Lecture 15

Recursion

Announcements for Today

Prelim 1

- Tonight at 7:30-9pm
  - **A–J** (Uris G01)
  - **K-Z** (Statler Auditorium)
- Graded by noon on Sun
  - Scores will be in CMS
  - In time for drop date
- Make-ups were e-mailed
  - If not, e-mail Jessica NOW

Other Announcements

- Reading: 5.8 – 5.10
- Assignment 3 now graded
  - **Mean** 94, **Median** 99
  - **Time**: 7 hrs, **StdDev**: 3 hrs
  - Unchanged from last year
- Assignment 4 posted Friday
  - Parts 1-3: Can do already
  - Part 4: material from today
  - Due two weeks from today

Recursion

- **Recursive Definition**:

  A definition that is defined in terms of itself
- **Recursive Function**:

  A function that calls itself (directly or indirectly)

- **Recursion**: If you understand the definition, stop; otherwise, see Recursion
- **Infinite Recursion**: See Infinite Recursion

A Mathematical Example: Factorial

- Non-recursive definition:

  $n! = n \times n\text{-}1 \times \ldots \times 2 \times 1$

  $= n\,(n\text{-}1 \times \ldots \times 2 \times 1)$

- Recursive definition:

  $n! = n\,(n\text{-}1)!$ for $n \geq 0$ **Recursive case**

  $0! = 1$ **Base case**

What happens if there is no base case?

Factorial as a Recursive Function

```
def factorial(n):
    """Returns: factorial of n.
    Pre: n ≥ 0 an int"""
    if n == 0:
        return 1

    return n*factorial(n-1)
```

- $n! = n\,(n-1)!$
- $0! = 1$

Base case(s)

Recursive case

What happens if there is no base case?

Example: Fibonnaci Sequence

- Sequence of numbers: 1, 1, 2, 3, 5, 8, 13, ...

  a_0 a_1 a_2 a_3 a_4 a_5 a_6

  - Get the next number by adding previous two
  - What is a_8?

A: $a_8 = 21$
B: $a_8 = 29$
C: $a_8 = 34$
D: None of these.

Example: Fibonnaci Sequence

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A: $a_8 = 21$
B: $a_8 = 29$
C: $a_8 = 34$ **correct**
D: None of these.

Example: Fibonnaci Sequence

- Sequence of numbers: 1, 1, 2, 3, 5, 8, 13, ...

  a_0 a_1 a_2 a_3 a_4 a_5 a_6

  - Get the next number by adding previous two
  - What is a_8?
- Recursive definition:
  - $a_n = a_{n-1} + a_{n-2}$ **Recursive Case**
  - $a_0 = 1$ **Base Case**
  - $a_1 = 1$ **(another) Base Case**

Why did we need two base cases this time?

Fibonacci as a Recursive Function

```
def fibonacci(n):
    """Returns: Fibonacci no. an
    Precondition: n ≥ 0 an int"""
    if n <= 1:
        return 1

    return (fibonacci(n-1)+
            fibonacci(n-2))
```

Base case(s)

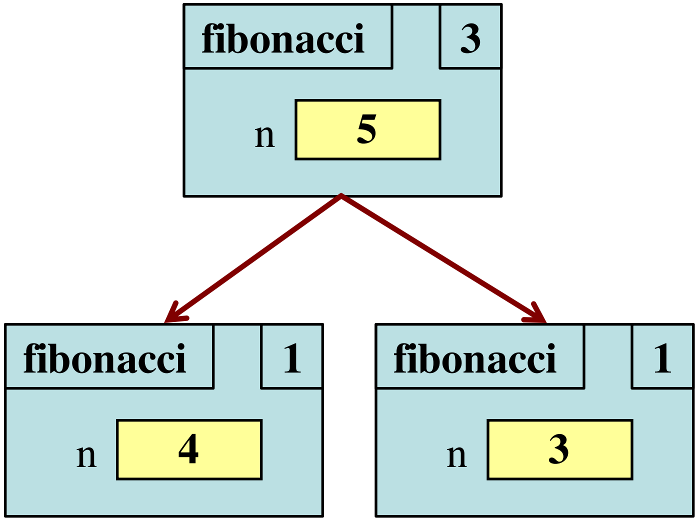
Recursive case

Note difference with base case conditional.

Fibonacci as a Recursive Function

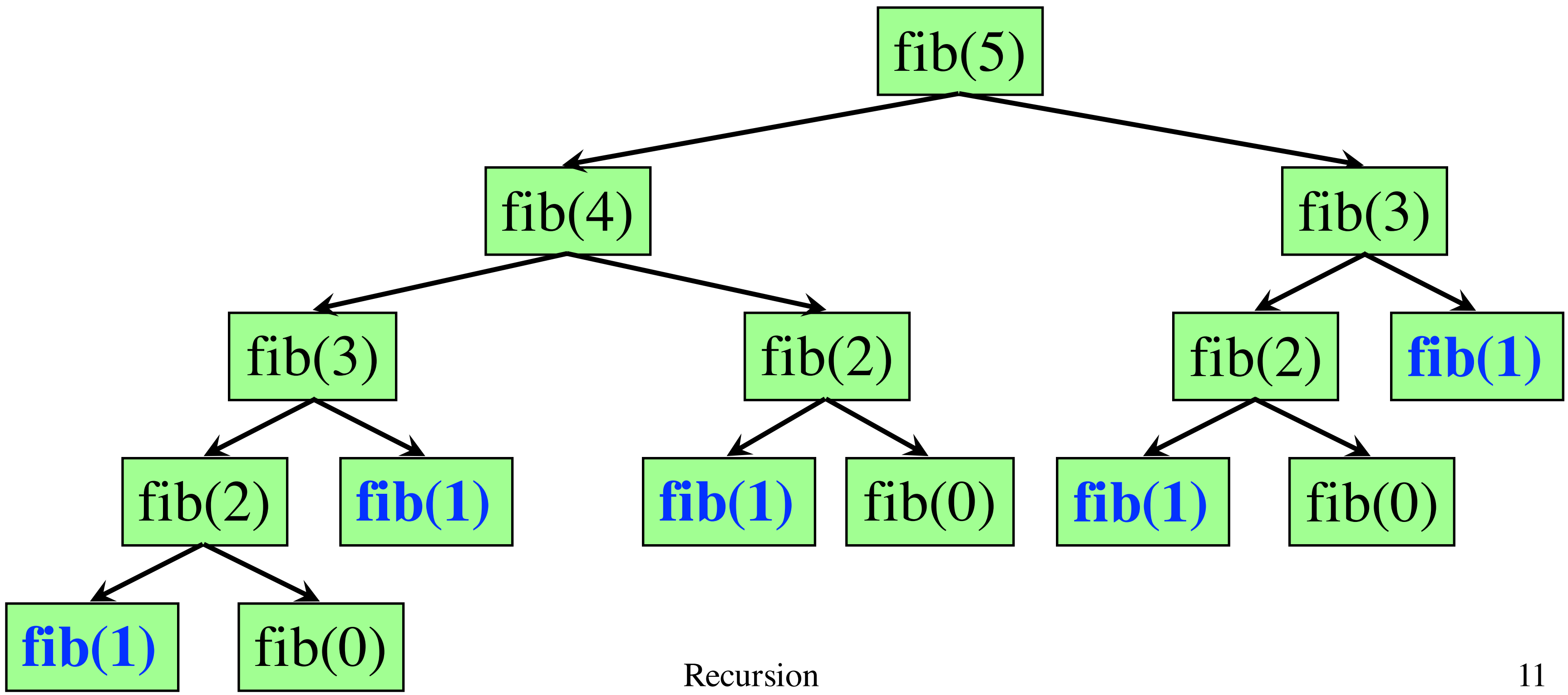
```
def fibonacci(n):
    """Returns: Fibonacci no. a_n
    Precondition: n ≥ 0 an int"""
    if n <= 1:
        return 1

    return (fibonacci(n-1)+
            fibonacci(n-2))
```

- Function that calls itself
  - Each call is new frame
  - Frames require memory
  - ∞ calls = ∞ memory



Fibonacci: # of Frames vs. # of Calls

- Fibonacci is very inefficient.
  - fib(n) has a stack that is always $\leq n$
  - But fib(n) makes a lot of redundant calls



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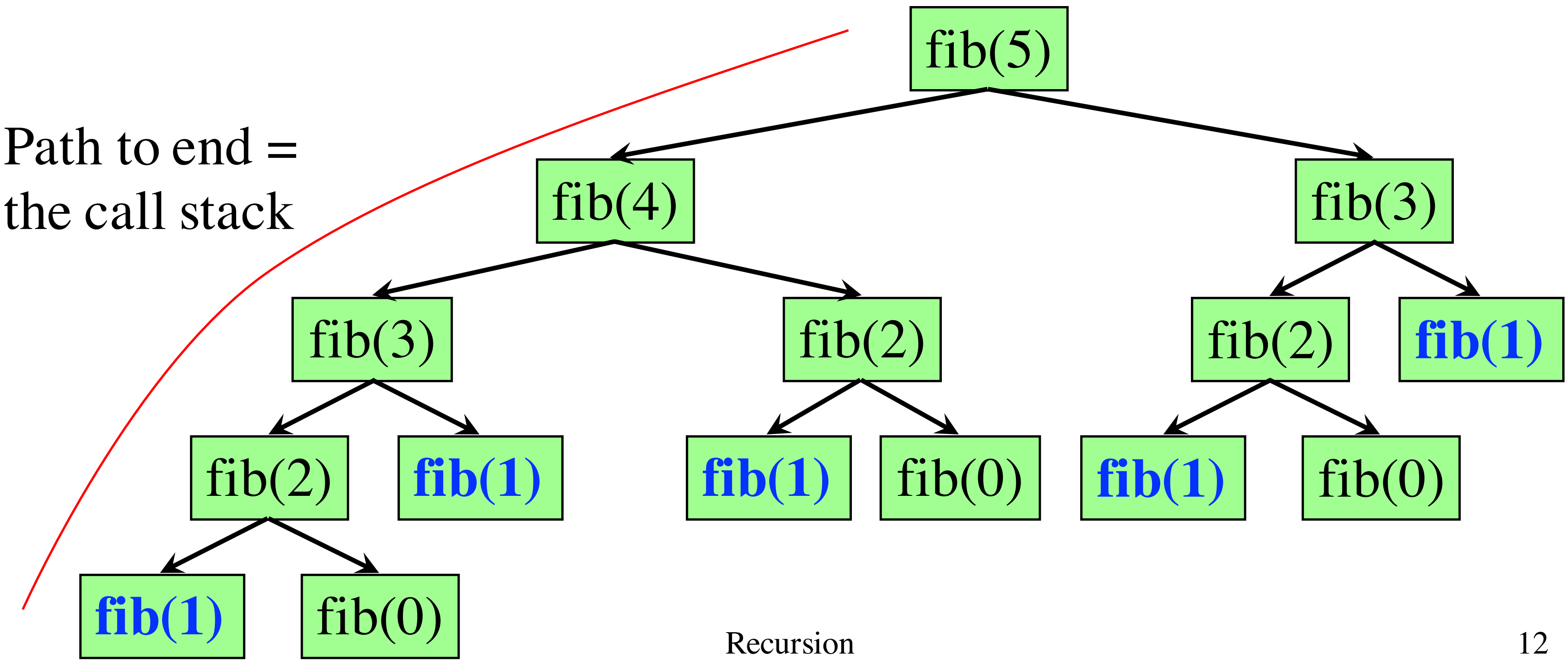

Two Major Issues with Recursion

- **How are recursive calls executed?**
  - We saw this with the Fibonacci example
  - Use the call frame model of execution
- **How do we understand a recursive function (and how do we create one)?**
  - You cannot trace the program flow to understand what a recursive function does – too complicated
  - You need to rely on the **function specification**

How to Think About Recursive Functions

1. **Have a precise function specification.**
2. **Base case(s):**
   - When the parameter values are as small as possible
   - When the answer is determined with little calculation.
3. **Recursive case(s):**
   - Recursive calls are used.
   - Verify recursive cases with the specification
4. **Termination:**
   - Arguments of calls must somehow get “smaller”
   - Each recursive call must get closer to a base case

Understanding the String Example

```
def num_es(s):
    """Returns: # of 'e's in s"""
    # s is empty
    if s == '':
        return 0

    # s has at least one 'e'
    if s[0] == 'e':
        return 1+num_es(s[1:])

    return num_es(s[1:]))
```

Base case

Recursive case

	0	1	len(s)
s	H	ello World!	

- Break problem into parts

> number of e's in s =
> number of e's in s[0]
> + number of e's in s[1:]

- Solve small part directly

> number of e's in s =
> number of e's in s[1:]
> (+1 if s[0] is an 'e')
> (+0 is s[0] not an 'e')

Understanding the String Example

- **Step 1:** Have a precise specification

```
def num_es(s):
    """Returns: # of 'e's in s"""
    # s is empty
    if s == '':
        return 0

    # return # of 'e's in s[0]+# of 'e's in s[1:]
    if s[0] == 'e':
        return 1+num_es(s[1:])

    return num_es(s[1:]))
```

Base case

Recursive case

"Write" your return statement using the **specification**

- **Step 2:** Check the base case
  - When s is the empty string, 0 is (correctly) returned.

Understanding the String Example

- **Step 3**: Recursive calls make progress toward termination

```
def num_es(s):
    """Returns: # of 'e's in s"""
    # s is empty
    if s == '':
        return 0

    # return # of 'e's in s[0]+# of 'e's in s[1:]
    if s[0] == 'e':
        return 1+num_es(s[1:])

    return num_es(s[1:]))
```

parameter s

argument s[1:]

argument s[1:] is smaller than parameter s, so there is progress toward reaching base case 0

- **Step 4**: Check the recursive case
  - Does it match the specification?

Exercise: Remove Blanks from a String

1. Have a precise specification

```
def deblank(s):
    """Returns: s but with its blanks removed"""
```

2. **Base Case**: the smallest String s is ''.

```
if s == '':
    return s
```

3. **Other Cases**: String s has at least 1 character.

```
return (s[0] with blanks removed) + (s[1:] with blanks removed)
```

Exercise: Remove Blanks from a String

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```
if s == '':
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```

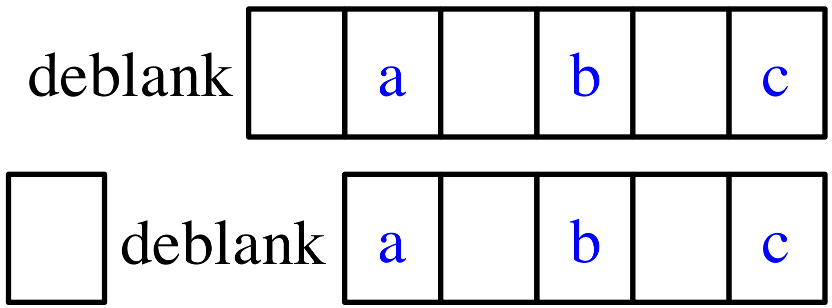
3. **Other Cases**: String s has at least 1 character.

$$\texttt{return } \underbrace{\texttt{(s[0] with blanks removed)}}_{\texttt{('' if s[0] == ' ' else s[0])}} \texttt{ + (s[1:] with blanks removed)}$$

What the Recursion Does

deblank		a		b		c

What the Recursion Does



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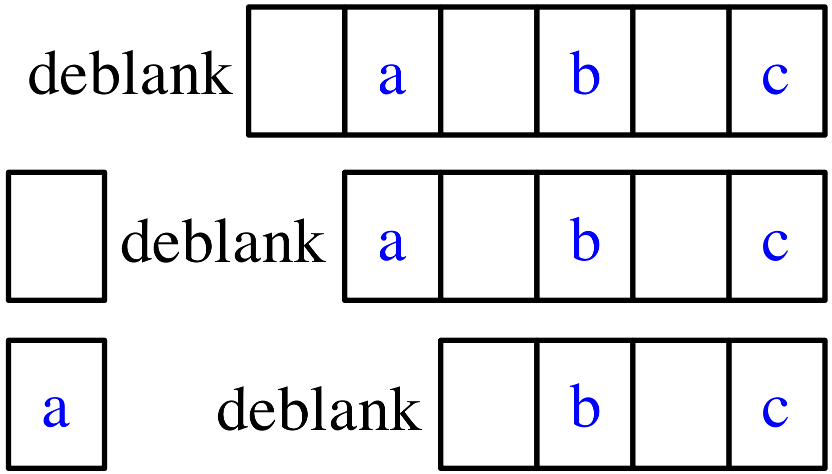

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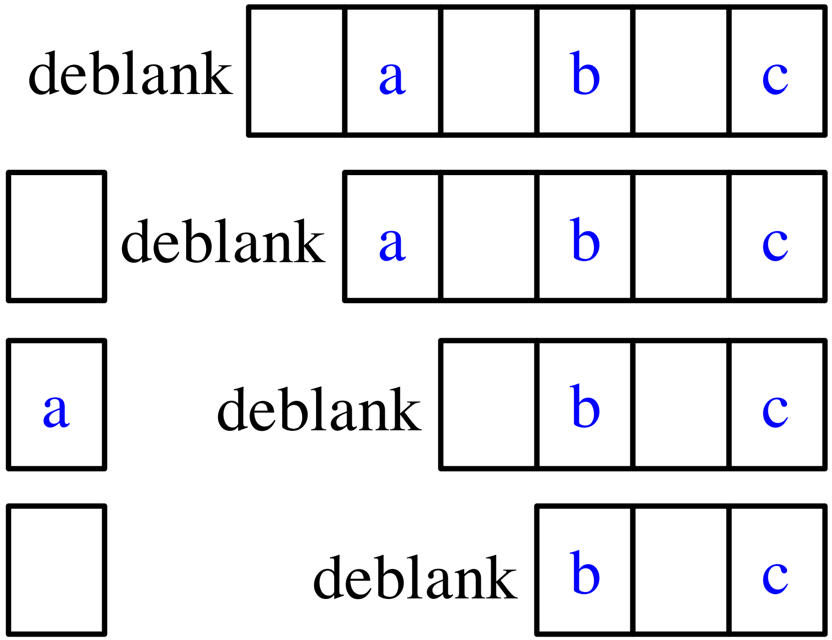

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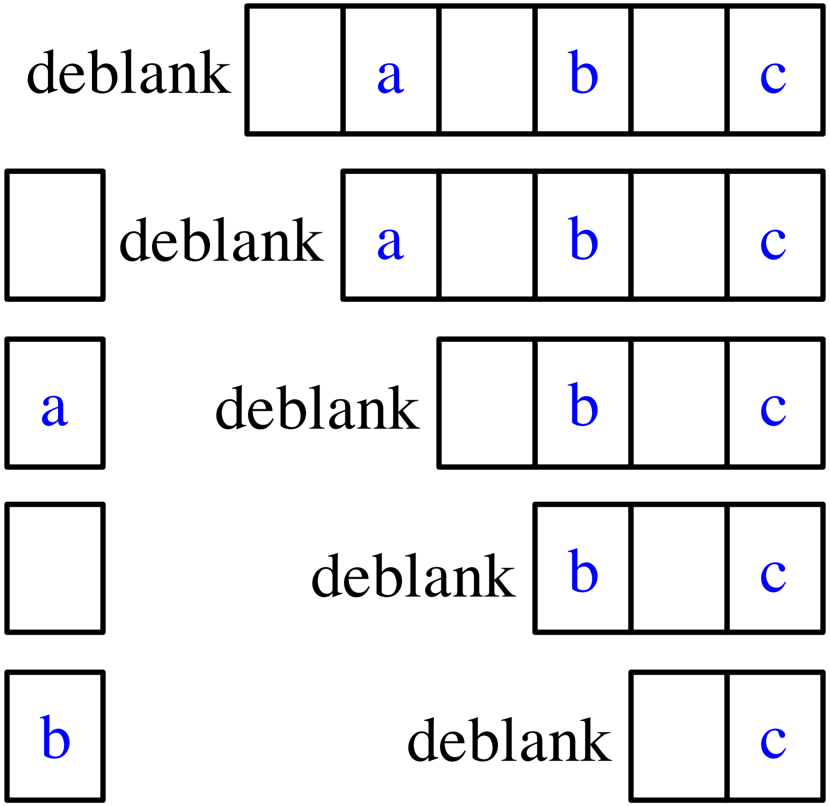

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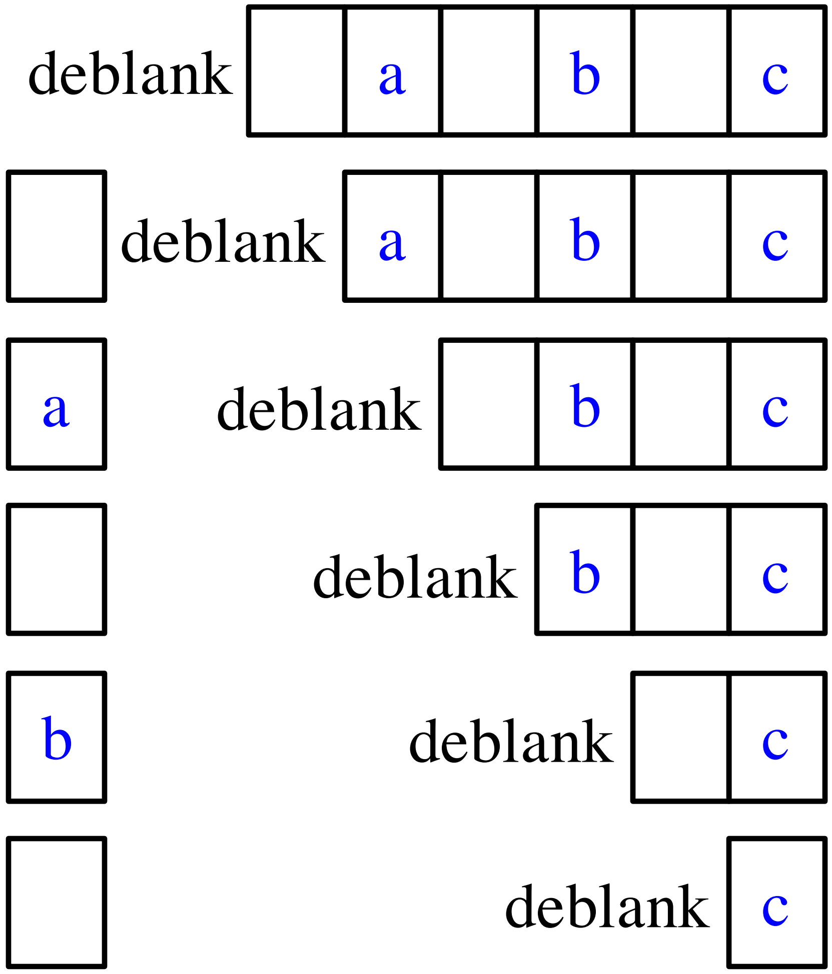

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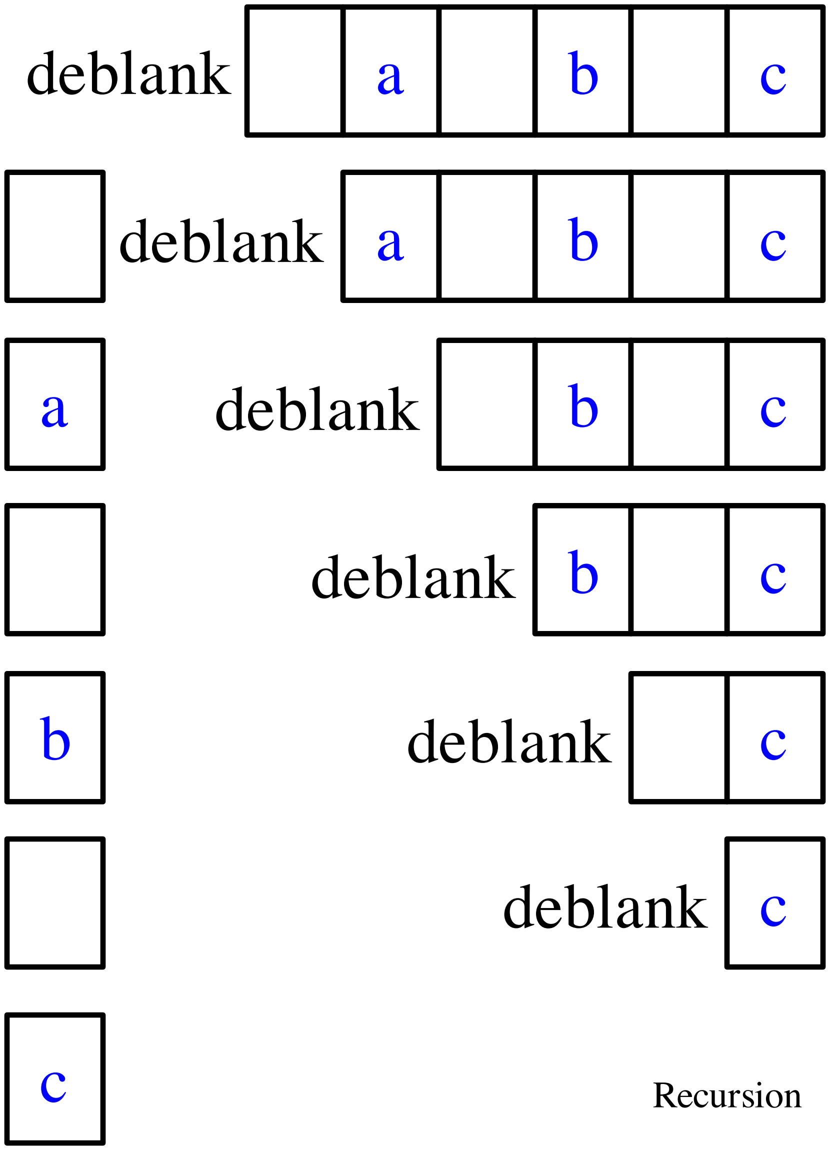

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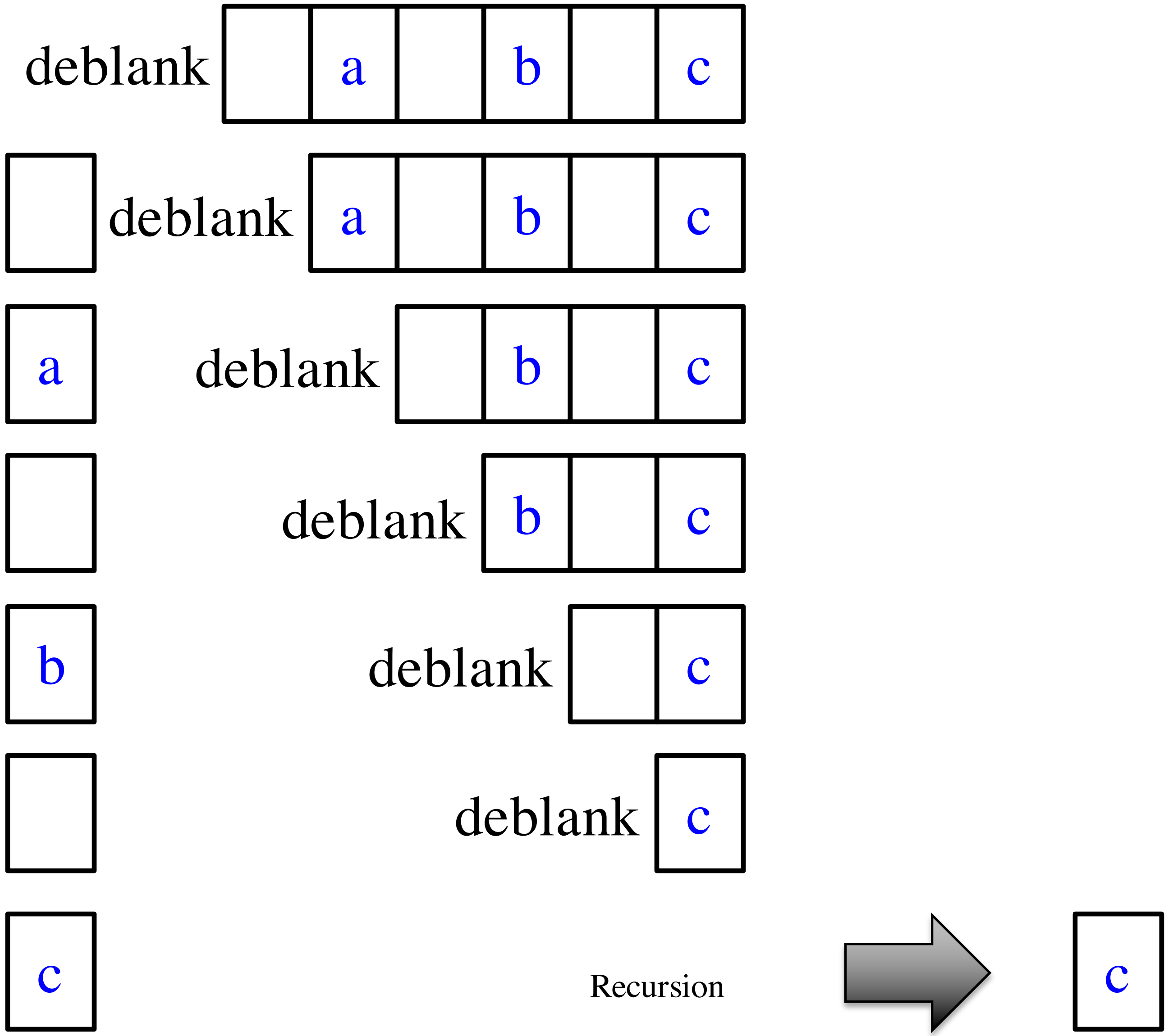

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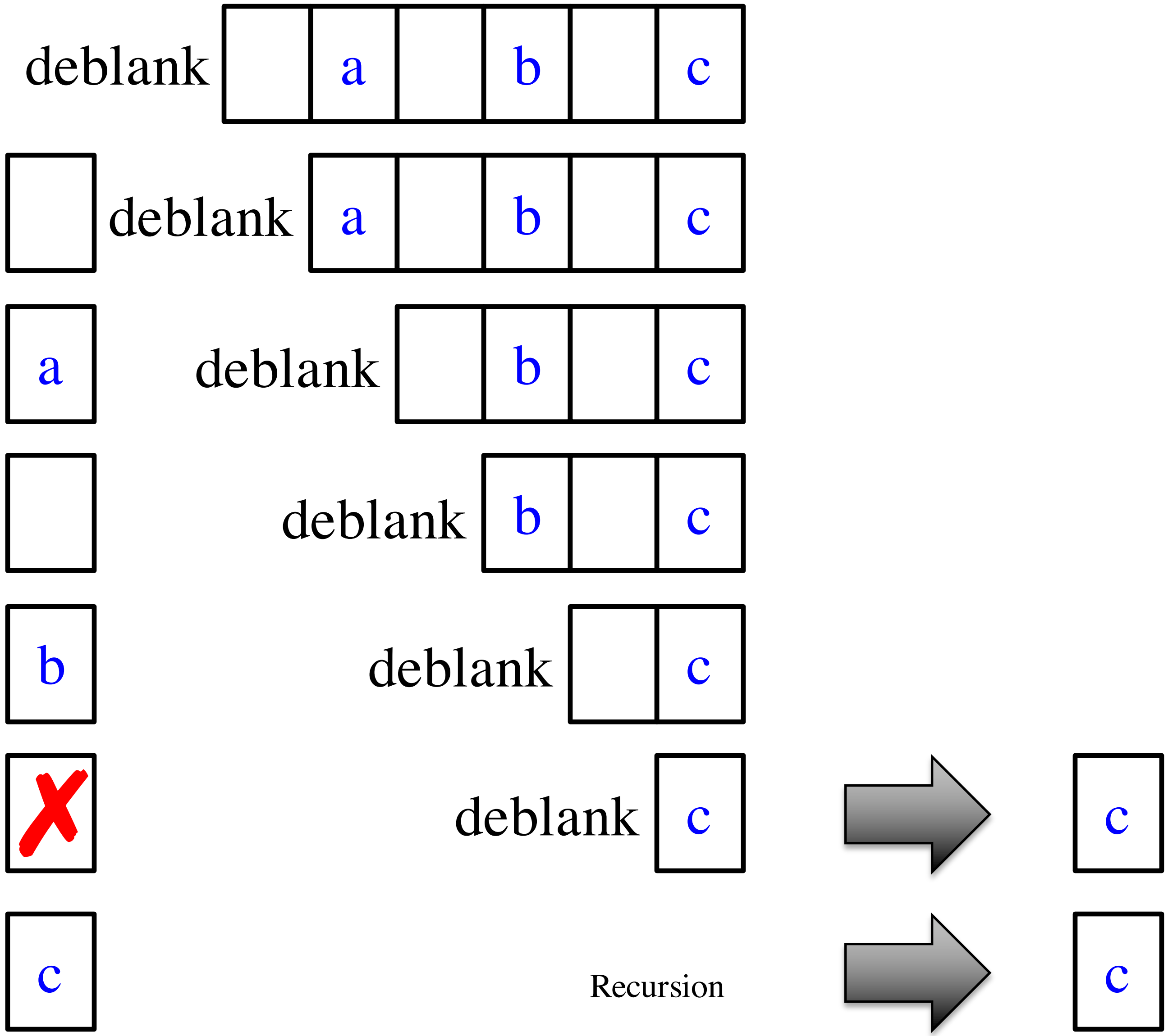

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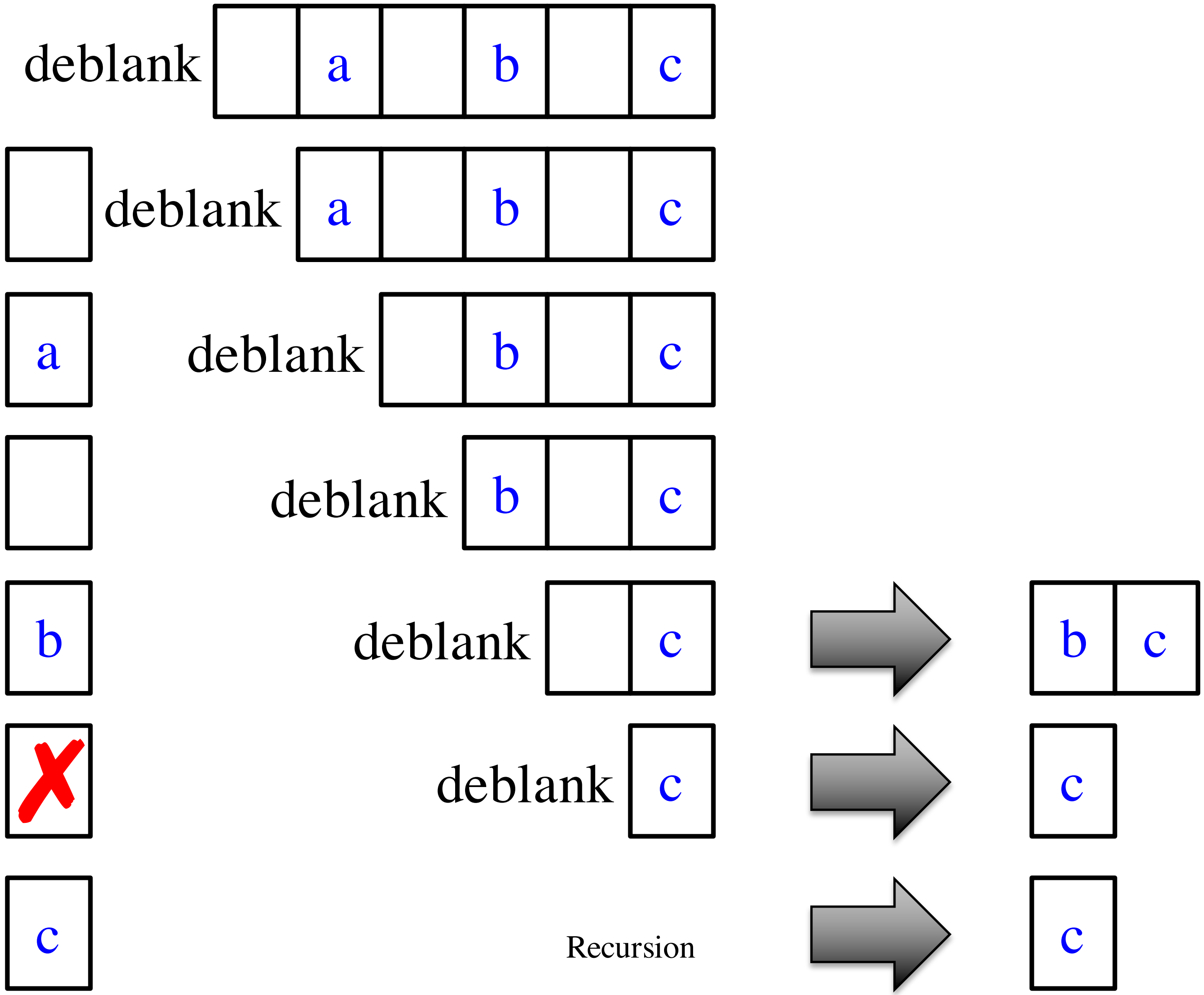

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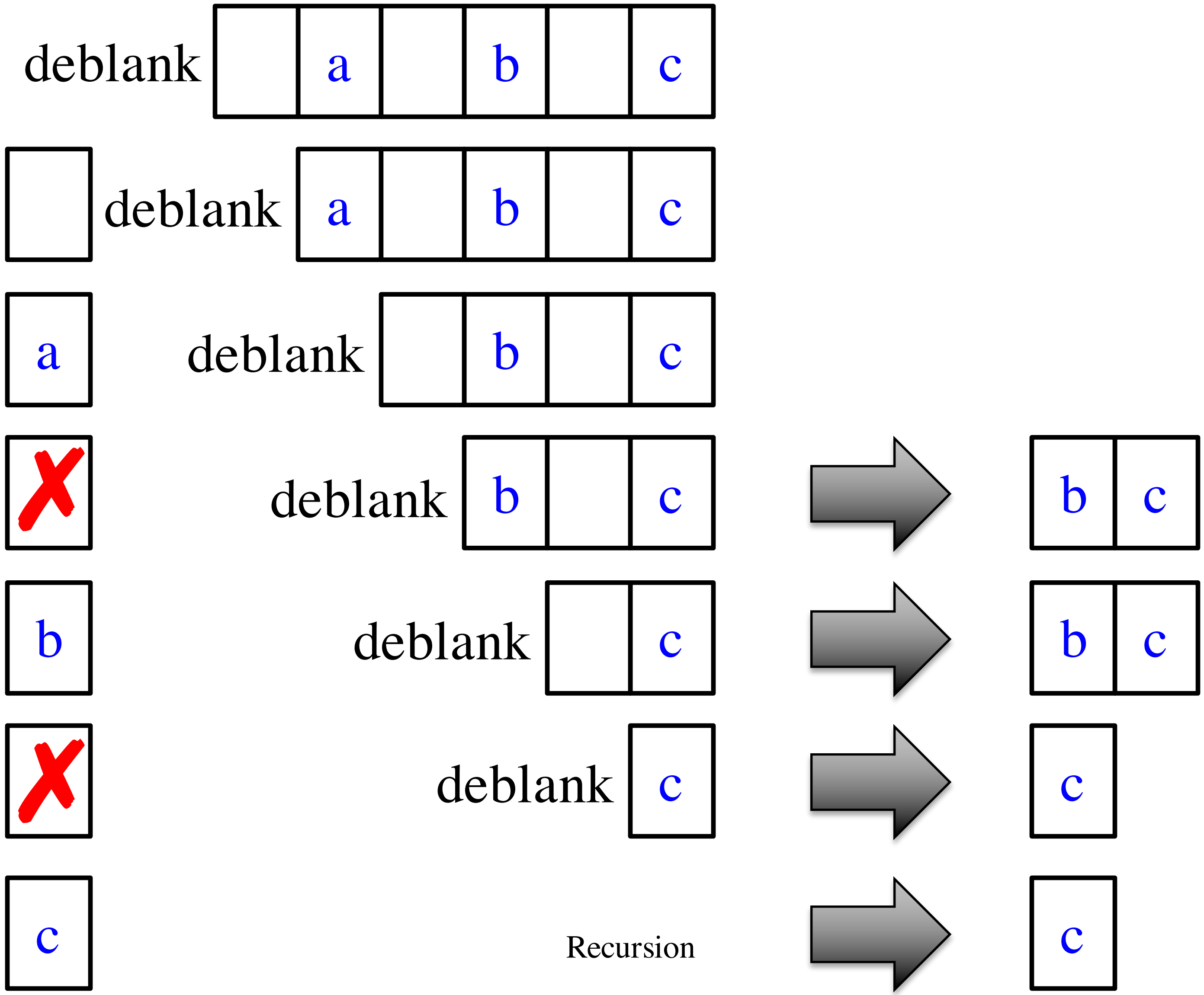

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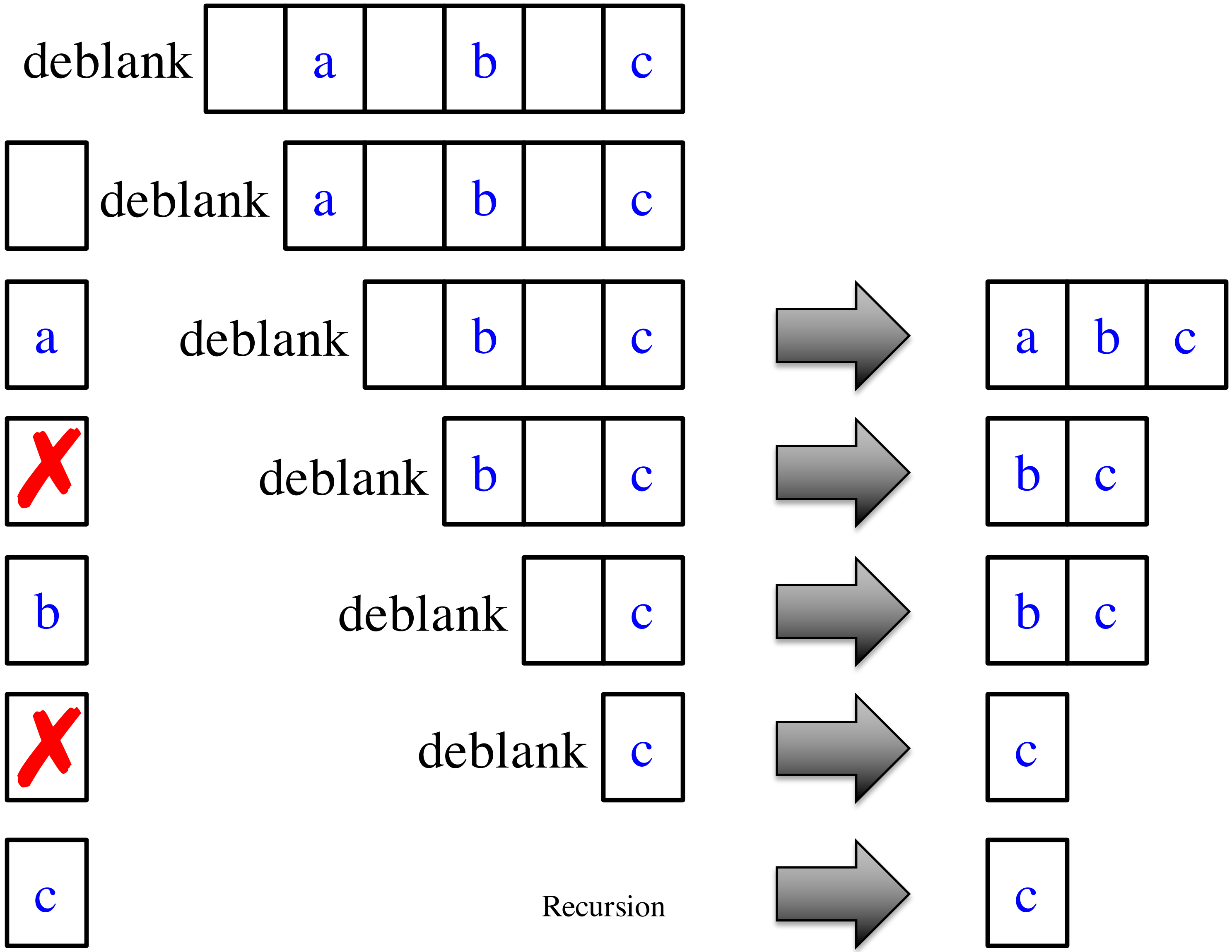

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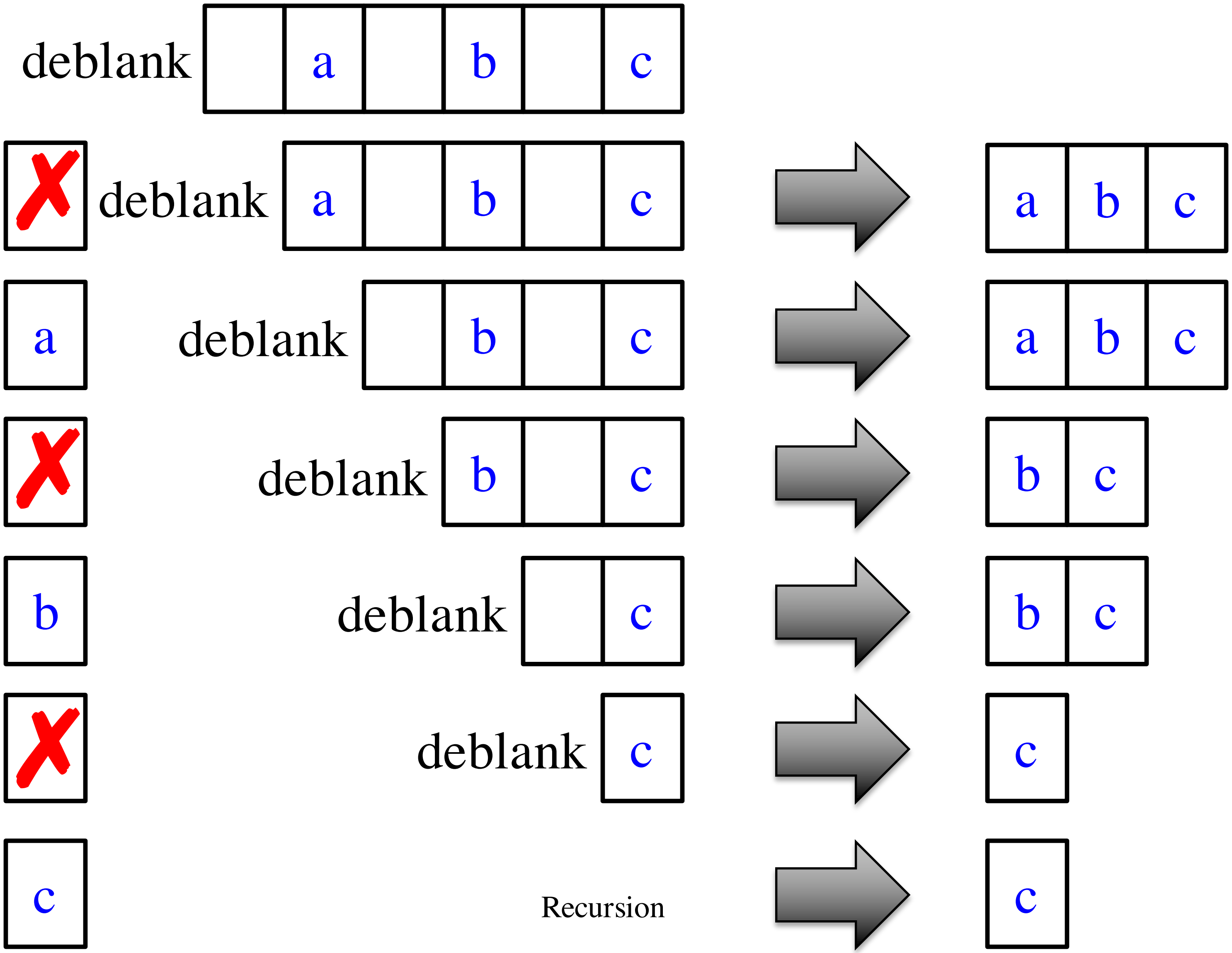

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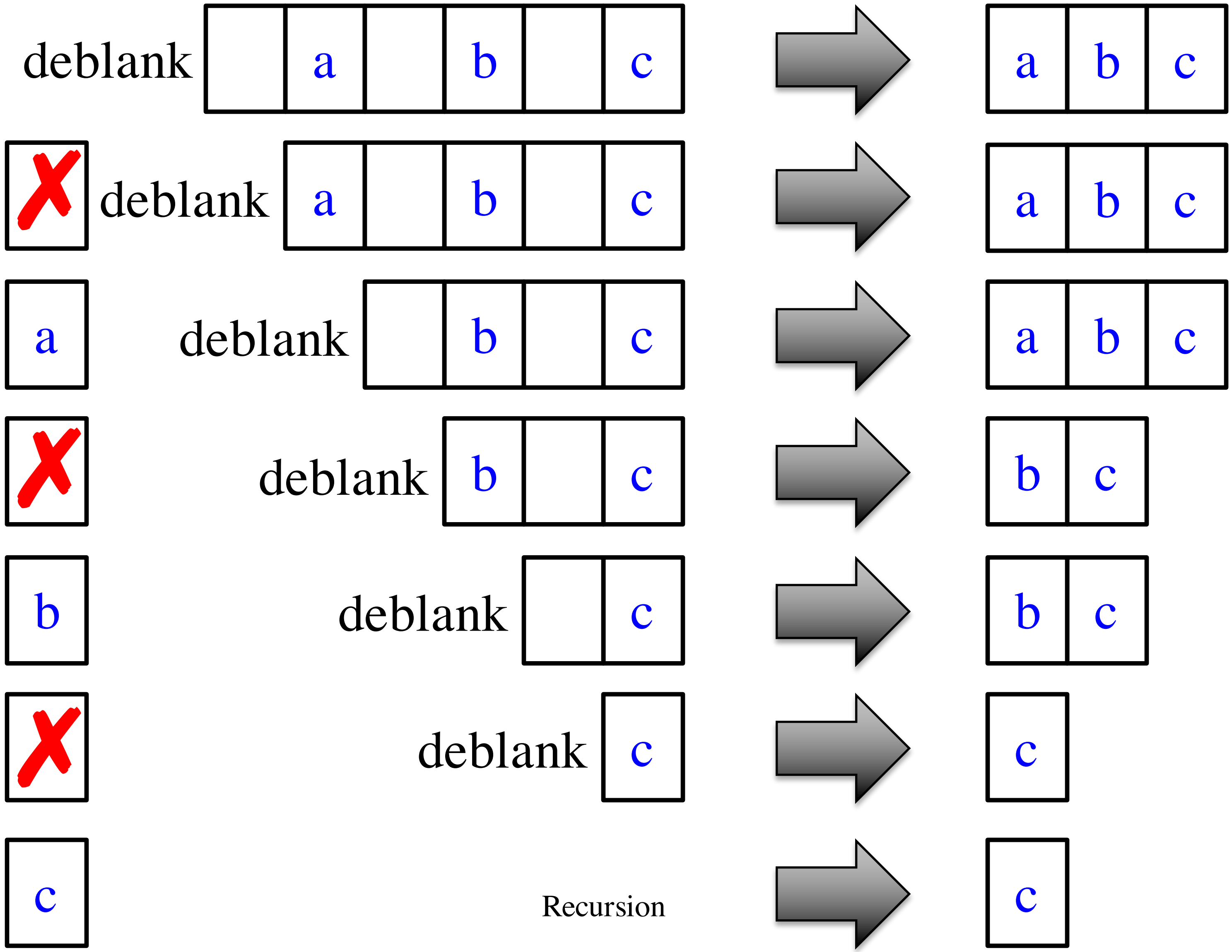


Exercise: Remove Blanks from a String

```
def deblank(s):
    """Returns: s with blanks removed"""
    if s == '':
        return s

    # s is not empty
    if s[0] is a blank:
        return s[1:] with blanks removed

    # s not empty and s[0] not blank
    return (s[0] +
            s[1:] with blanks removed)
```

- Sometimes easier to break up the recursive case
  - Particularly om small part
  - Write recursive case as a sequence of if-statements
- Write code in *pseudocode*
  - Mixture of English and code
  - Similar to top-down design
- Stuff in **red** looks like the function specification!
  - But on a smaller string
  - Replace with deblank(s[1:])

Exercise: Remove Blanks from a String

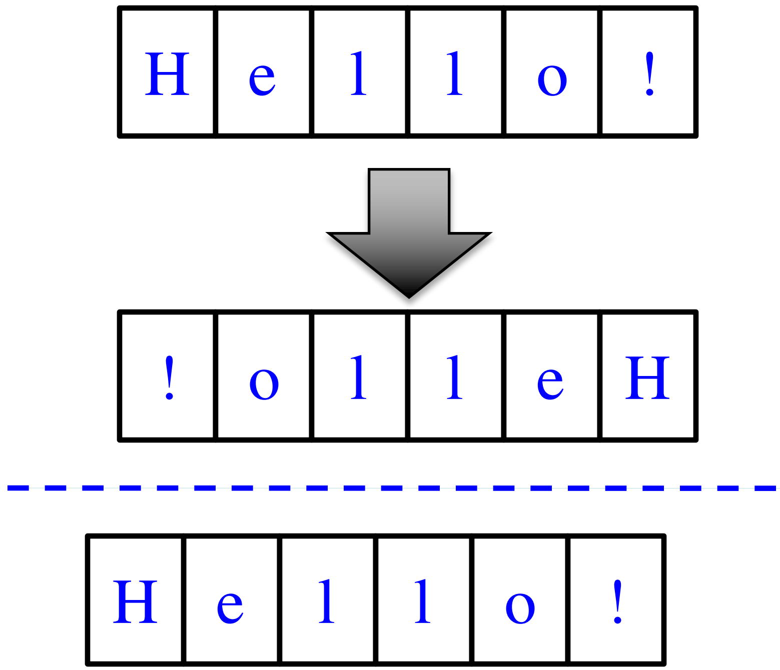
```
def deblank(s):
    """Returns: s with blanks removed"""
    if s == '':
        return s

    # s is not empty
    if s[0] in string.whitespace:
        return deblank(s[1:])

    # s not empty and s[0] not blank
    return (s[0] +
            deblank(s[1:]))
```

- Check the four points:
  1. Precise specification?
  2. **Base case**: correct?
  3. Progress towards termination?
  4. **Recursive case**: correct?

Module string has special constants to simplify detection of whitespace and other characters.

Example: Reversing a String

- **Precise Specification**:
  - Returns: reverse of s
- Solving with recursion
  - Suppose we can reverse a smaller string (e.g. less one character)
  - Can we use that solution to reverse whole string?
- Often easy to understand first without Python
  - Then sit down and code



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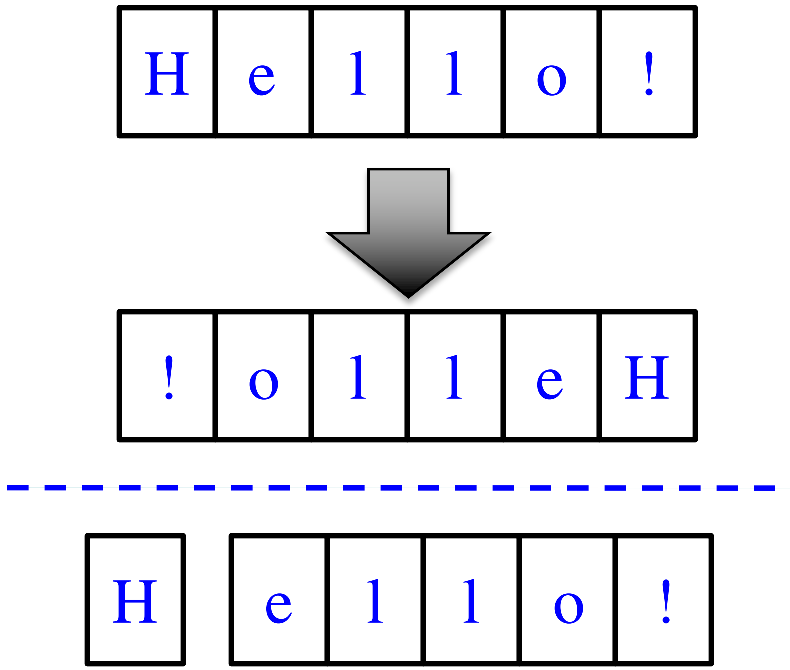

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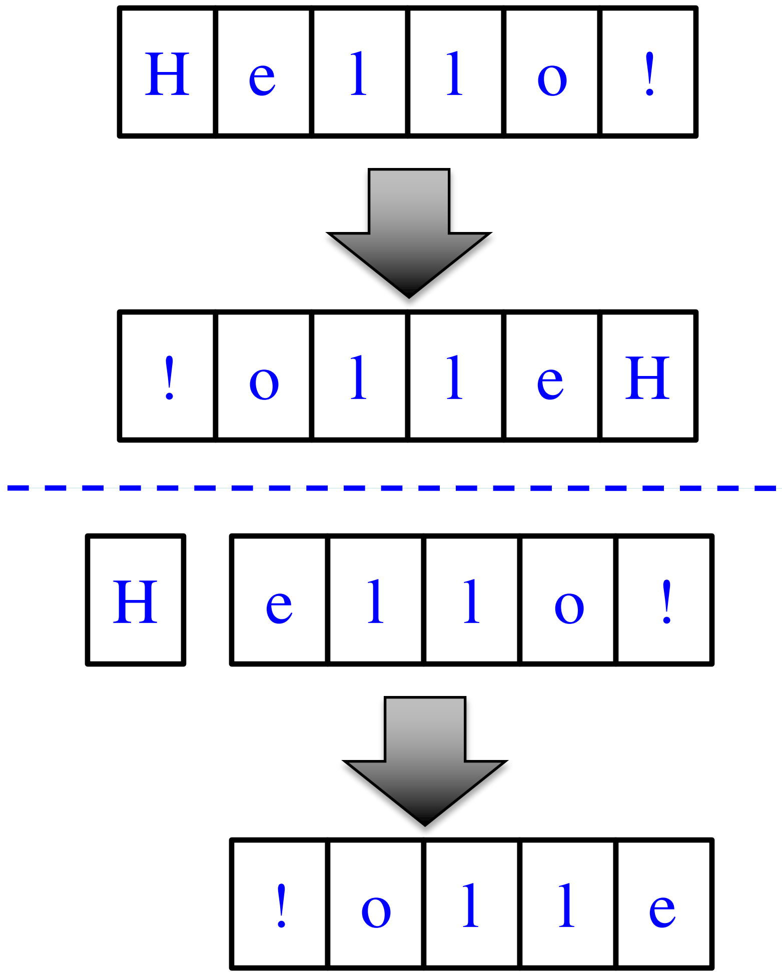

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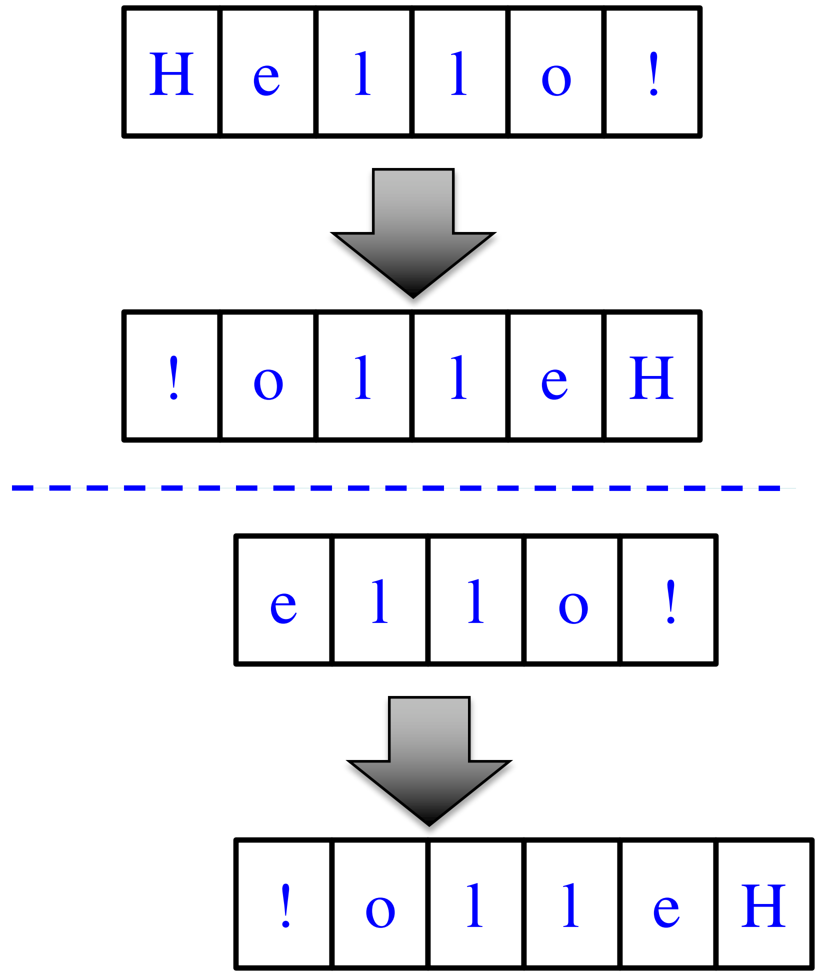
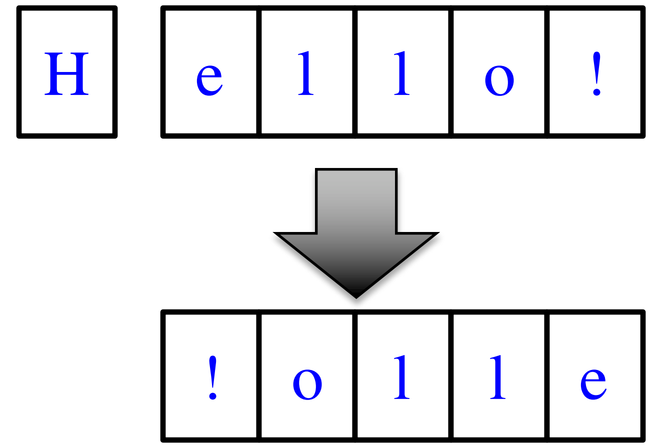


Example: Reversing a String

```
def reverse(s):
    """Returns: reverse of s

    Precondition: s a string"""
    # s is empty
    if s == '':
        return s

    # s has at least one char
    # (reverse of s[1:])+s[0]
    return reverse(s[1:])+s[0]
```



✔ 1. Precise specification?

✔ 2. Base case: correct?

✔ 3. Recursive case: progress to termination?

✔ 4. Recursive case: correct?

Next Time: Recursion vs. For-Loops