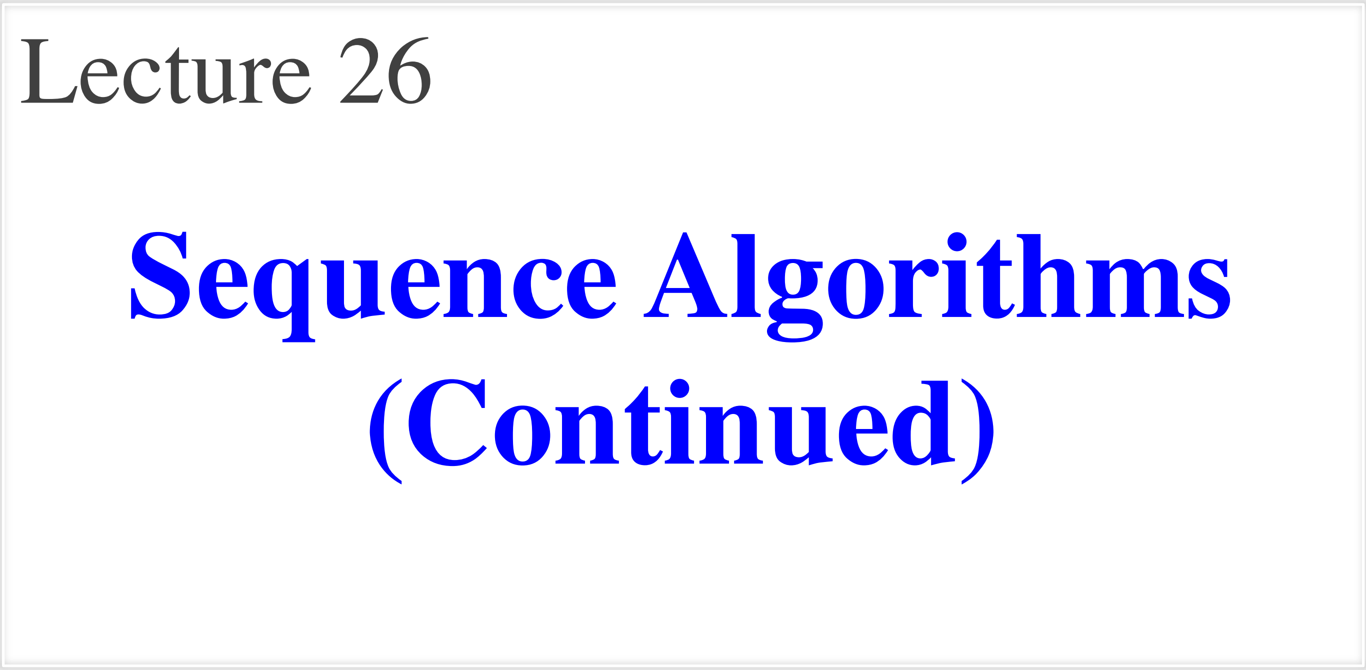


Lecture 26

Sequence Algorithms (Continued)

Announcements for This Lecture

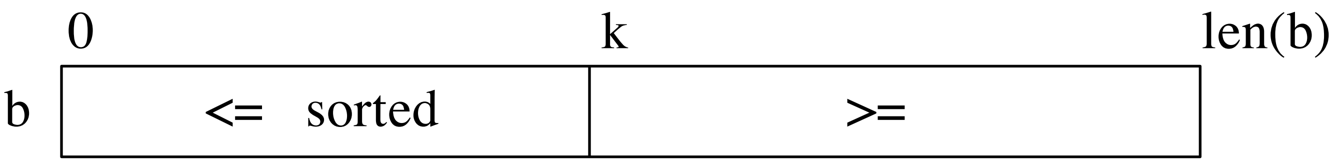
Assignment & Lab

- A6 is not graded yet
  - Done early next week
- A7 due **Fri, Dec. 11**
  - Friday after classes
  - Milestone not adjusted
  - Is your paddle moving?
- Lab Today: Office Hours
  - Get help on A7 paddle
  - Anyone can go to any lab

Next Week

- Last Week of Class!
  - Finish sorting algorithms
  - Special final lecture
- Lab held, but is optional
  - More invariant practice
  - Also use lab time on A7
- Details about the exam
  - Multiple review sessions

Recall: Horizontal Notation

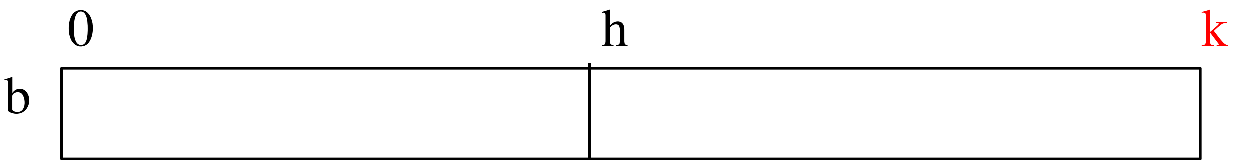


0 k len(b)

b <= sorted >=

Example of an assertion about an sequence b. It asserts that:

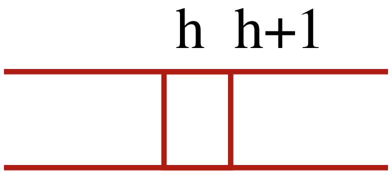
1. b[0..k–1] is sorted (i.e. its values are in ascending order)
2. Everything in b[0..k–1] is ≤ everything in b[k..len(b)–1]



0 h k

b

Given index h of the first element of a segment and index k of the element that follows that segment, the number of values in the segment is k – h.



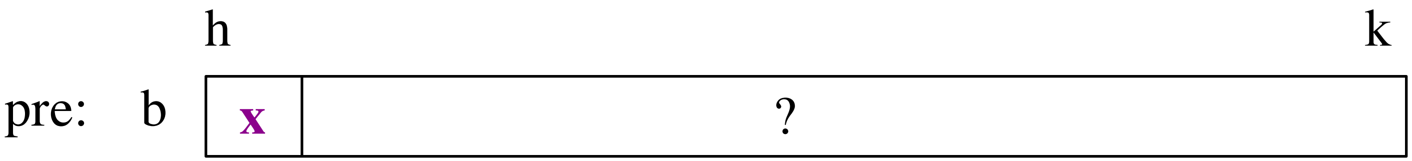
h h+1

(h+1) – h = 1

b[h .. k – 1] has k – h elements in it.

Partition Algorithm

- Given a sequence b[h..k] with some value x in b[h]:

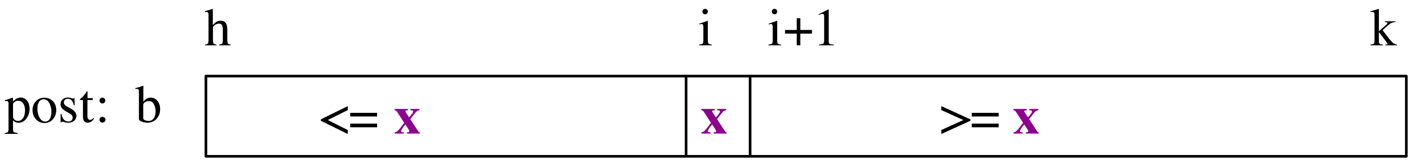


h k

pre: b

x ?

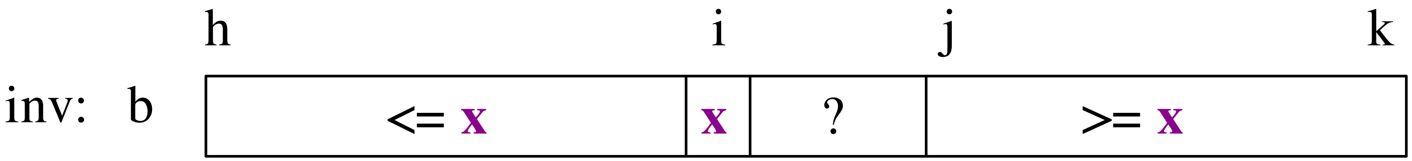
- Swap elements of b[h..k] and store in j to truthify post:



h i i+1 k

post: b

<= x x >= x



h i j k

inv: b

<= x x ? >= x

- Agrees with precondition when i = h, j = k+1
- Agrees with postcondition when j = i+1

Partition Algorithm Implementation

```
def partition(b, h, k):
    """Partition list  b[h..k] around a pivot x = b[h]"""
    i = h; j = k+1; x = b[h]
    # invariant:  b[h..i-1] < x, b[i] = x, b[j..k] >= x
    while i < j-1:
        if b[i+1] >= x:
            # Move to end of block.
            _swap(b,i+1,j-1)
            j = j - 1
        else:    # b[i+1] < x
            _swap(b,i,i+1)
            i = i + 1
    # post: b[h..i-1] < x, b[i] is  x, and b[i+1..k] >= x
    return i
```

partition(b,h,k), not partition(b[h:k+1])

Remember, slicing always copies the list!

We want to partition the **original** list

Partition Algorithm Implementation

```
def partition(b, h, k):
    """Partition list b[h..k] around a pivot x = b[h]"""
    i = h; j = k+1; x = b[h]
    # invariant: b[h..i-1] < x, b[i] = x, b[j..k] >= x
    while i < j-1:
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            # Move to end of block.
            _swap(b,i+1,j-1)
            j = j - 1
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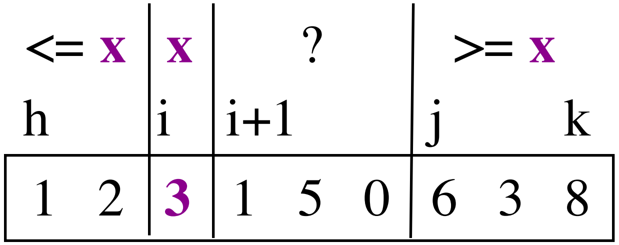
            _swap(b,i,i+1)
            i = i + 1
    # post: b[h..i-1] < x, b[i] is x, and b[i+1..k] >= x
    return i
```



<= x	x	?	>= x
h	i	i+1	j k
1 2	3	1 5 0	6 3 8

Partition Algorithm Implementation

```
def partition(b, h, k):
    """Partition list  b[h..k] around a pivot x = b[h]"""
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    # post: b[h..i-1] < x, b[i] is  x, and b[i+1..k] >= x
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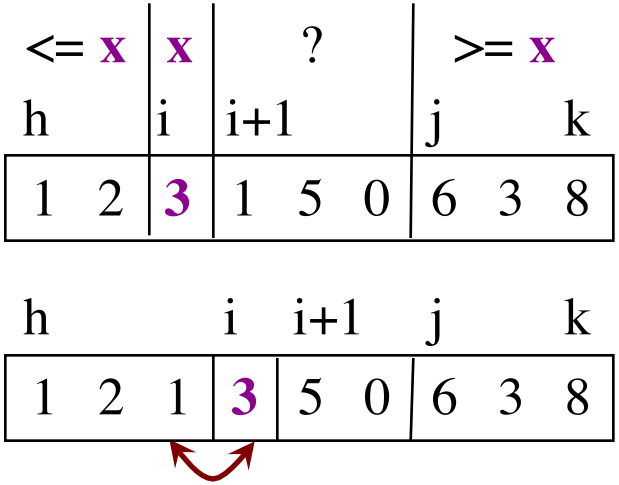
```



<= x	x	?	>= x
h	i	i+1	j k
1 2	3	1 5 0	6 3 8

h	i	i+1	j k
1 2 1	3	5 0	6 3 8

Partition Algorithm Implementation

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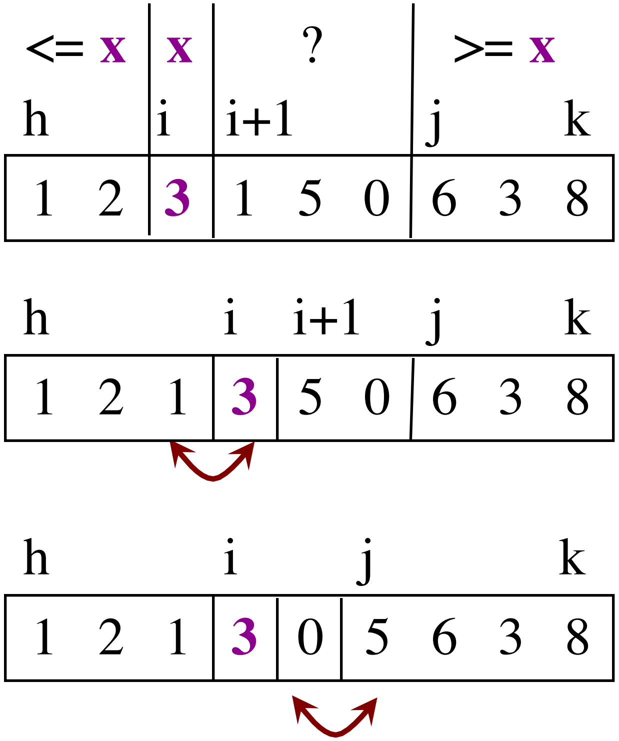
            # Move to end of block.
            _swap(b,i+1,j-1)
            j = j - 1
        else:   # b[i+1] < x
            _swap(b,i,i+1)
            i = i + 1
    # post: b[h..i-1] < x, b[i] is x, and b[i+1..k] >= x
    return i
```



<= x | x | ? | >= x

h i i+1 j k

1 2 3 1 5 0 6 3 8

h i i+1 j k

1 2 1 3 5 0 6 3 8

h i j k

1 2 1 3 0 5 6 3 8

Partition Algorithm Implementation

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def partition(b, h, k):
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            # Move to end of block.
            _swap(b,i+1,j-1)
            j = j - 1
        else:   # b[i+1] < x
            _swap(b,i,i+1)
            i = i + 1
    # post: b[h..i-1] < x, b[i] is x, and b[i+1..k] >= x
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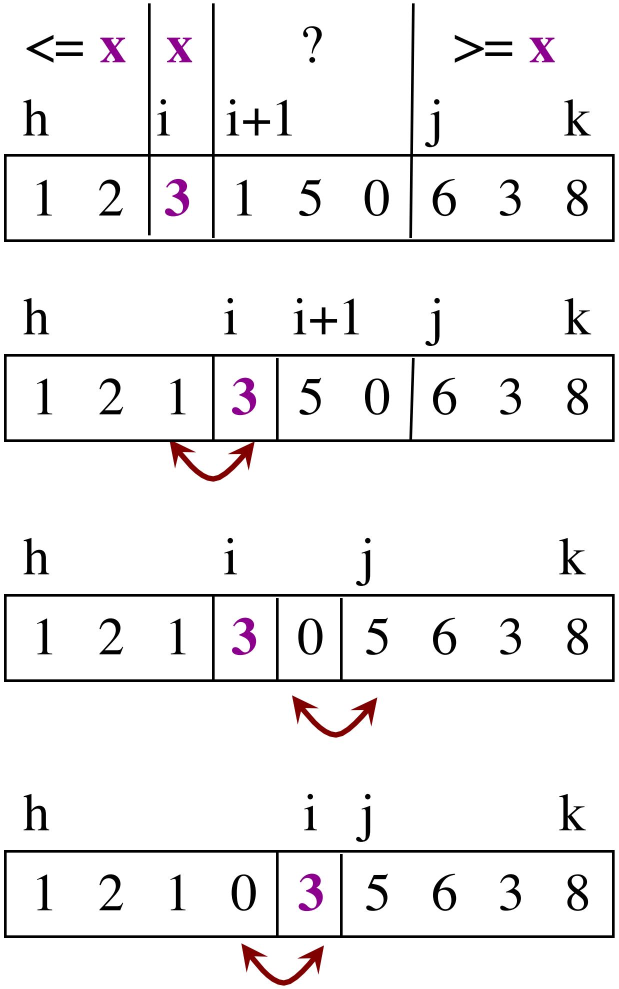
```



<= x x ? >= x

h i i+1 j k

1 2 3 1 5 0 6 3 8

h i i+1 j k

1 2 1 3 5 0 6 3 8

h i j k

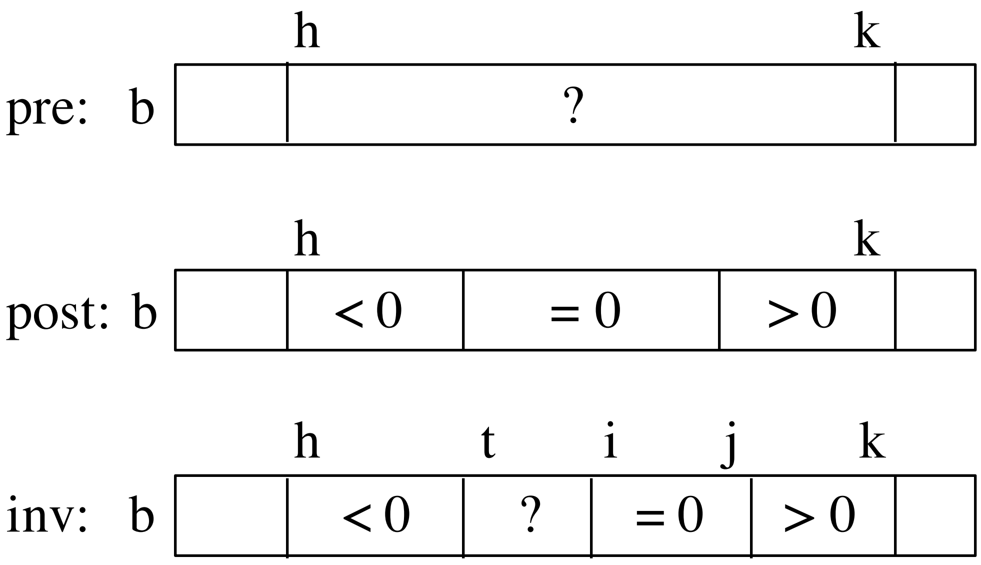
1 2 1 3 0 5 6 3 8

h i j k

1 2 1 0 3 5 6 3 8

Dutch National Flag Variant

- Sequence of integer values
  - ‘red’ = negatives, ‘white’ = 0, ‘blues’ = positive
  - Only rearrange part of the list, not all



h k

pre: b ?

h k

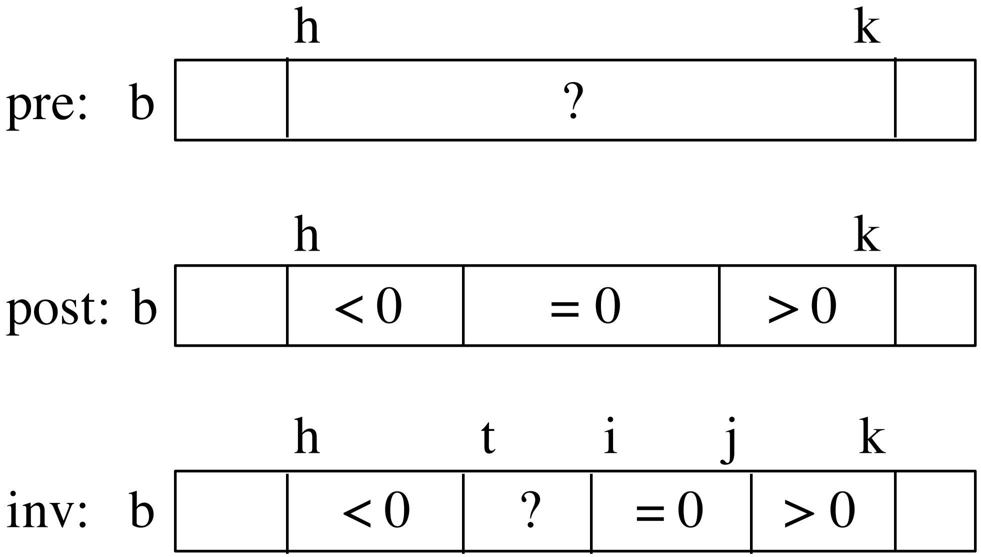
post: b < 0 = 0 > 0

h t i j k

inv: b < 0 ? = 0 > 0

Dutch National Flag Variant

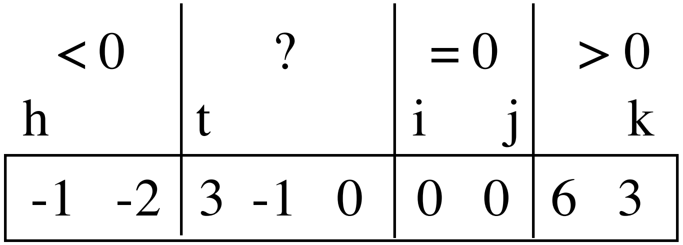
- Sequence of integer values
  - ‘red’ = negatives, ‘white’ = 0, ‘blues’ = positive
  - Only rearrange part of the list, not all



pre: t = h,
i = k+1,
j = k
post: t = i

Dutch National Flag Algorithm

```
def dnf(b, h, k):
    """Returns: partition points as a tuple (i,j)"""
    t = h; i = k+1, j = k;
    # inv: b[h..t-1] < 0, b[t..i-1] ?, b[i..j] = 0, b[j+1..k] > 0
    while t < i:
        if b[i-1] < 0:
            swap(b,i-1,t)
            t = t+1
        elif b[i-1] == 0:
            i = i-1
        else:
            swap(b,i-1,j)
            i = i-1; j = j-1
    # post: b[h..i-1] < 0, b[i..j] = 0, b[j+1..k] > 0
    return (i, j)
```



< 0 | ? | = 0 | > 0

h | t | i j | k

-1 -2 | 3 -1 0 | 0 0 | 6 3

Dutch National Flag Algorithm

```
def dnf(b, h, k):
    """Returns: partition points as a tuple (i,j)"""
    t = h; i = k+1, j = k;
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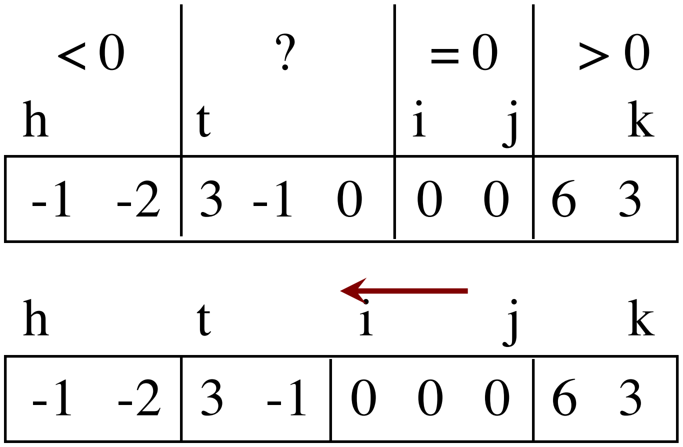
            t = t+1
        elif b[i-1] == 0:
            i = i-1
        else:
            swap(b,i-1,j)
            i = i-1; j = j-1
    # post: b[h..i-1] < 0, b[i..j] = 0, b[j+1..k] > 0
    return (i, j)
```



< 0	?	= 0	> 0
h	t	i j	k
-1 -2	3 -1 0	0 0	6 3

h	t	i	j	k
-1 -2	3 -1	0 0 0		6 3

Dutch National Flag Algorithm

```
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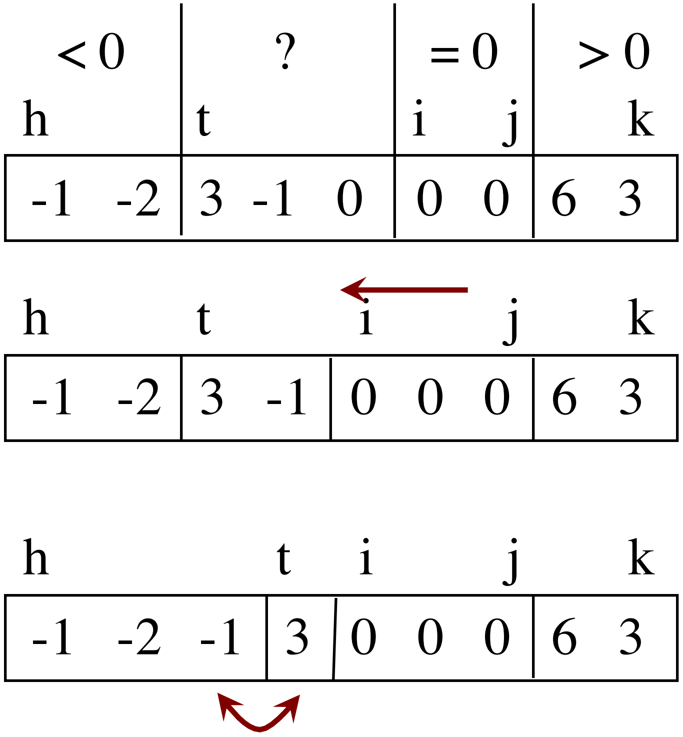
    while t < i:
        if b[i-1] < 0:
            swap(b,i-1,t)
            t = t+1
        elif b[i-1] == 0:
            i = i-1
        else:
            swap(b,i-1,j)
            i = i-1; j = j-1
    # post: b[h..i-1] < 0, b[i..j] = 0, b[j+1..k] > 0
    return (i, j)
```



Dutch National Flag Algorithm

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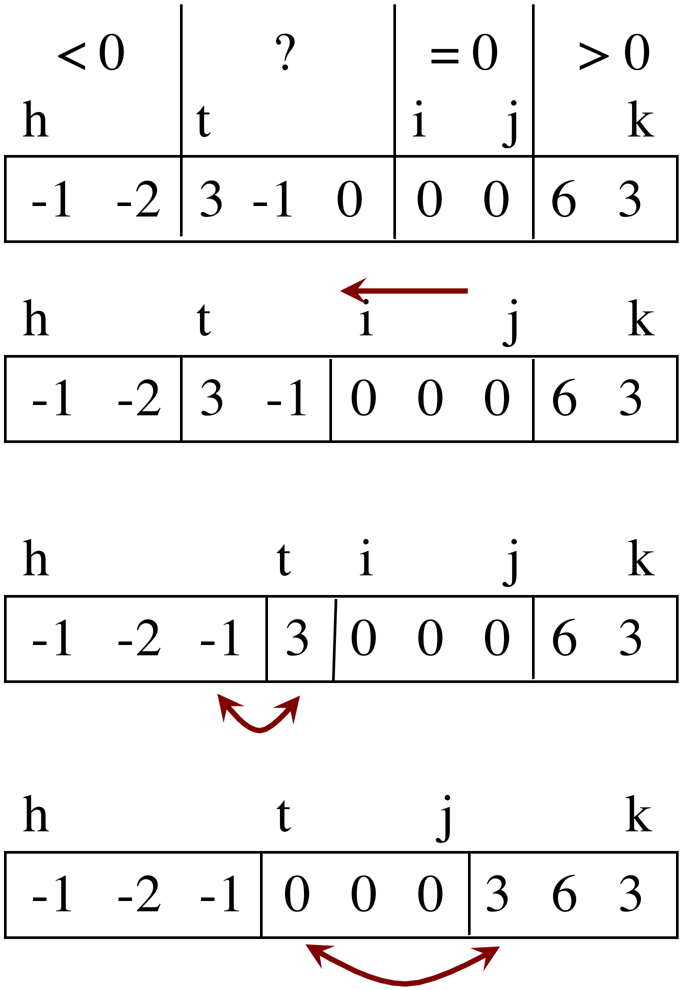
    while t < i:
        if b[i-1] < 0:
            swap(b,i-1,t)
            t = t+1
        elif b[i-1] == 0:
            i = i-1
        else:
            swap(b,i-1,j)
            i = i-1; j = j-1
    # post: b[h..i-1] < 0, b[i..j] = 0, b[j+1..k] > 0
    return (i, j)
```



< 0 | ? | = 0 | > 0

h t i j k

-1 -2 | 3 -1 0 | 0 0 | 6 3

h t i j k

-1 -2 | 3 -1 | 0 0 0 | 6 3

h t i j k

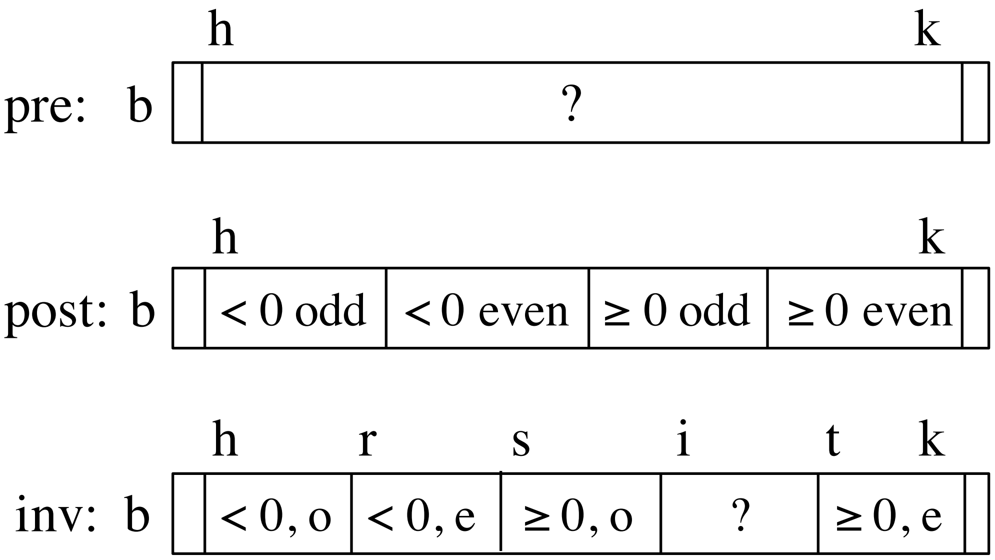
-1 -2 -1 | 3 | 0 0 0 | 6 3

h t j k

-1 -2 -1 | 0 0 0 | 3 6 3

Flag of Mauritius

- Now we have four colors!
  - Negatives: ‘red’ = odd, ‘purple’ = even
  - Positives: ‘yellow’ = odd, ‘green’ = even



h k

pre: b | ? |

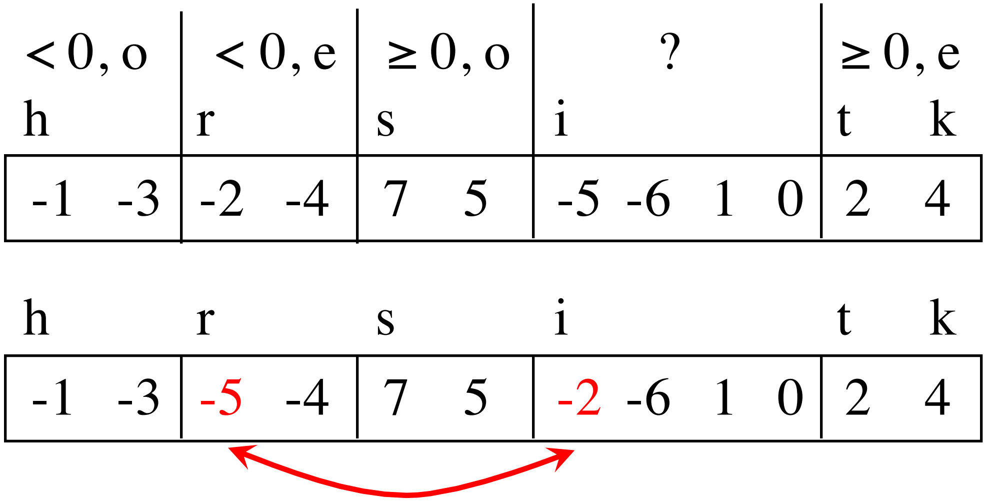
h k

post: b | < 0 odd | < 0 even | ≥ 0 odd | ≥ 0 even |

h r s i t k

inv: b | < 0, o | < 0, e | ≥ 0, o | ? | ≥ 0, e |

Flag of Mauritius

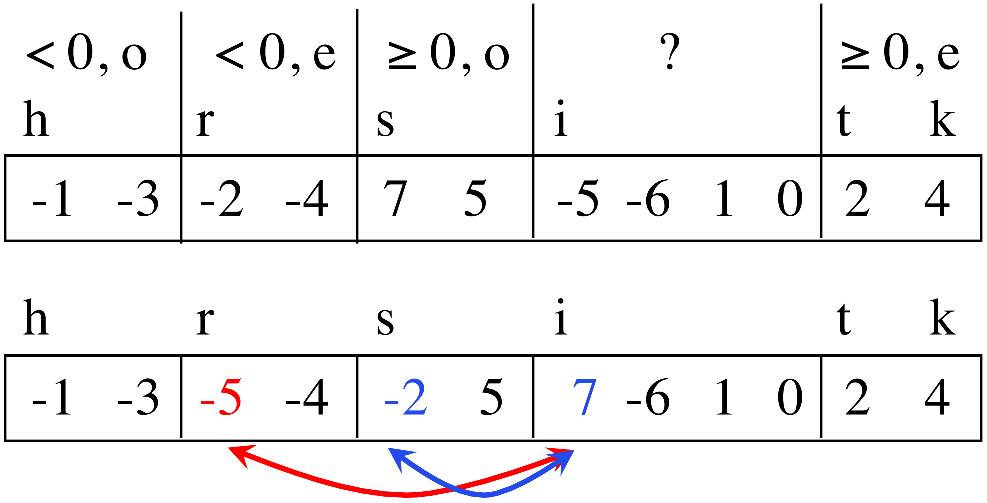


< 0, o		< 0, e		≥ 0, o		?				≥ 0, e	
h		r		s		i				t	k
-1	-3	-2	-4	7	5	-5	-6	1	0	2	4

h		r		s		i				t	k
-1	-3	-5	-4	7	5	-2	-6	1	0	2	4

One swap is not good enough

Flag of Mauritius

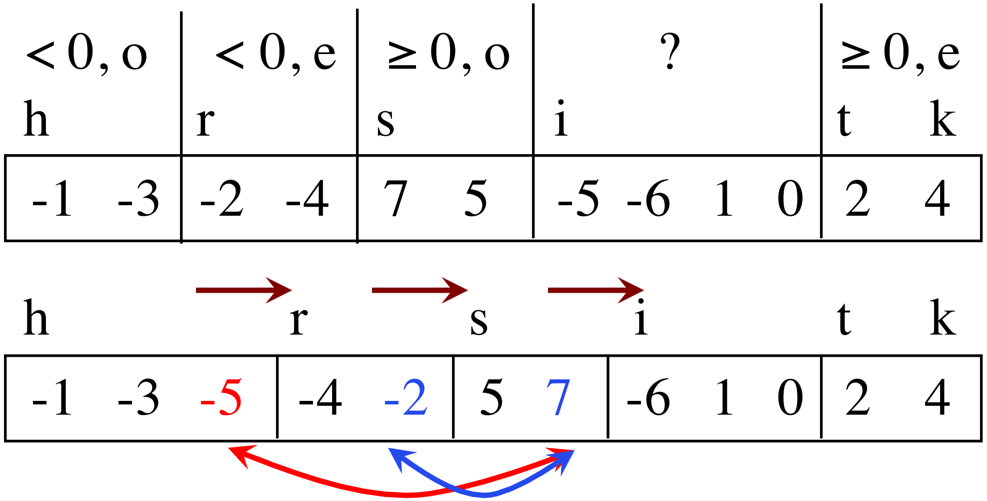


< 0, o	< 0, e	≥ 0, o	?	≥ 0, e
h	r	s	i	t k
-1 -3	-2 -4	7 5	-5 -6 1 0	2 4

h	r	s	i	t k
-1 -3	-5 -4	-2 5	7 -6 1 0	2 4

Need two swaps for two spaces

Flag of Mauritius

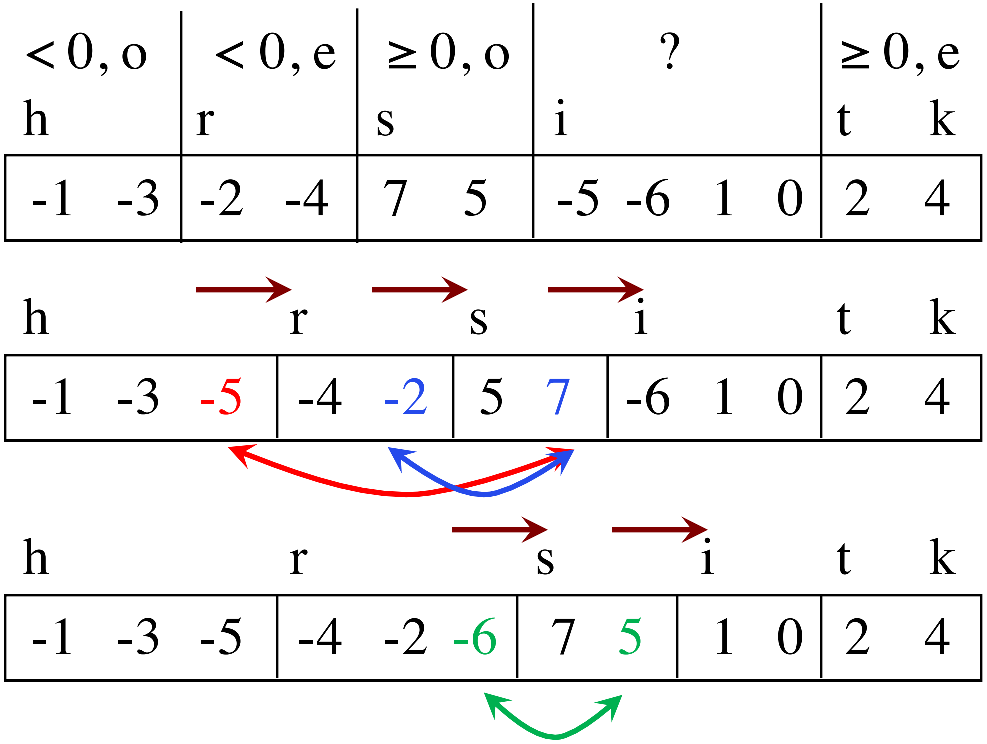


< 0, o	< 0, e	≥ 0, o	?	≥ 0, e
h	r	s	i	t k
-1 -3	-2 -4	7 5	-5 -6 1 0	2 4

h	r	s	i	t k
-1 -3 -5	-4 -2	5 7	-6 1 0	2 4

And adjust the loop variables

Flag of Mauritius



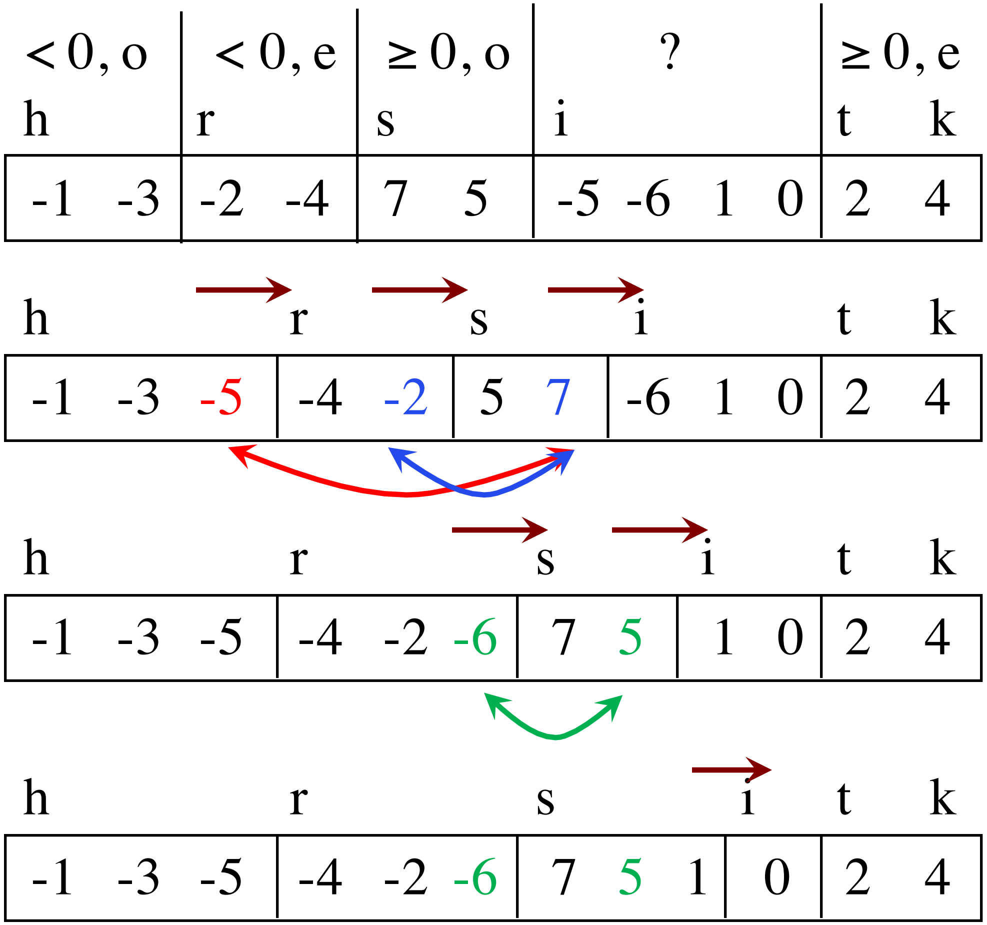
< 0, o	< 0, e	≥ 0, o	?	≥ 0, e
h	r	s	i	t k
-1 -3	-2 -4	7 5	-5 -6 1 0	2 4

h	r	s	i	t k
-1 -3 -5	-4 -2	5 7	-6 1 0	2 4

h	r	s	i	t k
-1 -3 -5	-4 -2 -6	7 5	1 0	2 4

See algorithms.py for Python code

Flag of Mauritius



< 0, o | < 0, e | ≥ 0, o | ? | ≥ 0, e

h r s i t k

-1 -3 | -2 -4 | 7 5 | -5 -6 1 0 | 2 4

h r s i t k

-1 -3 -5 | -4 -2 | 5 7 | -6 1 0 | 2 4

h r s i t k

-1 -3 -5 | -4 -2 -6 | 7 5 | 1 0 | 2 4

h r s i t k

-1 -3 -5 | -4 -2 -6 | 7 5 1 | 0 | 2 4

See algorithms.py
for Python code

Linear Search

- **Vague**: Find first occurrence of v in b[h..k-1].

Linear Search

- **Vague**: Find first occurrence of v in b[h..k-1].
- **Better**: Store an integer in i to truthify result condition post:

post: 1. v is not in b[h..i-1]

2. i = k OR v = b[i]

Linear Search

- **Vague**: Find first occurrence of v in b[h..k-1].
- **Better**: Store an integer in i to truthify result condition post:

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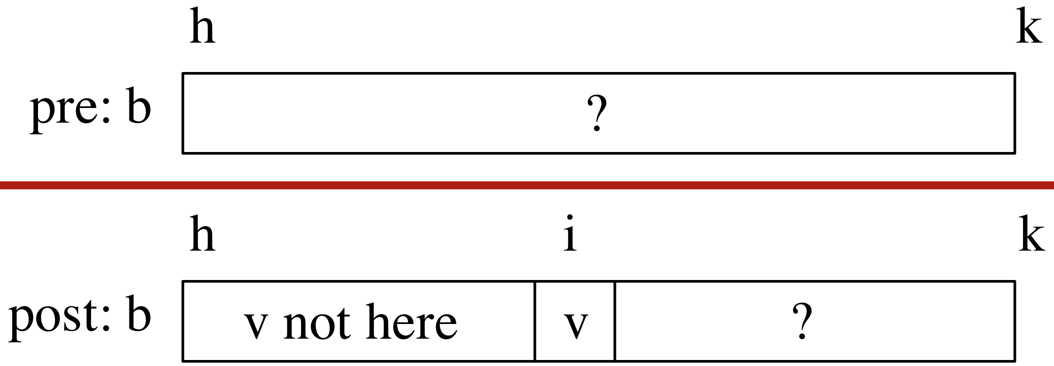


h k

pre: b ?

h i k

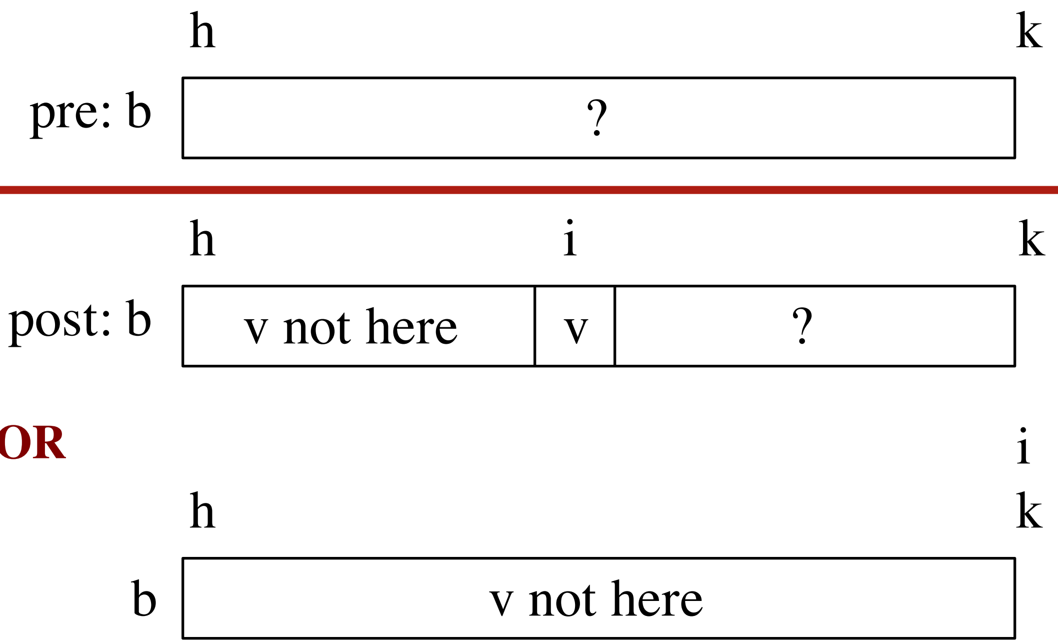
post: b v not here v ?

Linear Search

- **Vague**: Find first occurrence of v in b[h..k-1].
- **Better**: Store an integer in i to truthify result condition post:

post: 1. v is not in b[h..i-1]

2. i = k OR v = b[i]



h k

pre: b ?

h i k

post: b v not here v ?

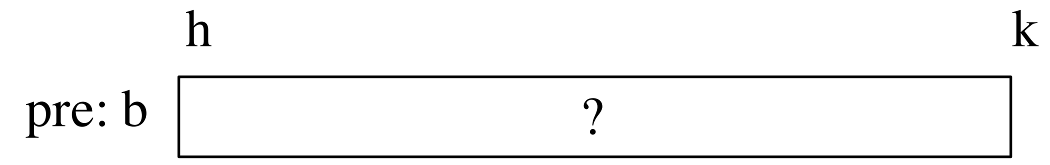
OR

i

h k

b v not here

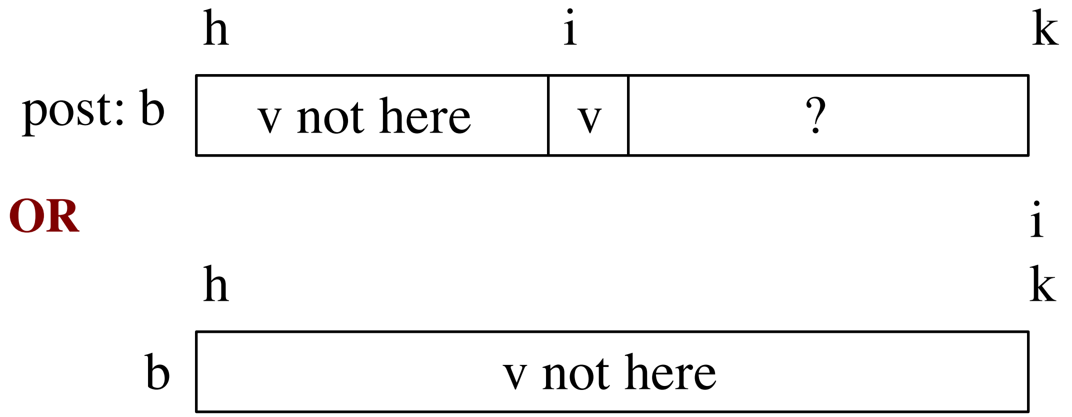
Linear Search



h k

pre: b

?



h i k

post: b

v not here | v | ?

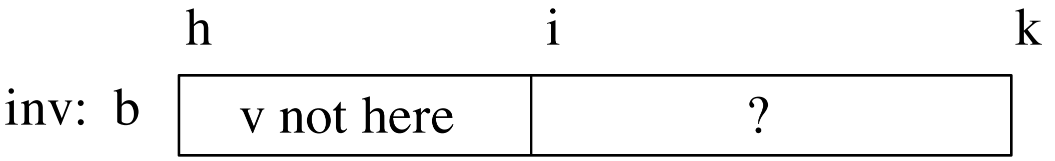
OR

i

h k

b

v not here



h i k

inv: b

v not here | ?

Linear Search

```
def linear_search(b,c,h):
    """Returns: first occurrence of c in b[h..]"""
    # Store in i the index of the first c in b[h..]
    i = h

    # invariant: c is not in b[0..i-1]
    while i < len(b) and b[i] != c:
        i = i + 1

    # post: c is not in b[h..i-1]
    #         i >= len(b) or b[i] == c
    return i if i < len(b) else -1
```

Analyzing the Loop

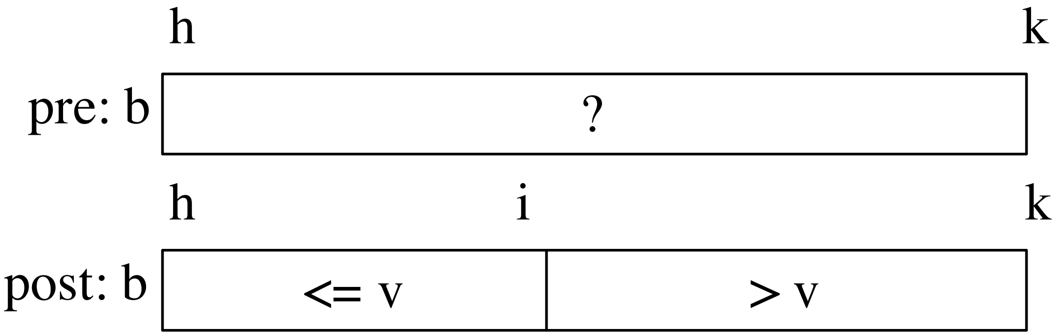
1. Does the initialization make **inv** true?
2. Is **post** true when **inv** is true and **condition** is false?
3. Does the repetend make progress?
4. Does the repetend keep the invariant **inv** true?

Binary Search

- **Vague:** Look for v in **sorted** sequence segment b[h..k].

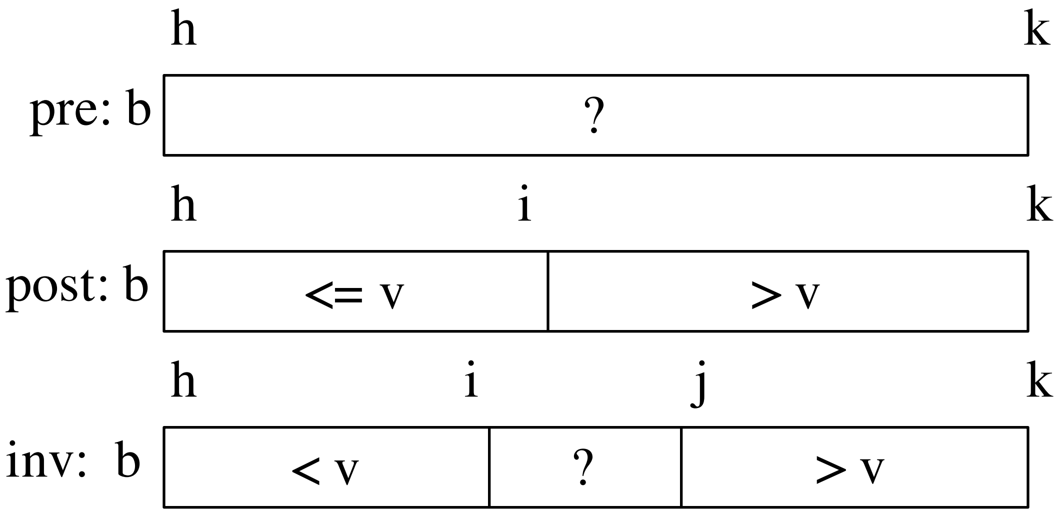
Binary Search

- **Vague:** Look for v in **sorted** sequence segment b[h..k].
- **Better:**
  - Precondition: b[h..k-1] is sorted (in ascending order).
  - Postcondition: b[h..i] <= v and v < b[i+1..k-1]
- Below, the array is in non-descending order:



Binary Search

- **Vague:** Look for v in **sorted** sequence segment b[h..k].
- **Better:**
  - Precondition: b[h..k-1] is sorted (in ascending order).
  - Postcondition: b[h..i] <= v and v < b[i+1..k-1]
- Below, the array is in non-descending order:



h k

pre: b ?

h i k

post: b <= v > v

h i j k

inv: b < v ? > v

Called binary search because each iteration of the loop cuts the array segment still to be processed in half

Extras Not Covered in Class

Loaded Dice

- Sequence p of length n represents n-sided die
  - Contents of p sum to 1
  - p[k] is probability die rolls the number k

1	2	3	4	5	6
0.1	0.1	0.1	0.1	0.3	0.3

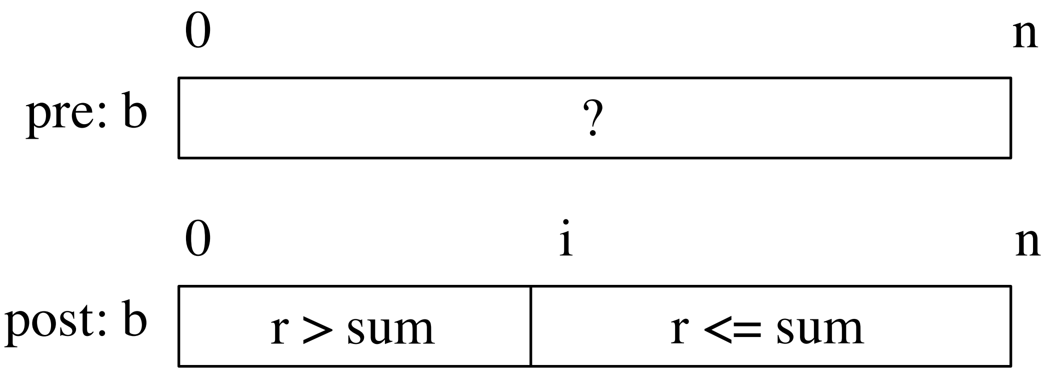
weighted d6, favoring 5, 6

- Goal: Want to “roll the die”
  - Generate random number r between 0 and 1
  - Pick p[i] such that $p[i-1] < r \leq p[i]$

0.1	0.1	0.1	0.1	0.3	0.3
0.1	0.2	0.3	0.4	0.7	1.0

Loaded Dice

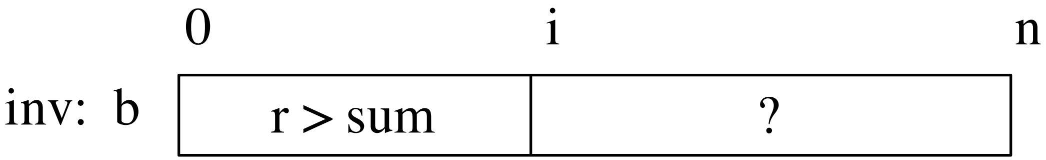
- **Want**: Value i such that p[i-1] < r <= p[i]


0 n

pre: b ?

0 i n

post: b r > sum r <= sum


0 i n

inv: b r > sum ?

- Same as precondition if i = 0
- Postcondition is invariant + false loop condition

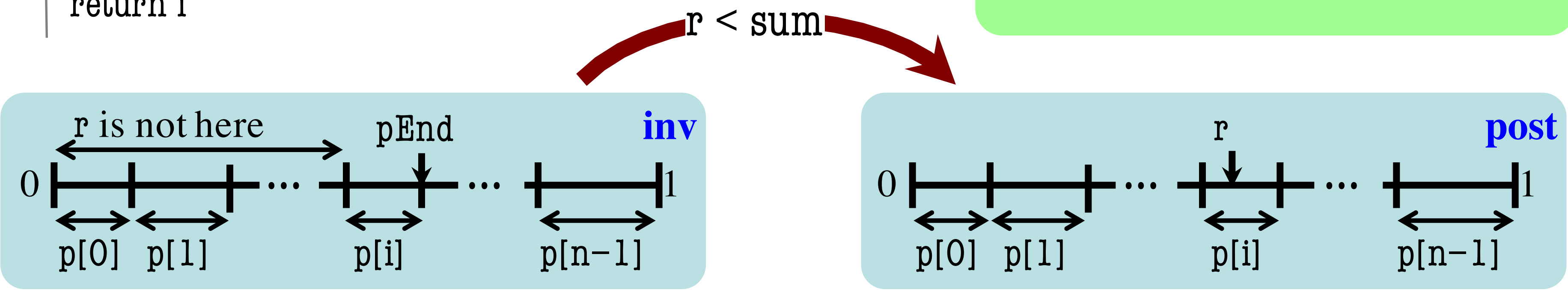
Loaded Dice

```
def roll(p):
    """Returns:  randint in 0..len(p)-1; i returned with prob. p[i]
    Precondition: p list of positive floats that sum to 1."""
    r = random.random()     # r in [0,1)
    # Think of interval [0,1] divided into segments of size p[i]
    # Store into i the segment number in which r falls.
    i = 0;    sum_of = p[0]
    # inv: r >= sum of p[0] .. p[i–1]; pEnd = sum of p[0] .. p[i]
    while r >= sum_of:
        sum_of = sum_of + p[i+1]
        i = i + 1

    # post: sum of p[0] .. p[i–1] <= r < sum of p[0] .. p[i]
    return i
```

Analyzing the Loop

1. Does the initialization make **inv** true?
2. Is **post** true when **inv** is true and **condition** is false?
3. Does the repetend make progress?
4. Does the repetend keep **inv** true?



Reversing a Sequence

h k

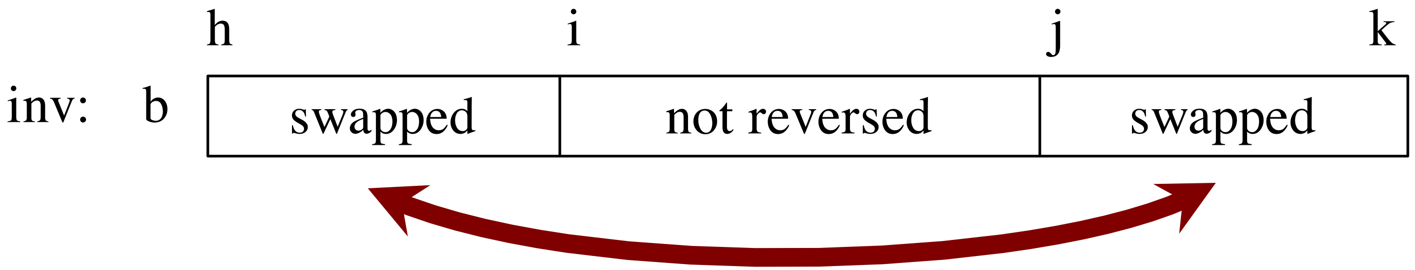
pre: b | not reversed |

h k

post: b | reversed |

change: b | 1 2 3 4 5 6 7 8 9 9 9 9 |

into b | 9 9 9 9 8 7 6 5 4 3 2 1 |



h i j k

inv: b | swapped | not reversed | swapped |