Lecture 24

Designing Sequence Algorithms

Announcements for This Lecture

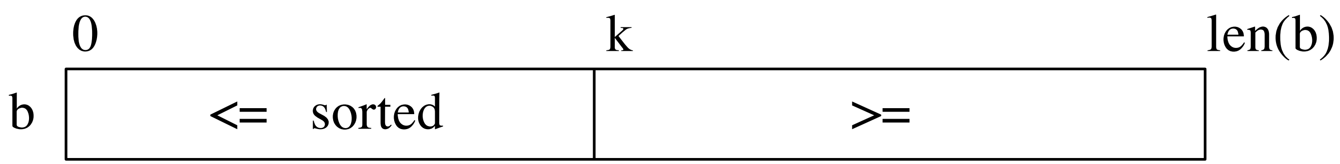
Exams

- Unfortunately, too easy
  - **Mean**: 83, **Median**: 87
  - Lacked a good A question
- What do grades mean?
  - **A**: 90s
  - **B**: 80s, mid 70s
  - **C**: Below 75
- Final will have to be harder
  - Not too hard, but 70 mean

Assignment & Lab

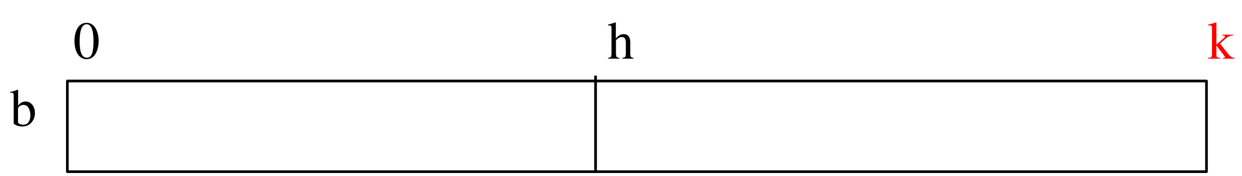
- A6 is due on Thursday
  - See consultants early!
  - Let us know about problems
- A7 is posted today
  - Piazza poll on due dates
- Today's lab is on invariants
  - Due after Thanksgiving
  - No official lab next week
  - But will be there on Tues

Horizontal Notation for Sequences

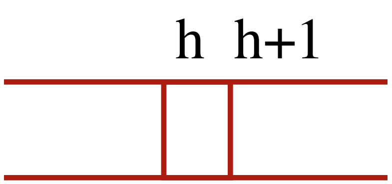


Example of an assertion about an sequence b. It asserts that:

1. b[0..k–1] is sorted (i.e. its values are in ascending order)
2. Everything in b[0..k–1] is ≤ everything in b[k..len(b)–1]



Given index h of the first element of a segment and index k of the element that follows that segment, the number of values in the segment is k – h.



(h+1) – h = 1

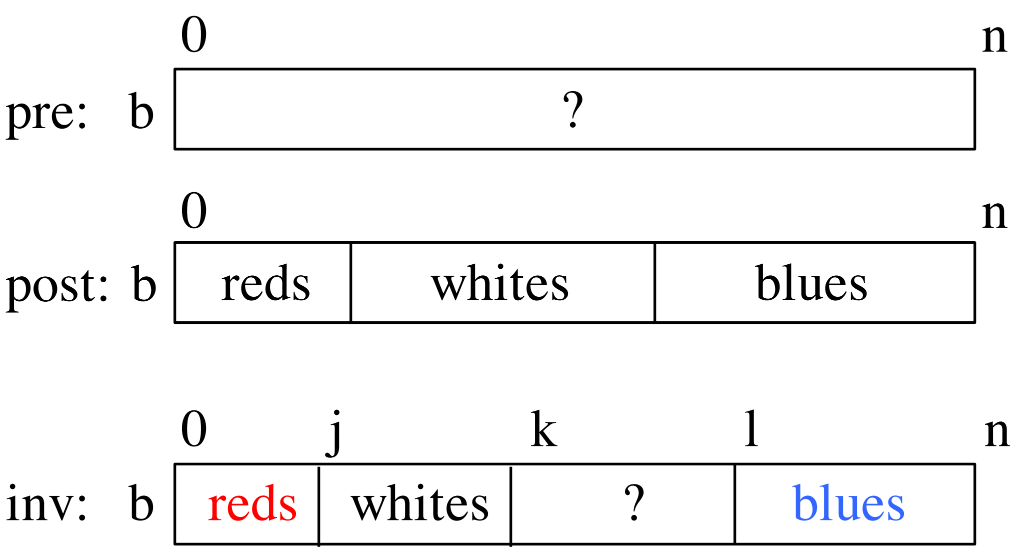
b[h .. k – 1] has k – h elements in it.

Developing Algorithms on Sequences

- Specify the algorithm by giving its precondition and postcondition as pictures.
- Draw the invariant by drawing another picture that "generalizes" the precondition and postcondition
  - The invariant is true at the beginning and at the end
- The four loop design questions (memorize them)
  1. How does loop start (how to make the invariant true)?
  2. How does it stop (is the postcondition true)?
  3. How does the body make progress toward termination?
  4. How does the body keep the invariant true?

Generalizing Pre- and Postconditions

- Dutch national flag: tri-color
  - Sequence of 0..n-1 of red, white, blue "pixels"
  - Arrange to put reds first, then whites, then blues



(values in 0..n-1 are unknown)

Make the red, white, blue sections initially **empty**:

- Range i..i-1 has 0 elements
- Main reason for this trick

Changing loop variables turns invariant into postcondition.

Generalizing Pre- and Postconditions

- Finding the minimum of a sequence.

0 n

pre: b | ? | and n >= 0

(values in 0..n are unknown)

0 n

post: b | x is the min of this segment |

- Put negative values before nonnegative ones.

0 n

pre: b | ? | and n >= 0

(values in 0..n are unknown)

0 k n

post: b | < 0 | >= 0 |

Generalizing Pre- and Postconditions

- Finding the minimum of a sequence.

	0		n	
pre: b	?			and n >= 0
	0		n	
post: b	x is the min of this segment			
	0	j	n	
inv: b	x is min of this segment	?		

(values in 0..n are unknown)

(values in j..n are unknown)

- Put negative values before nonnegative ones.

	0		n	
pre: b	?			and n >= 0
	0	k	n	
post: b	< 0	>= 0		

(values in 0..n are unknown)

Generalizing Pre- and Postconditions

- Finding the minimum of a sequence.

0 n
pre: b | ? | and n >= 0 (values in 0..n are unknown)

0 n
post: b | x is the min of this segment |

0 j n
inv: b | x is min of this segment | ? |

pre: j = 0
post: j = n

(values in j..n are unknown)

- Put negative values before nonnegative ones.

0 n
pre: b | ? | and n >= 0 (values in 0..n are unknown)

0 k n
post: b | < 0 | >= 0 |

Generalizing Pre- and Postconditions

- Finding the minimum of a sequence.

0 n
pre: b | ? | and n >= 0 (values in 0..n are unknown)

0 n
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0 j n
inv: b | x is min of this segment | ? | (values in j..n are unknown)

pre: j = 0
post: j = n

- Put negative values before nonnegative ones.

0 n
pre: b | ? | and n >= 0 (values in 0..n are unknown)

0 k n
post: b | < 0 | >= 0 |

0 k j n
inv: b | < 0 | ? | >= 0 | (values in k..j are unknown)

Generalizing Pre- and Postconditions

- Finding the minimum of a sequence.

pre: b [0 .. n] ? and n >= 0 (values in 0..n are unknown)

post: b [0 .. n] x is the min of this segment

inv: b [0 .. j] x is min of this segment [j .. n] ? (values in j..n are unknown)

pre: j = 0
post: j = n

- Put negative values before nonnegative ones.

pre: b [0 .. n] ? and n >= 0 (values in 0..n are unknown)

post: b [0 .. k] < 0 [k .. n] >= 0

inv: b [0 .. k] < 0 [k .. j] ? [j .. n] >= 0 (values in k..j are unknown)

pre: k = 0, j = n
post: k = j

Partition Algorithm

- Given a sequence b[h..k] with some value x in b[h]:

		h	k
pre:	b	x	?

- Swap elements of b[h..k] and store in j to truthify post:

		h	i	i+1 k
post:	b	<= x	x	>= x

		h								k
change:	b	**3**	5	4	1	6	2	3	8	1

		h			i					k
into	b	1	2	1	**3**	5	4	6	3	8

- x is called the pivot value
  - x is not a program variable
  - denotes value initially in b[h]

Partition Algorithm

- Given a sequence b[h..k] with some value x in b[h]:

pre: b

h	k
x	?

- Swap elements of b[h..k] and store in j to truthify post:

post: b

h	i	i+1 k
<= x	x	>= x

change: b

h								k
3	5	4	1	6	2	3	8	1

into b

h			i					k
1	2	1	**3**	5	4	6	3	8

or b

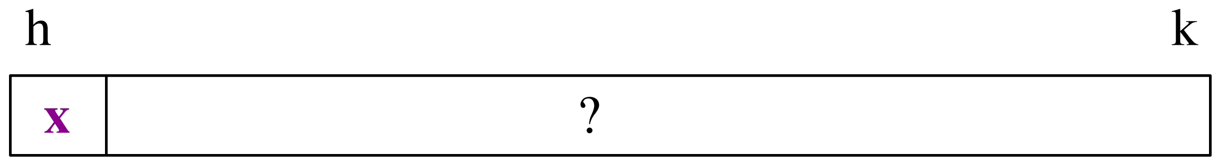
h				i				k
1	2	3	1	**3**	4	5	6	8

- x is called the pivot value
  - x is not a program variable
  - denotes value initially in b[h]

Partition Algorithm

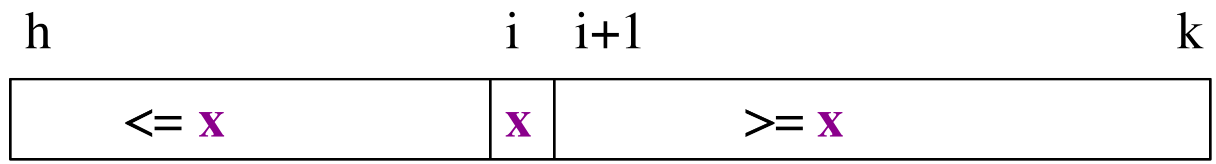
- Given a sequence b[h..k] with some value x in b[h]:

pre: b



- Swap elements of b[h..k] and store in j to truthify post:

post: b



Partition Algorithm

- Given a sequence b[h..k] with some value x in b[h]:

	h	k
pre: b	**x**	?

- Swap elements of b[h..k] and store in j to truthify post:

	h	i	i+1 k
post: b	<= **x**	**x**	>= **x**

	h	i	j	k
inv: b	<= **x**	**x**	?	>= **x**

- Agrees with precondition when i = h, j = k+1
- Agrees with postcondition when j = i+1

Partition Algorithm Implementation

```
def partition(b, h, k):
    """Partition list b[h..k] around a pivot x = b[h]"""
    i = h; j = k+1; x = b[h]
    # invariant: b[h..i-1] < x, b[i] = x, b[j..k] >= x
    while i < j-1:
        if b[i+1] >= x:
            # Move to end of block.
            _swap(b,i+1,j-1)
            j = j - 1
        else:   # b[i+1] < x
            _swap(b,i,i+1)
            i = i + 1
    # post: b[h..i-1] < x, b[i] is x, and b[i+1..k] >= x
    return i
```

partition(b,h,k), not partition(b[h:k+1])

Remember, slicing always copies the list!

We want to partition the **original** list

Partition Algorithm Implementation

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            j = j - 1
        else:   # b[i+1] < x
            _swap(b,i,i+1)
            i = i + 1
    # post: b[h..i-1] < x, b[i] is x, and b[i+1..k] >= x
    return i
```

<= **x**		**x**	?			>= **x**		
h		i	i+1			j		k
1	2	**3**	1	5	0	6	3	8

Partition Algorithm Implementation

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```

<= x		x	?			>= x		
h		i	i+1			j		k
1	2	3	1	5	0	6	3	8

h			i	i+1		j		k
1	2	1	3	5	0	6	3	8

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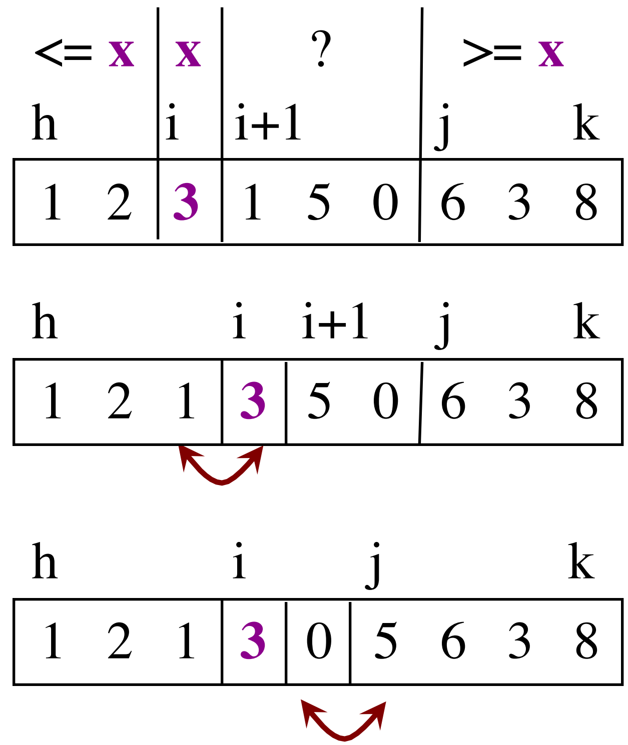

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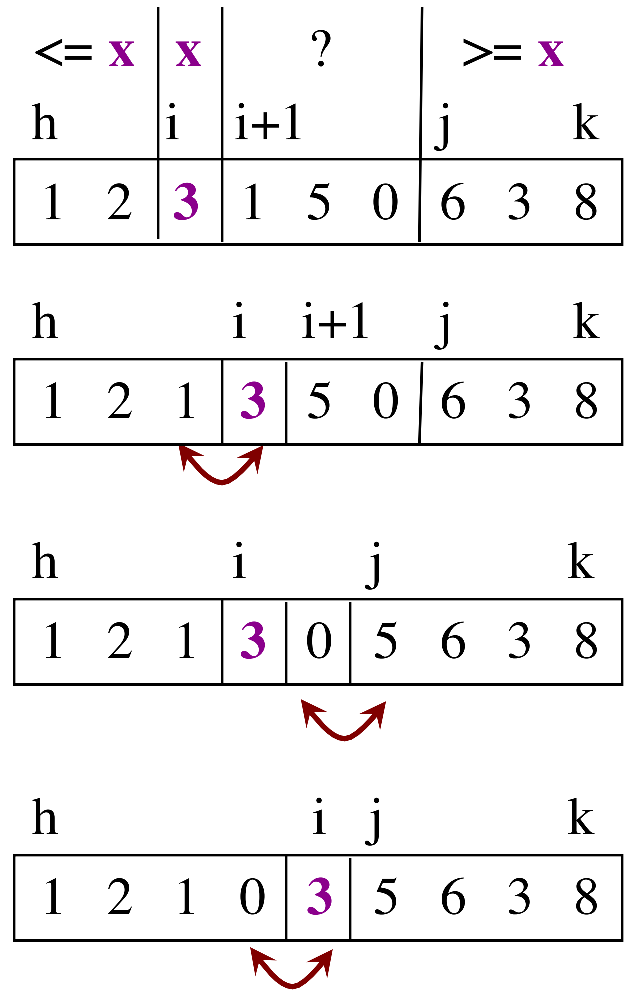
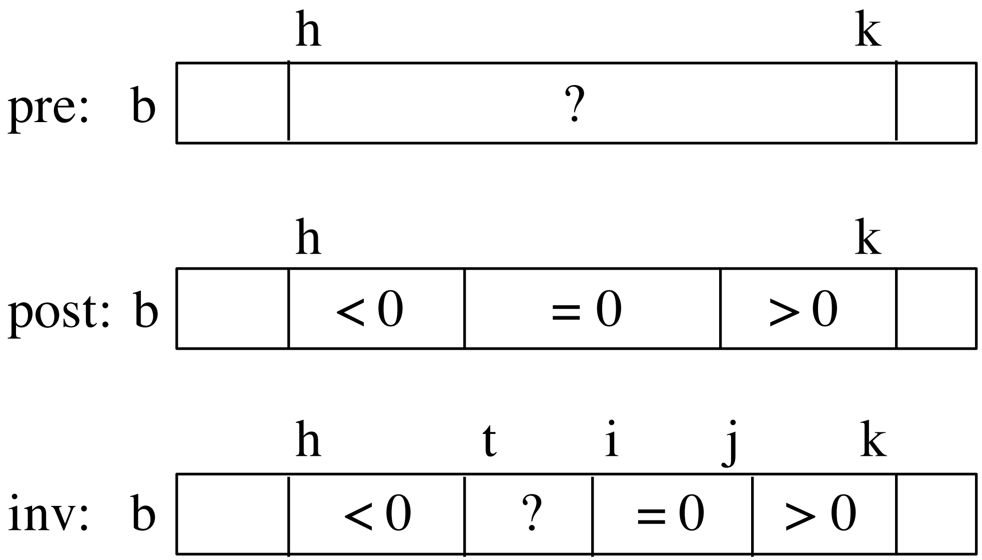


Dutch National Flag Variant

- Sequence of integer values
  - ‘red’ = negatives, ‘white’ = 0, ‘blues’ = positive
  - Only rearrange part of the list, not all



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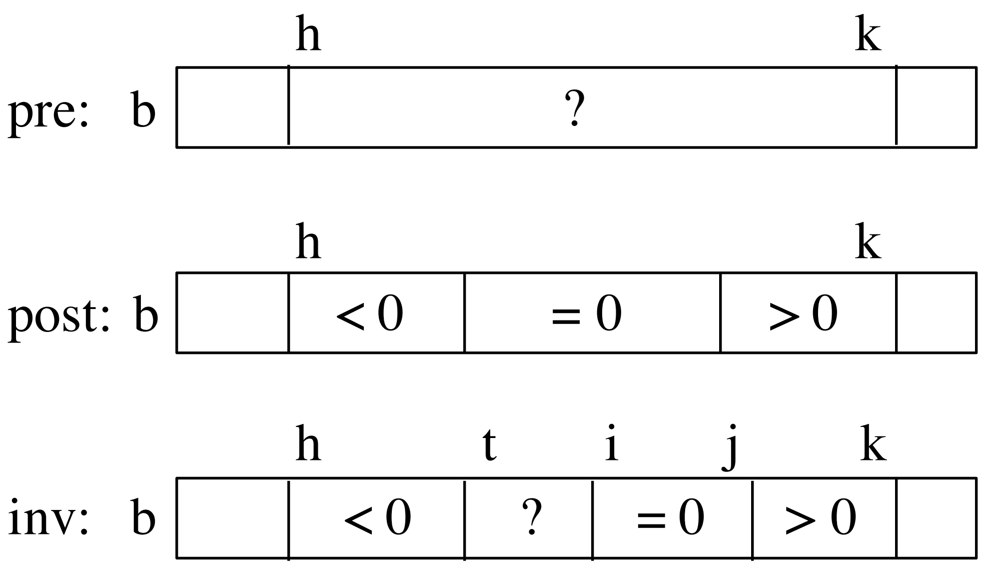

pre: $t = h$,
$i = k+1$,
$j = k$
post: $t = i$

Dutch National Flag Algorithm

```
def dnf(b, h, k):
    """Returns: partition points as a tuple (i,j)"""
    t = h; i = k+1, j = k;
    # inv: b[h..t-1] < 0, b[t..i-1] ?, b[i..j] = 0, b[j+1..k] > 0
    while t < i:
        if b[i-1] < 0:
            swap(b,i-1,t)
            t = t+1
        elif b[i-1] == 0:
            i = i-1
        else:
            swap(b,i-1,j)
            i = i-1; j = j-1
    # post: b[h..i-1] < 0, b[i..j] = 0, b[j+1..k] > 0
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```

< 0		?			$= 0$		> 0	
h		t			i	j		k
-1	-2	3	-1	0	0	0	6	3

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            swap(b,i-1,j)
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< 0		?			= 0		> 0	
h		t			i	j		k
-1	-2	3	-1	0	0	0	6	3

h		t		i		j		k
-1	-2	3	-1	0	0	0	6	3

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            swap(b,i-1,j)
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```

< 0	?	= 0	> 0
h	t	i j	k
-1 -2	3 -1 0	0 0	6 3

h	t	i j	k
-1 -2	3 -1	0 0 0	6 3

h	t	i	j	k
-1 -2 -1	3	0 0 0		6 3

Dutch National Flag Algorithm

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        if b[i-1] < 0:
            swap(b,i-1,t)
            t = t+1
        elif b[i-1] == 0:
            i = i-1
        else:
            swap(b,i-1,j)
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    # post: b[h..i-1] < 0, b[i..j] = 0, b[j+1..k] > 0
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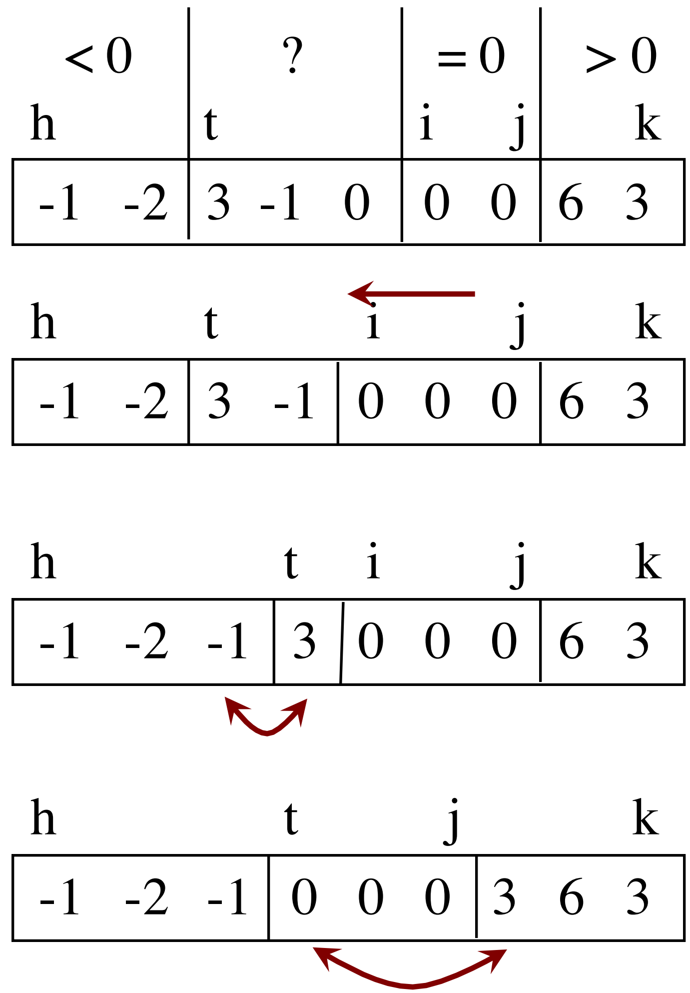

Will Finish This Next Week