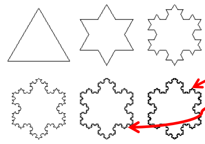
21. Recursion

Recursive Tiling
Recursive Evaluation of n!
Tracking a Recursive Function Call
Random Mondrian

What is Recursion?

A function is recursive if it calls itself.

A pattern is recursive if it is defined in terms of itself.



I can tell you what this is in terms of what that is.

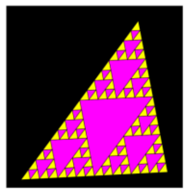
The Concept of Recursion Is Hard But VERY Important

Teaching Plan:

- Develop a recursive triangle-tiling procedure informally.
- Fully implement (in Python) a recursive rectangle-tiling procedure.
- Fully implement a recursive function for n!
- Fully implement a recursive function for sorting (in a later lecture).

Recursive Graphics

We will develop a graphics procedure that draws this:

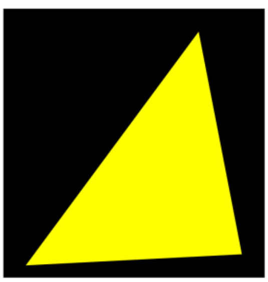


We are tiling a triangle with increasingly smaller triangles.

The procedure will call itself.

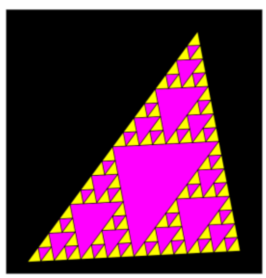
Tiling a Triangle

We start with one big triangle:



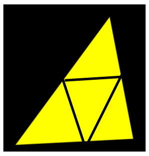
Tiling a Triangle

And are to end up with this:

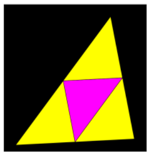


Requires Repetition

Given a yellow triangle

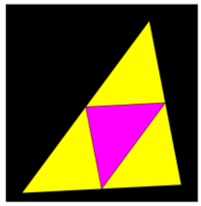


Define the inner triangle and the 3 corner triangles



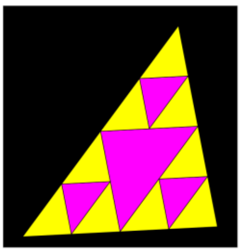
Color the inner triangle and repeat the process on the 3 corner triangles

"Repeat the Process"

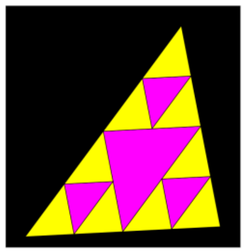


Visit every yellow triangle and replace it with this

We Get This...

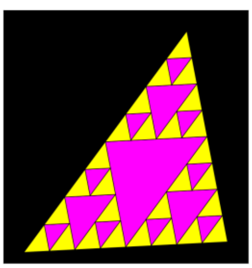


"Repeat the Process"

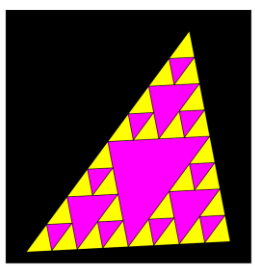


Visit every yellow triangle and replace it with

We Get This...

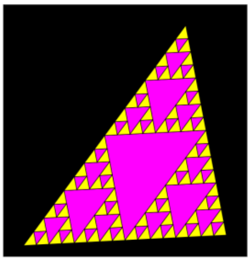


"Repeat the Process"



Visit every yellow triangle and replace it with

We Get This...

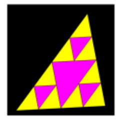


Etc.

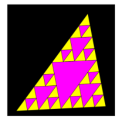
The Notion of Level

A level-0 tiling

A level-1 tiling



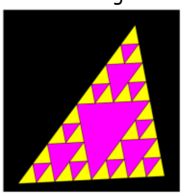
A level-2 tiling

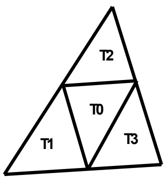


A level-3 tiling

The Connection Between Levels

A level-3 tiling

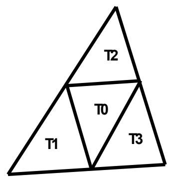




A level-2 tiling

To display a level-3 tiling you do this:
- display the inner triangle T0
- display a level-2 tiling of corner triangles T1, T2, and T3

The Connection Between Levels

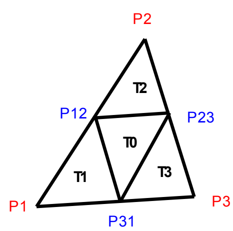


To display an level-L tiling you do this:
- display the inner triangle T0
- display an level-(L-1) tiling of triangles T1, T2, and T3

Some Tools to Pull This Off

```
class Point(object):
    def __init__(self,x,y):
        self.x = x
        self.y = y

    def Mid(self,other):
        """ Returns a point that encodes
        the  midpoint of theline segment
        that connects the Point self and
        the Point other.
        """
```

Midpoints from Vertices

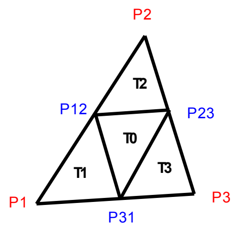



```
P12 = P1.Mid(P2)
P23 = P2.Mid(P3)
P31 = P3.Mid(P1)
```

Some Tools to Pull This Off

```
def DrawTriangle(P1,P2,P3,c):
     """ Draws a triangle with vertices
     P1, P2, and P3 and FillColor c

     PreC: P1, P2, and P3 are points
     and c is a rgb list.
     """
```

Coloring the Inner Triangle




```
DrawTriangle(P12,P23,P31,MAGENTA)
```

The Overall Procedure

```
def Tile(P1,P2,P3,L):
   if L ==0:
     # Base case. Draw a yellow triangle
     DrawTriangle(P1,P2,P3,YELLOW)
   else:
     # Compute side midpoints P12,P23,P31

     # Color inner triangle MAGENTA

     # Draw level L-1 tilings of T1,T2,T3

```

The Overall Procedure

```
# Compute side midpoints P12,P23,P31

   P12 = P1.Mid(P2)
   P23 = P2.Mid(P3)
   P31 = P3.Mid(P1)
```

The Overall Procedure

```
# Color the inner triangle magenta

   DrawTriangle(P12,P23,P31,MAGENTA)
```

The Overall Procedure

```
# Draw level-(L-1) tilings of T1, T2, T3

   Tile(P1,P31,P12,L-1)
   Tile(P2,P12,P23,L-1)
   Tile(P3,P23,P31,L-1)
```

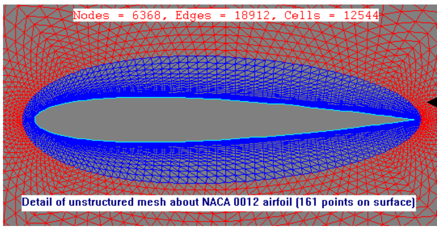
These are the recursive calls.

The Overall Procedure

```
def Tile(P1,P2,P3,L):
   if L ==0:
     # Base case. Draw a yellow triangle
     DrawTriangle(P1,P2,P3,YELLOW)
   else:
     # Compute side midpoints P12,P23,P31
     # Color inner triangle MAGENTA
     # Draw level L-1 tilings of T1,T2,T3
     Tile(P1,P31,P12,L-1)
     Tile(P2,P12,P23,L-1)
     Tile(P3,P23,P31,L-1)
```

A Note on Chopping up a Region into Triangles…

It is Important!



An Area Of Interest

Step One in simulating flow around an airfoil is to generate a triangular mesh and (say) estimate the velocity at each little triangle using physics and math.

Next Up

A Non-Graphics Example of Recursion: The Factorial Function

Recursive Evaluation of Factorial

Recall the factorial function:

```
def F(n):
    x = 1
    for k in range(1,n+1):
       x = x*k
    return x
```

5! = 1x2x3x4x5

Recursive Evaluation of Factorial

Q. How would you compute 6! given that you have computed 5! = 120 ?

A. 6! = 120 x 6

5! = 1x2x3x4x5

Recursive Evaluation of Factorial

```
def F(n):
    if n<=1:
        return 1
    else:
        a = F(n-1)
        return n*a
```

How does this work?

Executing F(3)

```
m = 3
x = F(m)
print x
```

```
m --> 3
x -->
```

We are in the calling script

Executing F(3)

```
m = 3
x = F(m)
print
```

```
def F(n):
  if n<=1:
    return 1
  else:
    a = F(n-1)
    return n*a
```

```
m --> 3
x -->
```

```
n --> 3
a -->
return
```

The function F is called with argument 3. We open up a call frame.

Executing F(3)

```
m = 3
x = F(m)
print x
```

```
def F(n):
  if n<=1:
    return 1
  else:
    a = F(n-1)
    return n*a
```

```
m --> 3
x -->
```

```
n --> 3
a -->
return
```

We encounter a function call. F is called with argument equal to 2.

Executing F(3)

```
m = 3
x = F(m)
print x
```

```
def F(n):
  if n<=1:
    return 1
  else:
    a = F(n-1)
    return n*a
```

```
def F(n):
  if n<=1:
    return 1
  else:
    a = F(n-1)
    return n*a
```

```
m --> 3
x -->
```

```
n --> 3
a -->
return
```

```
n --> 2
a -->
return
```

We open up a call frame.

Executing F(3)

```
m = 3
x = F(m)
print x
```

```
def F(n):
  if n<=1:
    return 1
  else:
    a = F(n-1)
    return n*a
```

```
def F(n):
  if n<=1:
    return 1
  else:
    a = F(n-1)
    return n*a
```

```
m --> 3
x -->
```

```
n --> 3
a -->
return
```

```
n --> 2
a -->
return
```

We encounter a function call. F is called with argument 1

Executing F(3)

```
m = 3
x = F(m)
print x
```

```
def F(n):
  if n<=1:
    return 1
  else:
    a = F(n-1)
    return n*a
```

```
def F(n):
  if n<=1:
    return 1
  else:
    a = F(n-1)
    return n*a
```

```
def F(n):
  if n<=1:
    return 1
  else:
    a = F(n-1)
    return n*a
```

m --> 3	n --> 3	n --> 2	n --> 1
x -->	a -->	a -->	a -->
	return	return	return

We open up a call frame.

Executing F(3)

```
m = 3
x = F(m)
print x
```

```
def F(n):
  if n<=1:
    return 1
  else:
    a = F(n-1)
    return n*a
```

```
def F(n):
  if n<=1:
    return 1
  else:
    a = F(n-1)
    return n*a
```

```
def F(n):
  if n<=1:
    return 1
  else:
    a = F(n-1)
    return n*a
```

m --> 3	n --> 3	n --> 2	n --> 1
x -->	a -->	a -->	a -->
	return	return	return 1

The value of 1 is “assigned” to return

Executing F(3)

```
m = 3
x = F(m)
print x
```

```
def F(n):
  if n<=1:
    return 1
  else:
    a = F(n-1)
    return n*a
```

```
def F(n):
  if n<=1:
    return 1
  else:
    a = F(n-1)
    return n*a
```

```
def F(n):
  if n<=1:
    return 1
  else:
    a = F(n-1)
    return n*a
```

m --> 3	n --> 3	n --> 2	n --> 1
x -->	a -->	a --> 1	a -->
	return	return	return 1

The value is sent back to the caller.

Executing F(3)

```
m = 3
x = F(m)
print x
```

```
def F(n):
  if n<=1:
    return 1
  else:
    a = F(n-1)
    return n*a
```

```
def F(n):
  if n<=1:
    return 1
  else:
    a = F(n-1)
    return n*a
```

```
def F(n):
  if n<=1:
    return 1
  else:
    a = F(n-1)
    return n*a
```

m --> 3	n --> 3	n --> 2	n --> 1
x -->	a -->	a --> 1	a -->
	return	return	return 1

That function call is over

Executing F(3)

```
m = 3
x = F(m)
print x
```

```
def F(n):
  if n<=1:
    return 1
  else:
    a = F(n-1)
    return n*a
```

```
def F(n):
  if n<=1:
    return 1
  else:
    a = F(n-1)
    return n*a
```

m --> 3	n --> 3	n --> 2
x -->	a -->	a --> 1
	return	return

Control now passes to this “edition” of F

Executing F(3)

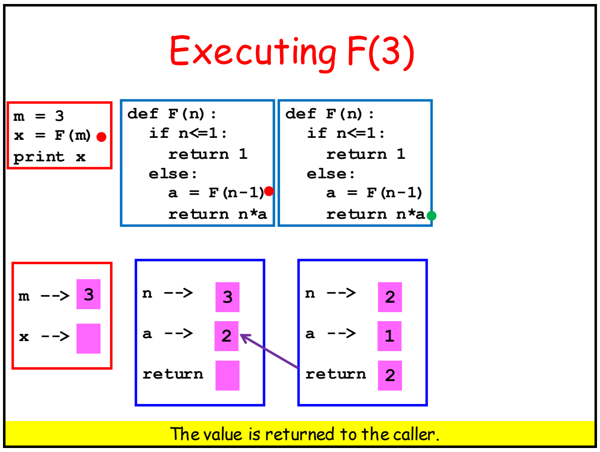
```
m = 3
x = F(m)
print x
```

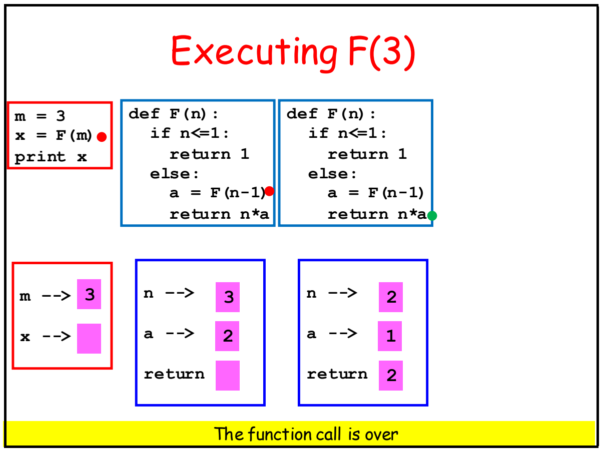
```
def F(n):
  if n<=1:
    return 1
  else:
    a = F(n-1)
    return n*a
```

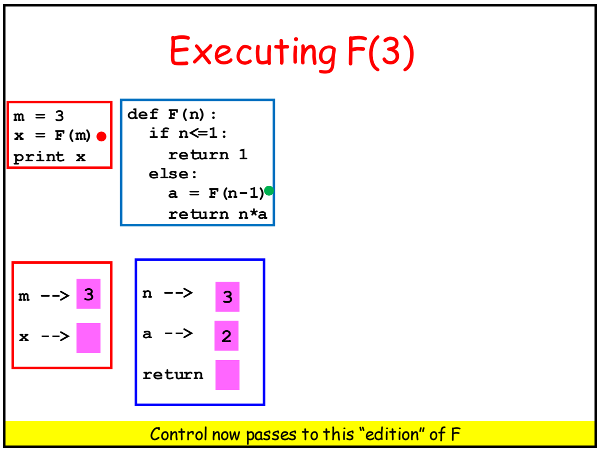
```
def F(n):
  if n<=1:
    return 1
  else:
    a = F(n-1)
    return n*a
```

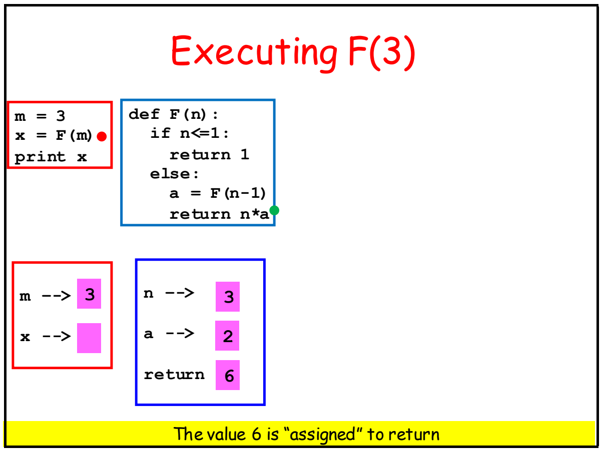
m --> 3	n --> 3	n --> 2
x -->	a -->	a --> 1
	return	return 2

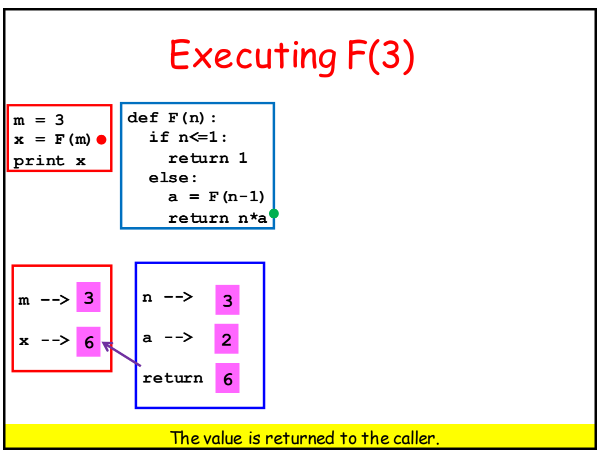
Control passes to this “edition” of F. The value 2 is “assigned” to return

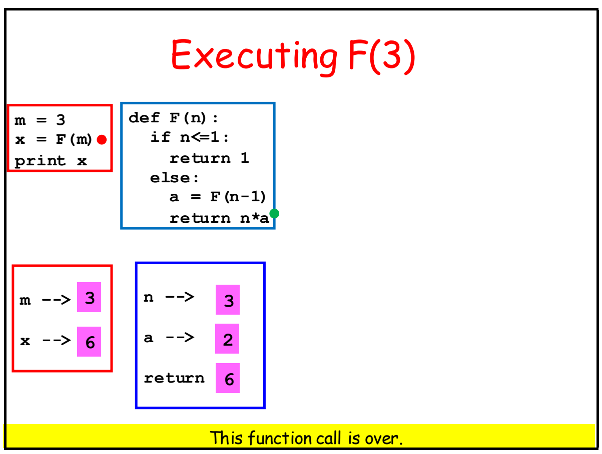

Executing F(3)
m = 3
x = F(m)
print x
def F(n):
if n<=1:
return 1
else:
a = F(n-1)
return n*a
def F(n):
if n<=1:
return 1
else:
a = F(n-1)
return n*a
m --> 3
x -->
n --> 3
a --> 2
return
n --> 2
a --> 1
return 2
The value is returned to the caller.


Executing F(3)
m = 3
x = F(m)
print x
def F(n):
if n<=1:
return 1
else:
a = F(n-1)
return n*a
def F(n):
if n<=1:
return 1
else:
a = F(n-1)
return n*a
m --> 3
x -->
n --> 3
a --> 2
return
n --> 2
a --> 1
return 2
The function call is over


Executing F(3)
m = 3
x = F(m)
print x
def F(n):
if n<=1:
return 1
else:
a = F(n-1)
return n*a
m --> 3
x -->
n --> 3
a --> 2
return
Control now passes to this "edition" of F


Executing F(3)
m = 3
x = F(m)
print x
def F(n):
if n<=1:
return 1
else:
a = F(n-1)
return n*a
m --> 3
x -->
n --> 3
a --> 2
return 6
The value 6 is "assigned" to return


Executing F(3)
m = 3
x = F(m)
print x
def F(n):
if n<=1:
return 1
else:
a = F(n-1)
return n*a
m --> 3
x --> 6
n --> 3
a --> 2
return 6
The value is returned to the caller.


Executing F(3)
m = 3
x = F(m)
print x
def F(n):
if n<=1:
return 1
else:
a = F(n-1)
return n*a
m --> 3
x --> 6
n --> 3
a --> 2
return 6
This function call is over.

Executing F(3)

```
m = 3
x = F(m)
print x
```

```
m --> 3
x --> 6
```

Control passes to the script that asked for F(3)

Executing F(3)

```
m = 3
x = F(m)
print x
```

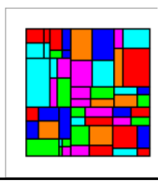
```
m --> 3
x --> 6
```

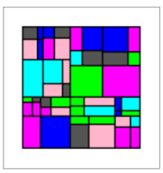
Output: 6

All Done!

Another Example: Random Mondrians

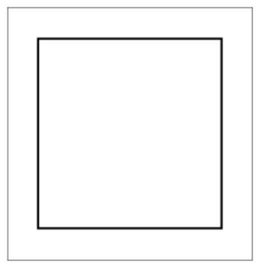
Using Python:





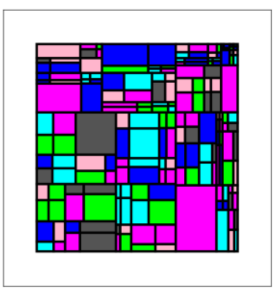
Random Mondrian

Given This:

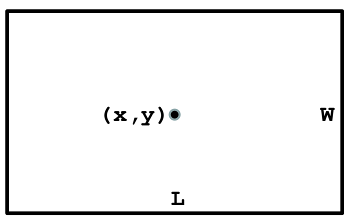


Random Mondrian

Draw This:

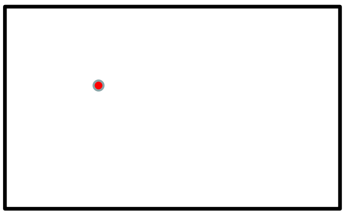


The Subdivide Process Applies to a Rectangle

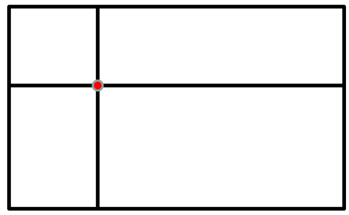


Given a rectangle specified by its length, width, and center, either randomly color it or randomly subdivide it.

Subdivision Starts with a Random Dart Throw

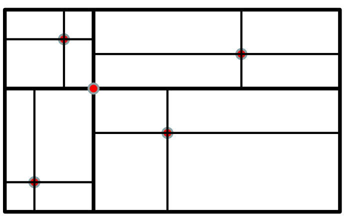


This Defines 4 Smaller Rectangles



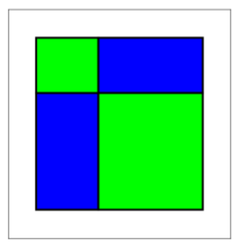
Repeat the process on each of the 4 smaller rectangles…

This Defines 4 Smaller Rectangles

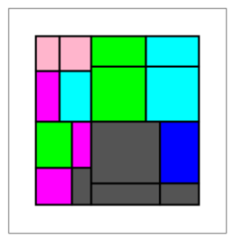


We can again repeat the process on each of the 16 smaller rectangles. Etc.

The Notion of Level



A 1-level Partitioning



A 2-level Partitioning

Pseudocode

```
def Mondrian(x,y,L,W,level):
   if level ==0:
      c = RandomColor())
      DrawRect(x,y,L,W,FillColor=c)
   else:
     # Subdivide into 4 smaller rectangles
      Mondrian(upper  left rectangle info,level-1)
      Mondrian(upper  right rectangle info,level-1)
      Mondrian(lower  left rectangle info,level-1)
      Mondrian(lower  right rectangle info,level-1)
```

We look at a few details. Complete implementation online

How to Generate Random Colors

We need some new technology to organize the selection random colors.

We need lists whose entries are lists.

Lists with Entries that Are Lists

An Example:

```
cyan      = [0.0,1.0,1.0]
magenta   = [1.0,0.0,1.0]
yellow    = [1.0,1.0,0.0]
colorList = [cyan,magenta,yellow]
```

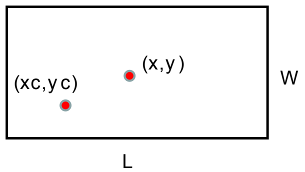
Pick a Color at Random

```
cyan      = [0.0,1.0,1.0]
magenta   = [1.0,0.0,1.0]
yellow    = [1.0,1.0,0.0]
colorList = [cyan,magenta,yellow]
r = randi(0,2)
randomColor = colorList[r]
```

Package the Idea…

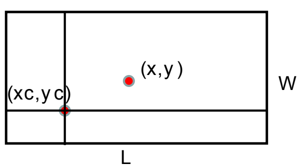
```
from simpleGraphics import *
from random import randint as randi

def RandomColor():
    """ Returns a randomly selected
    rgb list."""
    c = [RED,GREEN,BLUE,ORANGE,CYAN]
    i = randi(0,len(c)-1)
    return c[i]
```

How to Randomly Subdivide a Rectangle




```
xc = randu(x-L/2,x+L/2)
yc = randu(y-W/2,y+W/2)
```

The Math Behind the Little Rectangles



The upper right rectangle is typical:

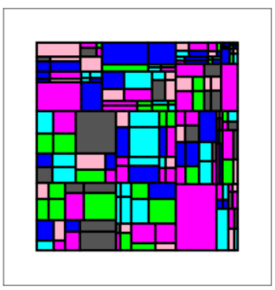
Length: `L1 = (x+L/2)-xc`
Width: `W1 = (y+W/2)-yc`
Center: `(xc+L1/2,yc+W1/2)`

The Procedure `Mondrian`

A couple of features to make the design more interesting:

(1) The dart throw that determines the subdivision can't land too near the edge. No super skinny tiles!

(2) Randomly decide whether or not to subdivide. This creates a nice diversity in size.



Overall Conclusions

Recursion is sometimes the simplest way to organize a computation.

It would be next to impossible to do the triangle tiling problem any other way.

On the other hand, factorial computation is easier via for-loop iteration.

Overall Conclusions

Infinite recursion (like infinite loops) can happen so careful reasoning is required.

Will we reach the "base case"?

Graphics examples: We will reach Level==0
Factorial: We will reach n==1