Lecture 24

Designing Sequence Algorithms

Announcements for This Lecture

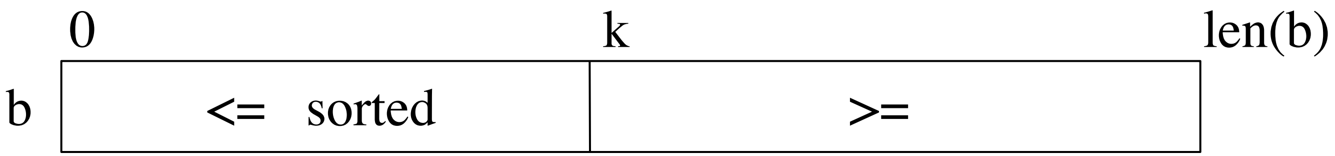
Assignments

- A5 will be slow in grading
  - Want to be fair in grading
  - Hard to grade over T-Day
  - Done by last day of class
- **Survey** still up for A5
  - All new questions!
  - Each person must answer
- A6 due **Monday, Dec. 9**
  - 2.5 weeks including T-Day
  - 2 weeks without the break

Labs

- Lab 11 due before break
  - Show it next Tuesday
  - Show it in consultant hours
- No lab next week
  - But Tuesday hours are open
  - Go for help on lab or A6
- Lab 12 is the last lab
  - Due before final exam
  - Consultant hours still open

Horizontal Notation for Sequences

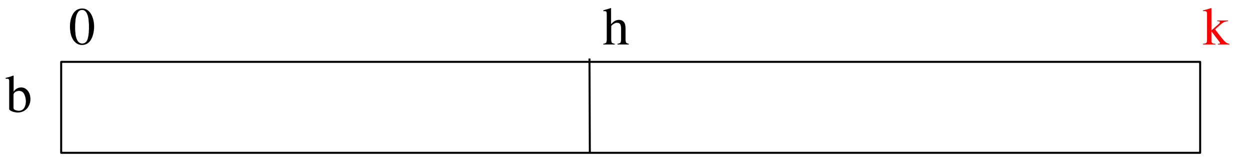


0 k len(b)

b <= sorted >=

Example of an assertion about an sequence b. It asserts that:

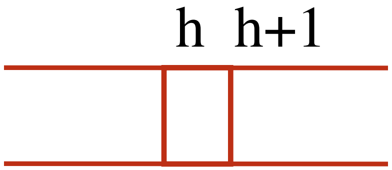
1. b[0..k–1] is sorted (i.e. its values are in ascending order)
2. Everything in b[0..k–1] is ≤ everything in b[k..len(b)–1]



0 h k

b

Given index h of the first element of a segment and index k of the element that follows that segment, the number of values in the segment is k – h.



h h+1

(h+1) – h = 1

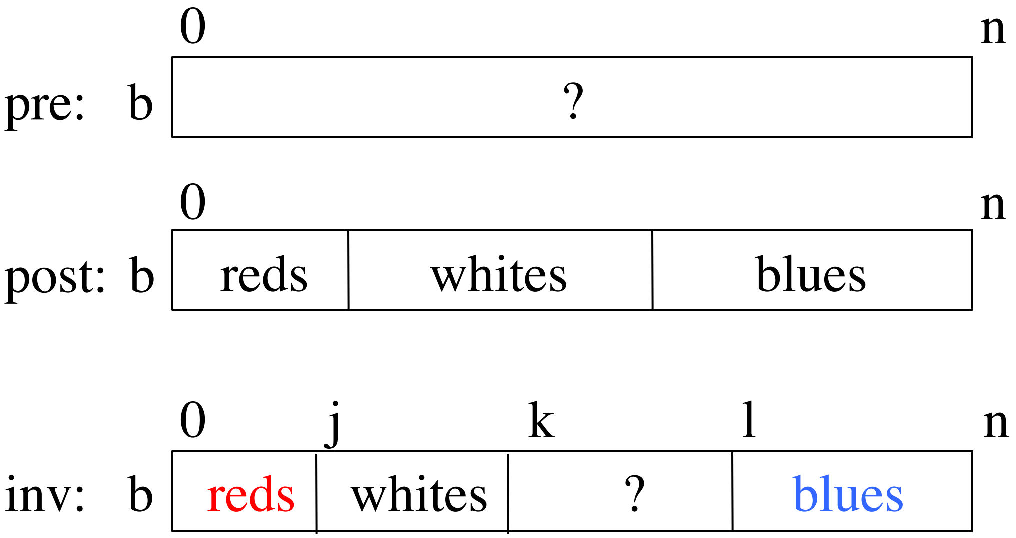
b[h .. k – 1] has k – h elements in it.

Developing Algorithms on Sequences

- Specify the algorithm by giving its precondition and postcondition as pictures.
- Draw the invariant by drawing another picture that "generalizes" the precondition and postcondition
  - The invariant is true at the beginning and at the end
- The four loop design questions (memorize them)
  1. How does loop start (how to make the invariant true)?
  2. How does it stop (is the postcondition true)?
  3. How does repetend make progress toward termination?
  4. How does repetend keep the invariant true?

Generalizing Pre- and Postconditions

- Dutch national flag: tri-color
  - Sequence of 0..n-1 of red, white, blue "pixels"
  - Arrange to put reds first, then whites, then blues



0 n

pre: b ?

0 n

post: b reds whites blues

0 j k l n

inv: b reds whites ? blues

(values in 0..n-1 are unknown)

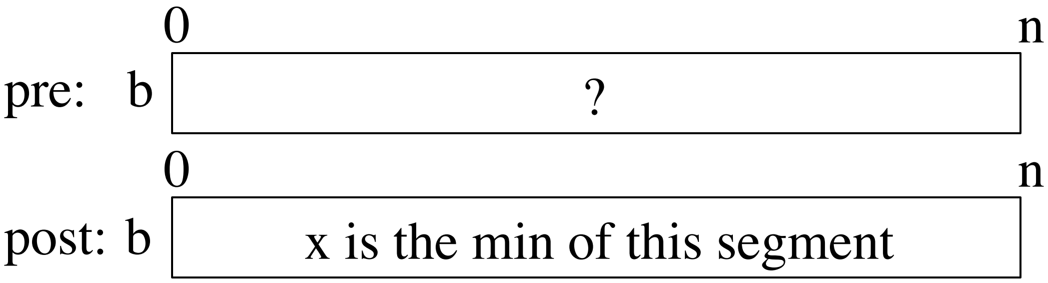
Make the red, white, blue sections initially **empty**:

- Range i..i-1 has 0 elements
- Main reason for this trick

Changing loop variables turns invariant into postcondition.

Generalizing Pre- and Postconditions

- Finding the minimum of a sequence.



0 n

pre: b ?

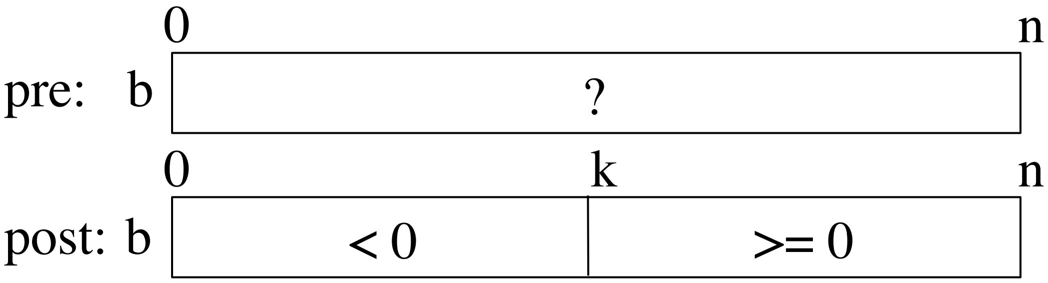
0 n

post: b x is the min of this segment

and n >= 0

(values in 0..n
are unknown)

- Put negative values before nonnegative ones.



0 n

pre: b ?

0 k n

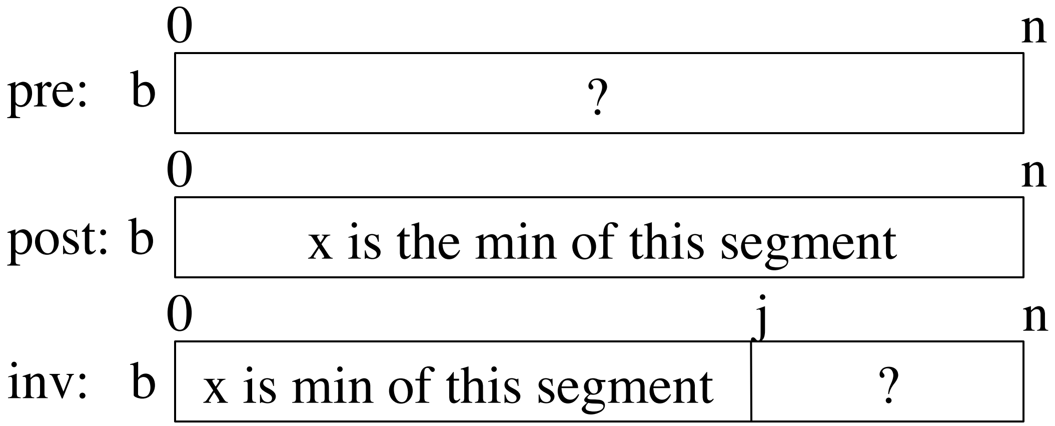
post: b < 0 >= 0

and n >= 0

(values in 0..n
are unknown)

Generalizing Pre- and Postconditions

- Finding the minimum of a sequence.



pre: b [0 ? n] and n >= 0

post: b [0 x is the min of this segment n]

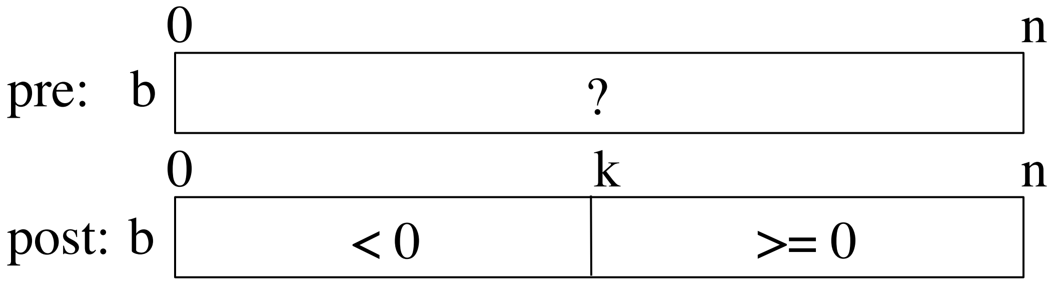
inv: b [0 x is min of this segment j ? n]

and n >= 0

(values in 0..n are unknown)

(values in j..n are unknown)

- Put negative values before nonnegative ones.



pre: b [0 ? n] and n >= 0

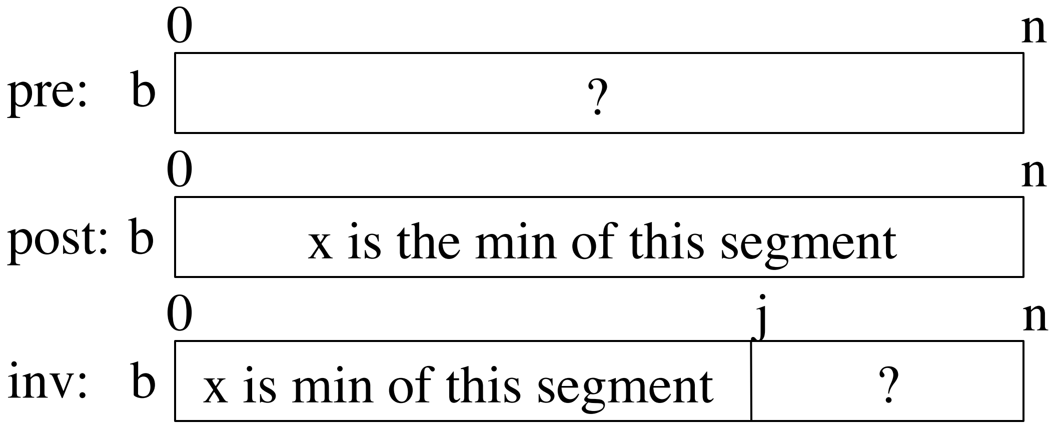
post: b [0 < 0 k >= 0 n]

and n >= 0

(values in 0..n are unknown)

Generalizing Pre- and Postconditions

- Finding the minimum of a sequence.



pre: b [0 … n] ? and n >= 0

post: b [0 … n] x is the min of this segment

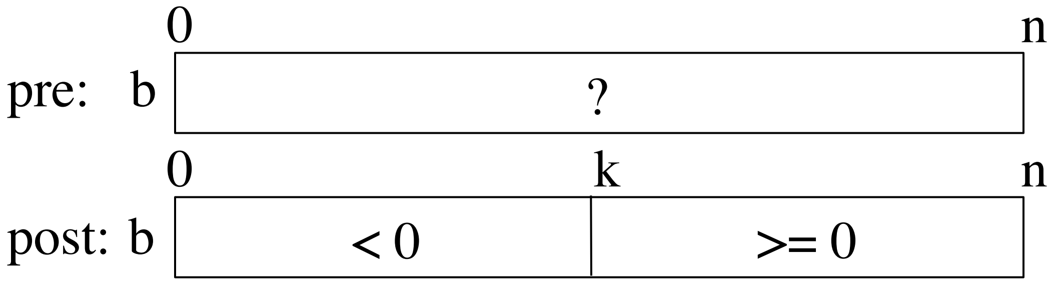
inv: b [0 … j] x is min of this segment [j … n] ?

(values in 0..n are unknown)

pre: j = 0
post: j = n

(values in j..n are unknown)

- Put negative values before nonnegative ones.



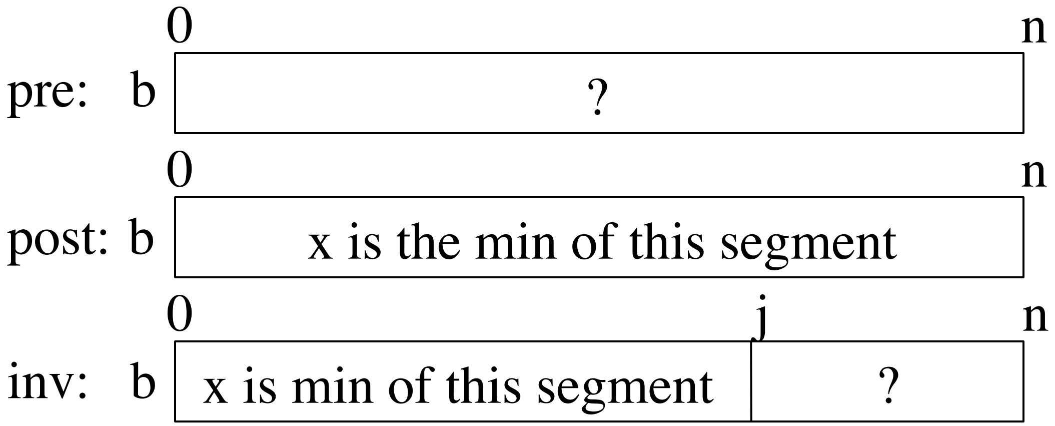
pre: b [0 … n] ? and n >= 0

post: b [0 … k] < 0 [k … n] >= 0

(values in 0..n are unknown)

Generalizing Pre- and Postconditions

- Finding the minimum of a sequence.



0 n

pre: b ?

0 n

post: b x is the min of this segment

0 j n

inv: b x is min of this segment ?

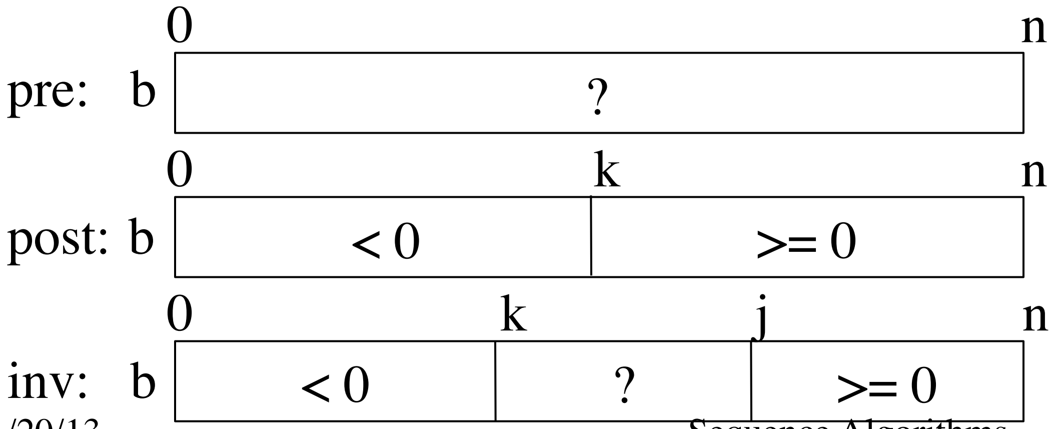
and n >= 0

(values in 0..n are unknown)

pre: j = 0
post: j = n

(values in j..n are unknown)

- Put negative values before nonnegative ones.



0 n

pre: b ?

0 k n

post: b < 0 >= 0

0 k j n

inv: b < 0 ? >= 0

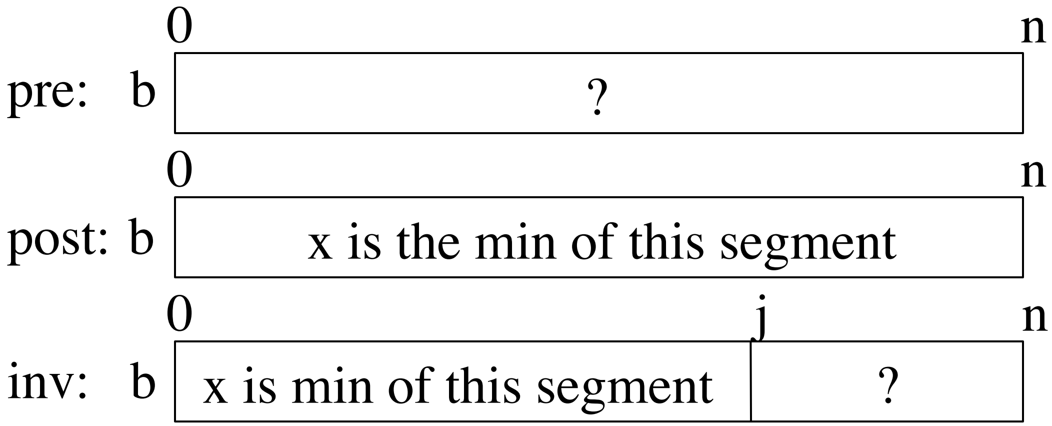
and n >= 0

(values in 0..n are unknown)

(values in k..j are unknown)

Generalizing Pre- and Postconditions

- Finding the minimum of a sequence.



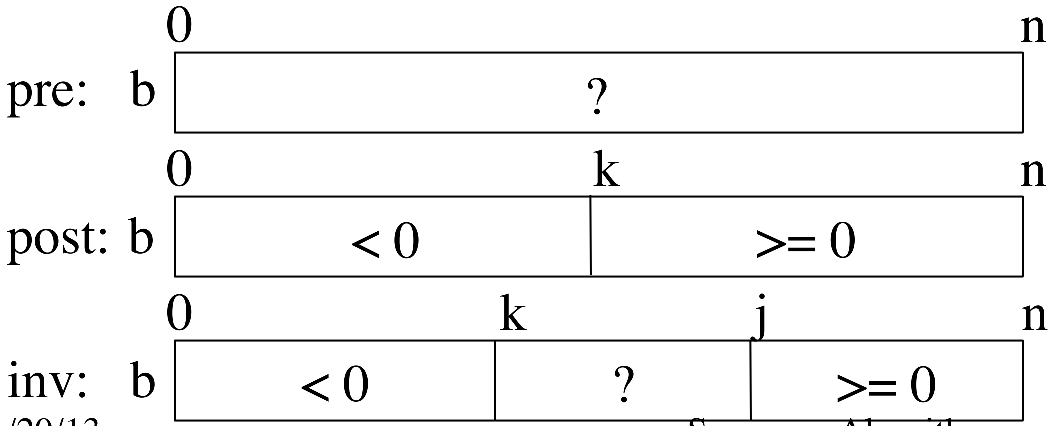
and n >= 0

(values in 0..n are unknown)

pre: j = 0
post: j = n

(values in j..n are unknown)

- Put negative values before nonnegative ones.



and n >= 0

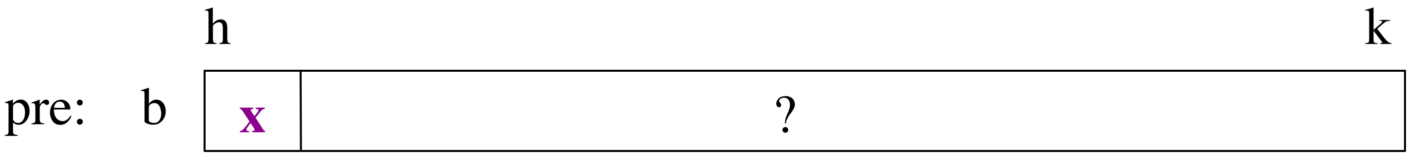
(values in 0..n are unknown)

pre: k = 0,
j = n
post: k = j

(values in k..j are unknown)

Partition Algorithm

- Given a sequence b[h..k] with some value x in b[h]:



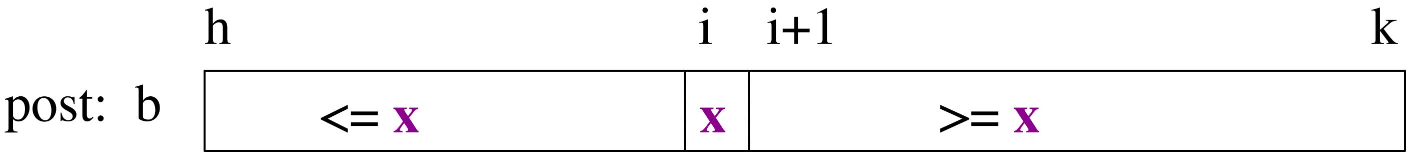
h k

pre: b

x

?

- Swap elements of b[h..k] and store in j to truthify post:



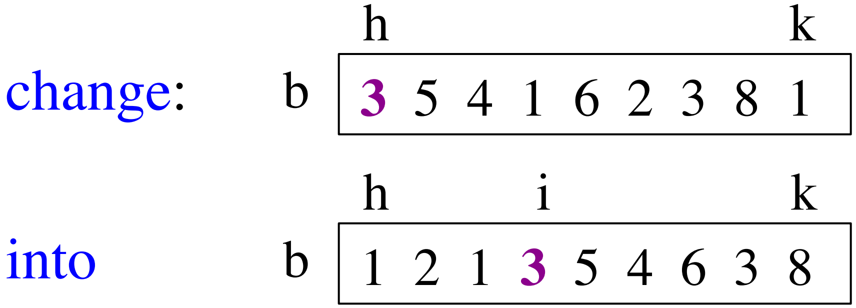
h i i+1 k

post: b

<= x

x

>= x



change: b

h k

3 5 4 1 6 2 3 8 1

into b

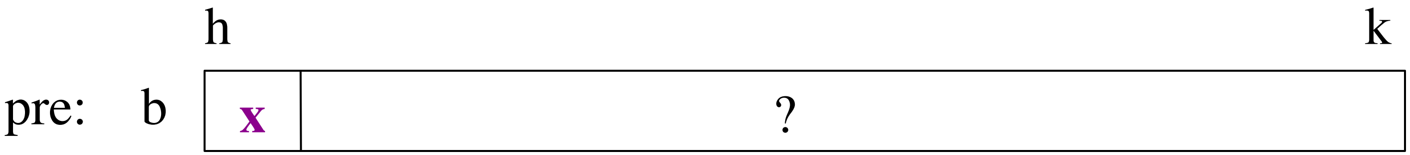
h i k

1 2 1 3 5 4 6 3 8

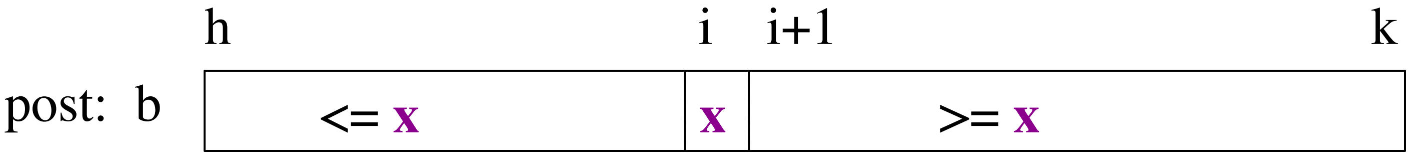
- x is called the pivot value
  - x is not a program variable
  - denotes value initially in b[h]

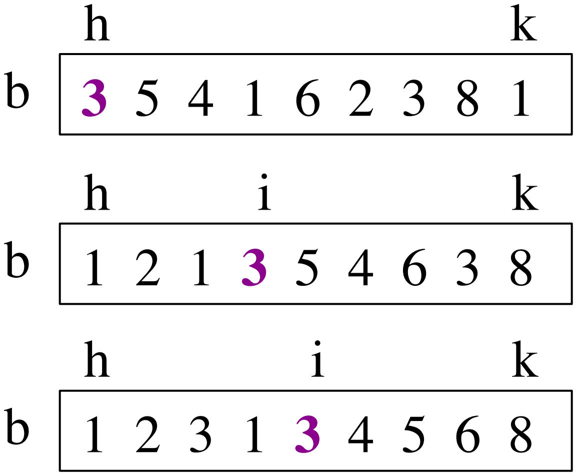
Partition Algorithm

- Given a sequence b[h..k] with some value x in b[h]:



- Swap elements of b[h..k] and store in j to truthify post:





change: b 3 5 4 1 6 2 3 8 1

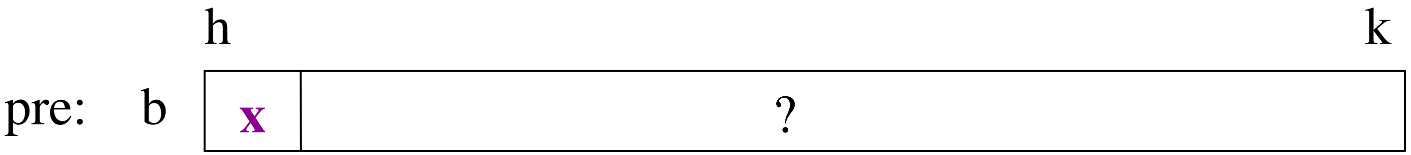
into b 1 2 1 3 5 4 6 3 8

or b 1 2 3 1 3 4 5 6 8

- x is called the pivot value
  - x is not a program variable
  - denotes value initially in b[h]

Partition Algorithm

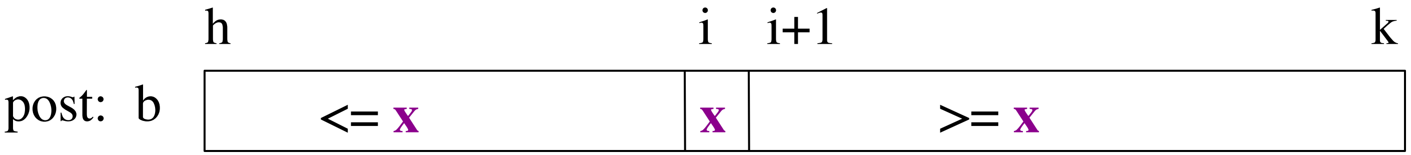
- Given a sequence b[h..k] with some value x in b[h]:



h k

pre: b x ?

- Swap elements of b[h..k] and store in j to truthify post:



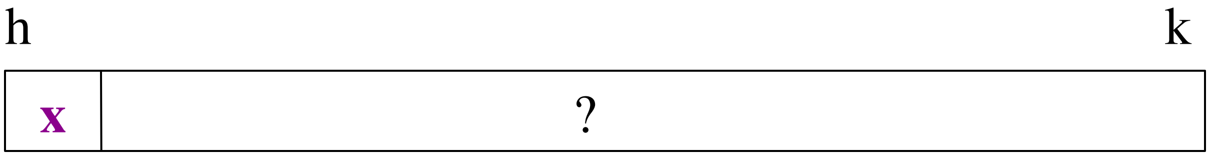
h i i+1 k

post: b <= x x >= x

Partition Algorithm

- Given a sequence b[h..k] with some value x in b[h]:

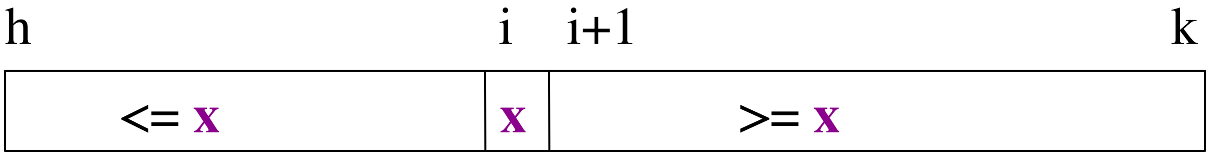
pre: b


h k

x ?

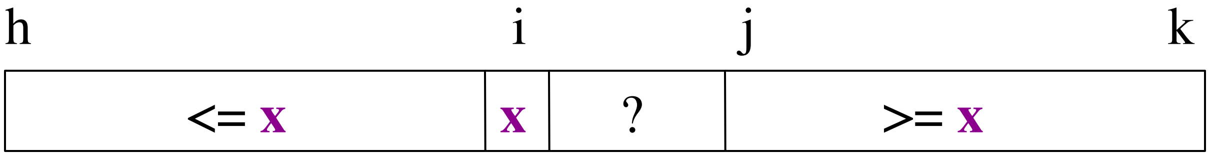
- Swap elements of b[h..k] and store in j to truthify post:

post: b


h i i+1 k

<= x x >= x

inv: b


h i j k

<= x x ? >= x

- Agrees with precondition when i = h, j = k+1
- Agrees with postcondition when j = i+1

Partition Algorithm Implementation

```
def partition(b, h, k):
    """Partition list b[h..k] around a pivot x = b[h]"""
    i = h; j = k+1; x = b[h]
    # invariant: b[h..i-1] < x, b[i] = x, b[j..k] >= x
    while i < j-1:
        if b[i+1] >= x:
            # Move to end of block.
            _swap(b,i+1,j-1)
            j = j - 1
        else:   # b[i+1] < x
            _swap(b,i,i+1)
            i = i + 1
    # post: b[h..i-1] < x, b[i] is x, and b[i+1..k] >= x
    return i
```

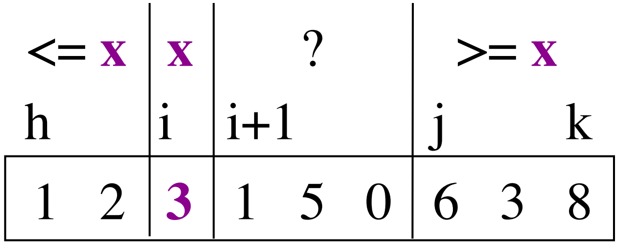
partition(b,h,k), not partition(b[h:k+1])

Remember, slicing always copies the list!

We want to partition the **original** list

Partition Algorithm Implementation

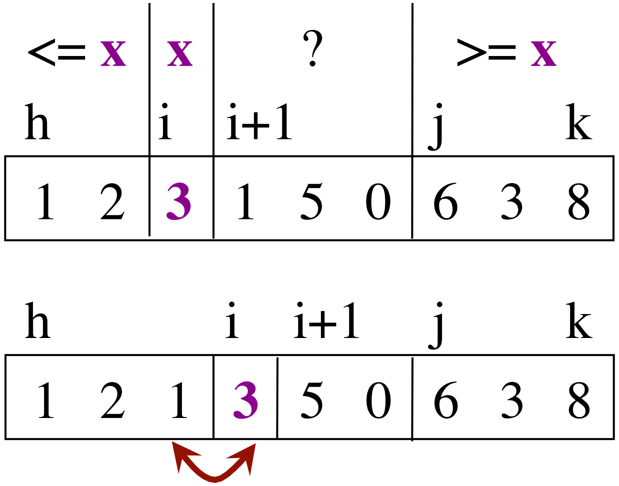
```
def partition(b, h, k):
    """Partition list b[h..k] around a pivot x = b[h]"""
    i = h; j = k+1; x = b[h]
    # invariant: b[h..i-1] < x, b[i] = x, b[j..k] >= x
    while i < j-1:
        if b[i+1] >= x:
            # Move to end of block.
            _swap(b,i+1,j-1)
            j = j - 1
        else:   # b[i+1] < x
            _swap(b,i,i+1)
            i = i + 1
    # post: b[h..i-1] < x, b[i] is x, and b[i+1..k] >= x
    return i
```



<= x	x	?	>= x
h	i	i+1	j k
1 2	3	1 5 0	6 3 8

Partition Algorithm Implementation

```
def partition(b, h, k):
    """Partition list b[h..k] around a pivot x = b[h]"""
    i = h; j = k+1; x = b[h]
    # invariant: b[h..i-1] < x, b[i] = x, b[j..k] >= x
    while i < j-1:
        if b[i+1] >= x:
            # Move to end of block.
            _swap(b,i+1,j-1)
            j = j - 1
        else:   # b[i+1] < x
            _swap(b,i,i+1)
            i = i + 1
    # post: b[h..i-1] < x, b[i] is x, and b[i+1..k] >= x
    return i
```

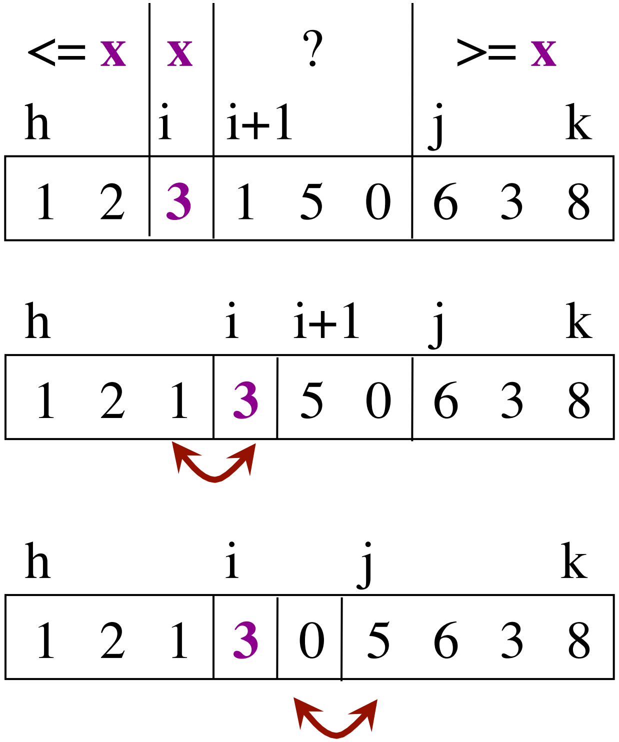


<= x	x	?	>= x
h	i	i+1	j k
1 2	3	1 5 0	6 3 8

h		i	i+1	j k
1 2 1		3	5 0	6 3 8

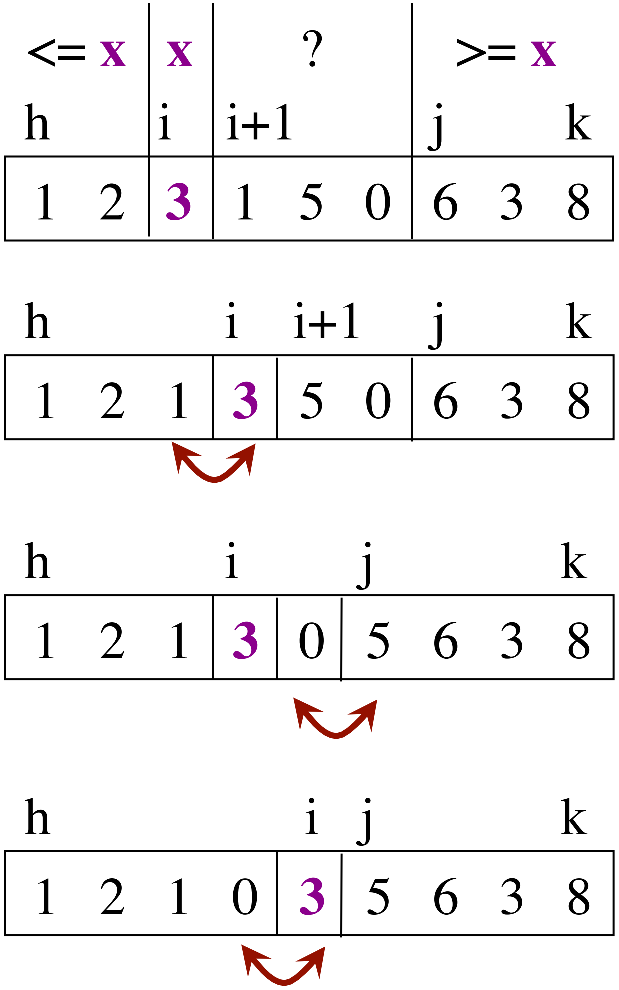
Partition Algorithm Implementation

```
def partition(b, h, k):
    """Partition list b[h..k] around a pivot x = b[h]"""
    i = h; j = k+1; x = b[h]
    # invariant: b[h..i-1] < x, b[i] = x, b[j..k] >= x
    while i < j-1:
        if b[i+1] >= x:
            # Move to end of block.
            _swap(b,i+1,j-1)
            j = j - 1
        else:   # b[i+1] < x
            _swap(b,i,i+1)
            i = i + 1
    # post: b[h..i-1] < x, b[i] is x, and b[i+1..k] >= x
    return i
```



Partition Algorithm Implementation

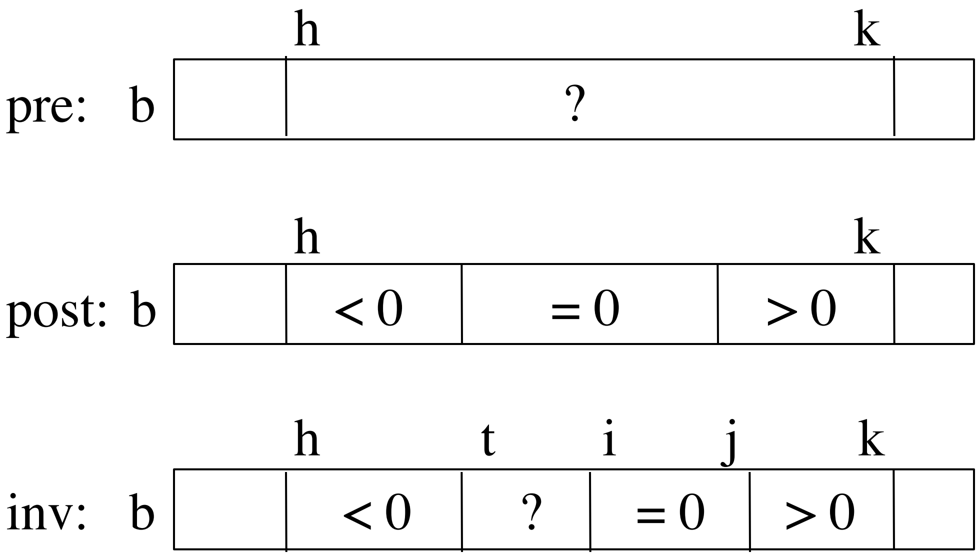
```
def partition(b, h, k):
    """Partition list b[h..k] around a pivot x = b[h]"""
    i = h; j = k+1; x = b[h]
    # invariant: b[h..i-1] < x, b[i] = x, b[j..k] >= x
    while i < j-1:
        if b[i+1] >= x:
            # Move to end of block.
            _swap(b,i+1,j-1)
            j = j - 1
        else:   # b[i+1] < x
            _swap(b,i,i+1)
            i = i + 1
    # post: b[h..i-1] < x, b[i] is x, and b[i+1..k] >= x
    return i
```



Dutch National Flag Variant

- Sequence of integer values
  - ‘red’ = negatives, ‘white’ = 0, ‘blues’ = positive
  - Only rearrange part of the list, not all

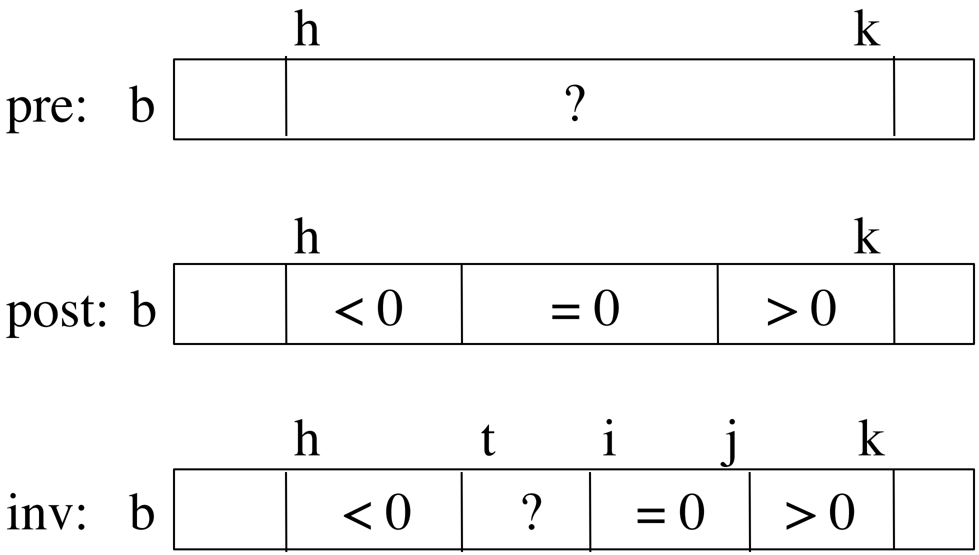
Final Exam:
Be prepared for variants



Dutch National Flag Variant

- Sequence of integer values
  - 'red' = negatives, 'white' = 0, 'blues' = positive
  - Only rearrange part of the list, not all

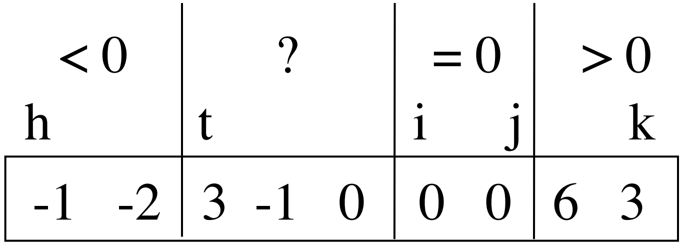
Final Exam:
Be prepared for variants



pre: t = h,
i = k+1,
j = k
post: t = i

Dutch National Flag Algorithm

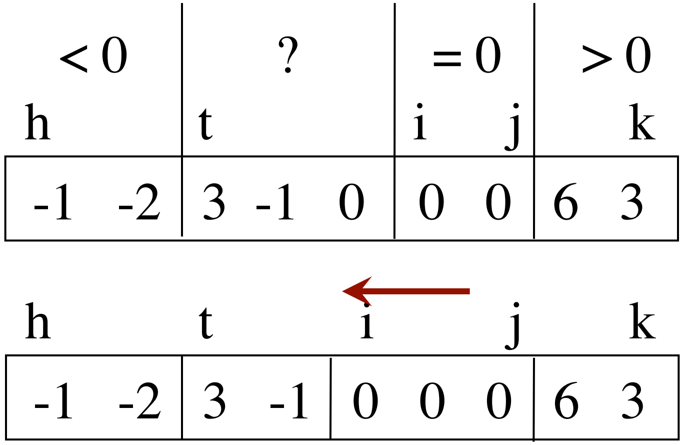
```
def dnf(b, h, k):
    """Returns: partition points as a tuple (i,j)"""
    t = h; i = k+1, j = k;
    # inv: b[h..t-1] < 0, b[t..i-1] ?, b[i..j] = 0, b[j+1..k] > 0
    while t < i:
        if b[i-1] < 0:
            swap(b,i-1,t)
            t = t+1
        elif b[i-1] == 0:
            i = i-1
        else:
            swap(b,i-1,j)
            i = i-1; j = j-1
    # post: b[h..i-1] < 0, b[i..j] = 0, b[j+1..k] > 0
    return (i, j)
```



< 0	?	= 0	> 0
h	t	i j	k
-1 -2	3 -1 0	0 0	6 3

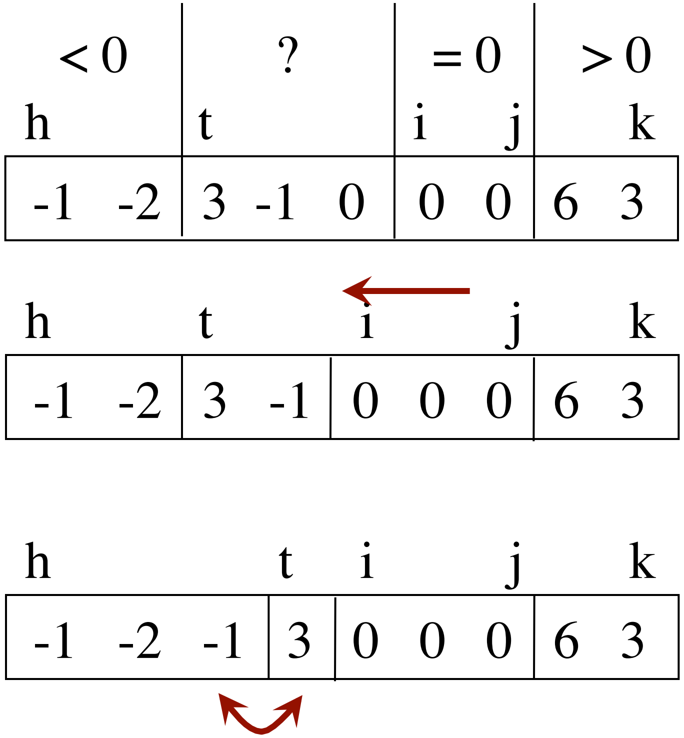
Dutch National Flag Algorithm

```
def dnf(b, h, k):
    """Returns: partition points as a tuple (i,j)"""
    t = h; i = k+1, j = k;
    # inv: b[h..t-1] < 0, b[t..i-1] ?, b[i..j] = 0, b[j+1..k] > 0
    while t < i:
        if b[i-1] < 0:
            swap(b,i-1,t)
            t = t+1
        elif b[i-1] == 0:
            i = i-1
        else:
            swap(b,i-1,j)
            i = i-1; j = j-1
    # post: b[h..i-1] < 0, b[i..j] = 0, b[j+1..k] > 0
    return (i, j)
```



Dutch National Flag Algorithm

```
def dnf(b, h, k):
    """Returns: partition points as a tuple (i,j)"""
    t = h; i = k+1, j = k;
    # inv: b[h..t-1] < 0, b[t..i-1] ?, b[i..j] = 0, b[j+1..k] > 0
    while t < i:
        if b[i-1] < 0:
            swap(b,i-1,t)
            t = t+1
        elif b[i-1] == 0:
            i = i-1
        else:
            swap(b,i-1,j)
            i = i-1; j = j-1
    # post: b[h..i-1] < 0, b[i..j] = 0, b[j+1..k] > 0
    return (i, j)
```



< 0 | ? | = 0 | > 0

h t i j k

-1 -2 | 3 -1 0 | 0 0 | 6 3

h t i j k

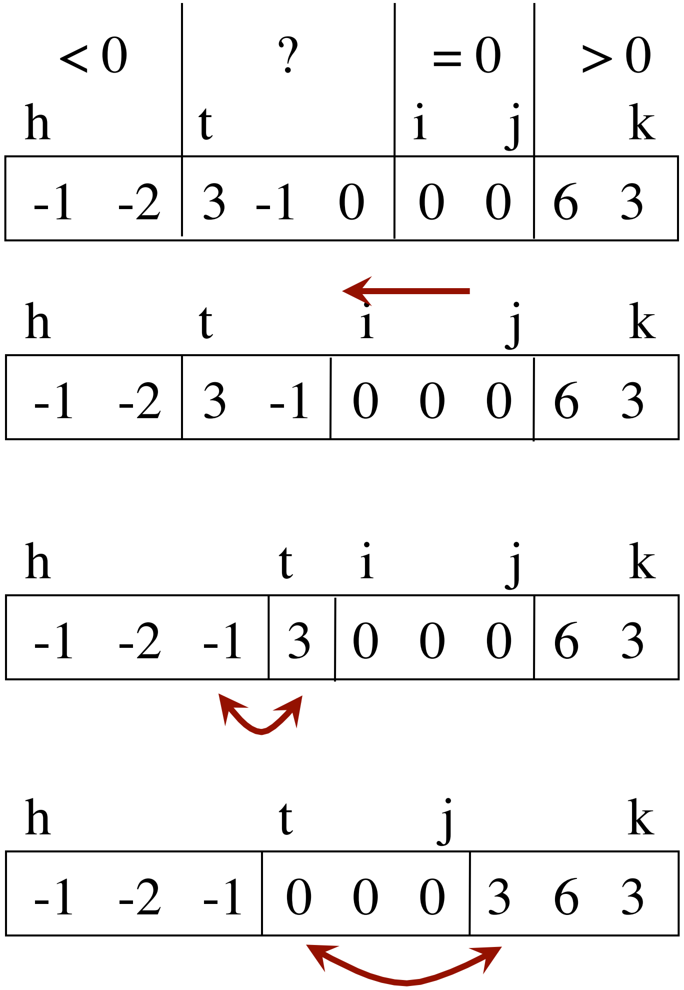
-1 -2 | 3 -1 | 0 0 0 | 6 3

h t i j k

-1 -2 -1 | 3 | 0 0 0 | 6 3

Dutch National Flag Algorithm

```
def dnf(b, h, k):
    """Returns: partition points as a tuple (i,j)"""
    t = h; i = k+1, j = k;
    # inv: b[h..t-1] < 0, b[t..i-1] ?, b[i..j] = 0, b[j+1..k] > 0
    while t < i:
        if b[i-1] < 0:
            swap(b,i-1,t)
            t = t+1
        elif b[i-1] == 0:
            i = i-1
        else:
            swap(b,i-1,j)
            i = i-1; j = j-1
    # post: b[h..i-1] < 0, b[i..j] = 0, b[j+1..k] > 0
    return (i, j)
```



-1	-2	3	-1	0	0	0	6	3

-1	-2	3	-1	0	0	0	6	3

-1	-2	-1	3	0	0	0	6	3

-1	-2	-1	0	0	0	3	6	3

Linear Search

- **Vague**: Find first occurrence of v in b[h..k-1].

Linear Search

- **Vague**: Find first occurrence of v in b[h..k-1].
- **Better**: Store an integer in i to truthify result condition post:

post: 1. v is not in b[h..i-1]

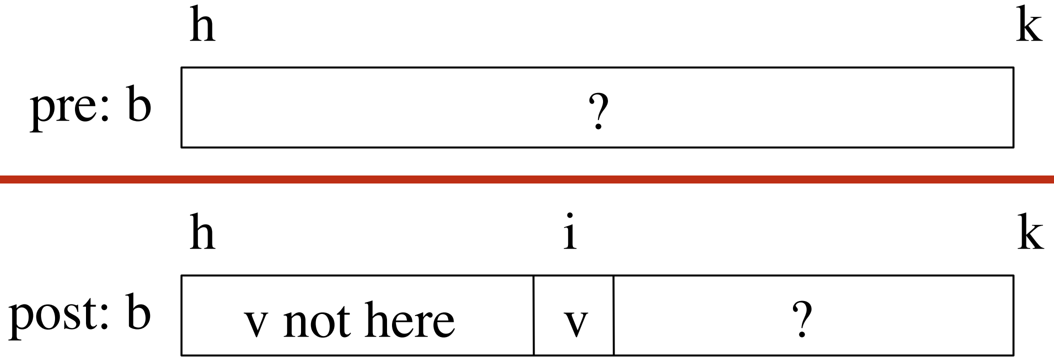
2. i = k OR v = b[i]

Linear Search

- **Vague**: Find first occurrence of v in b[h..k-1].
- **Better**: Store an integer in i to truthify result condition post:

  post: 1. v is not in b[h..i-1]

  2. i = k OR v = b[i]

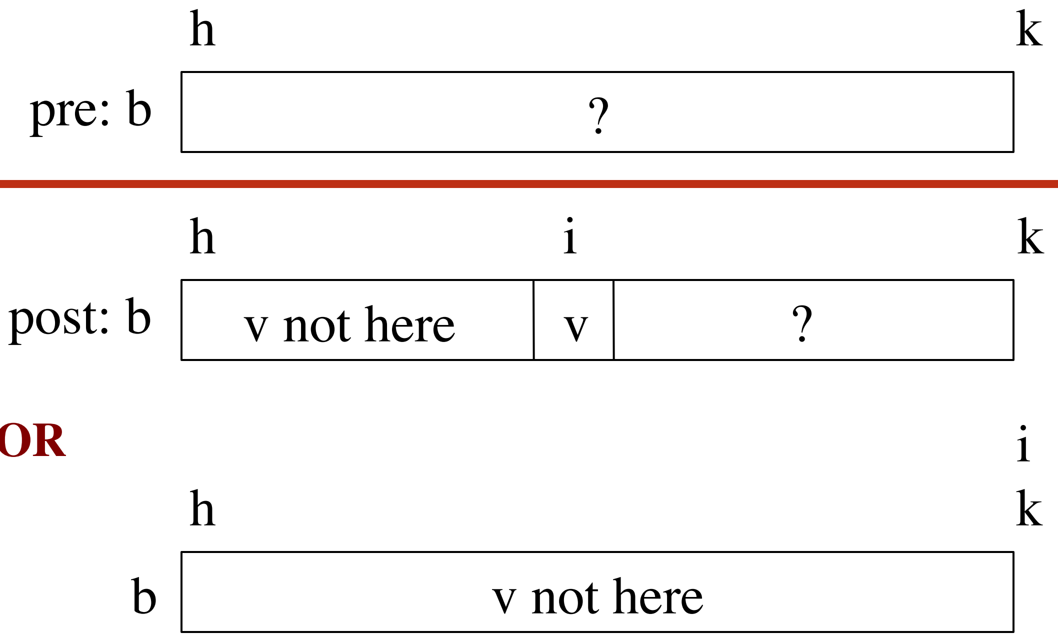


Linear Search

- **Vague**: Find first occurrence of v in b[h..k-1].
- **Better**: Store an integer in i to truthify result condition post:

  post: 1. v is not in b[h..i-1]

  2. i = k OR v = b[i]



h k

pre: b ?

h i k

post: b v not here v ?

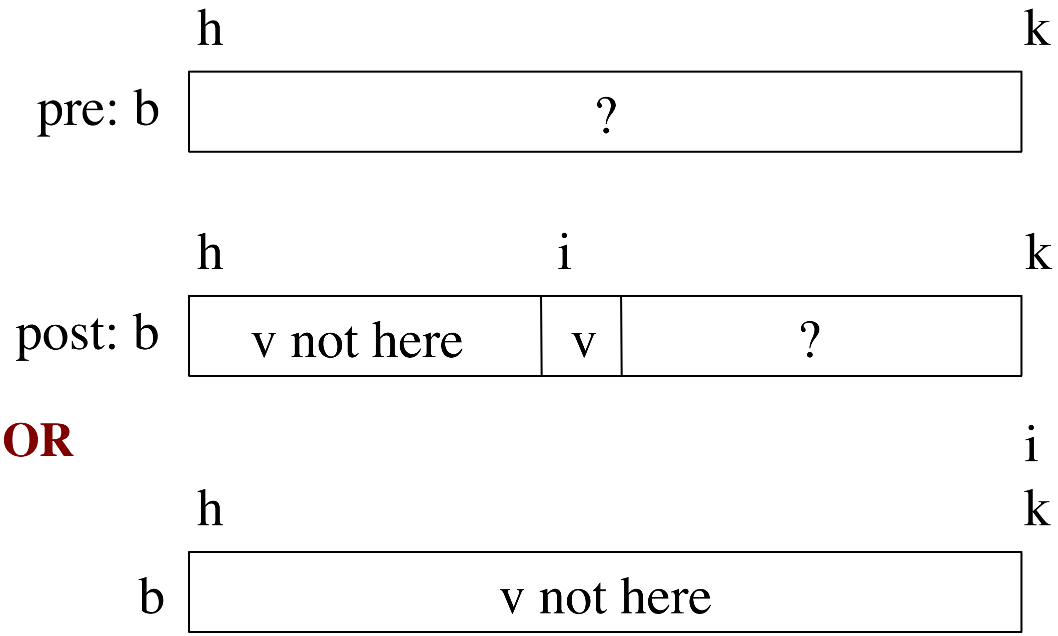
OR

i

h k

b v not here

Linear Search



h k

pre: b ?

h i k

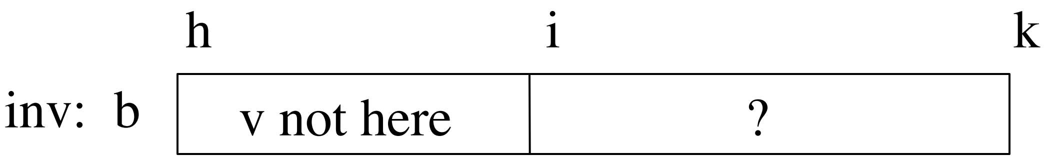
post: b v not here v ?

OR

i

h k

b v not here



h i k

inv: b v not here ?

Linear Search

```
def linear_search(b,c,h):
    """Returns: first occurrence of c in b[h..]"""
    # Store in i the index of the first c in b[h..]
    i = h

    # invariant: c is not in b[0..i-1]
    while i < len(b) and b[i] != c:
        i = i + 1

    # post: c is not in b[h..i-1]
    #          i >= len(b) or b[i] == c
    return i if i < len(b) else -1
```

Analyzing the Loop

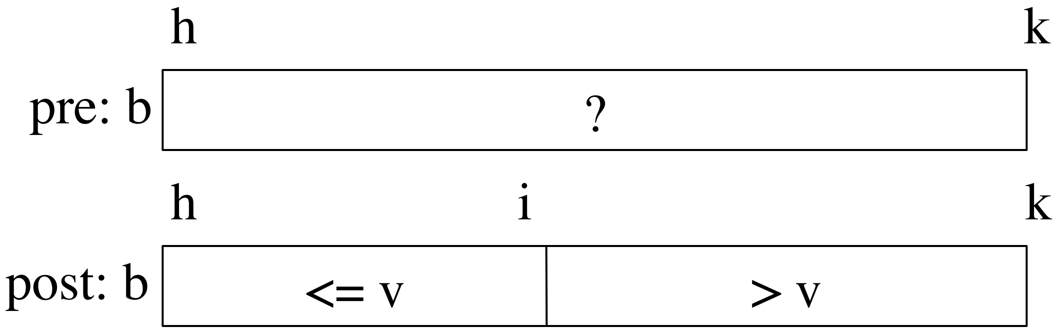
1. Does the initialization make **inv** true?
2. Is **post** true when **inv** is true and **condition** is false?
3. Does the repetend make progress?
4. Does the repetend keep the invariant **inv** true?

Binary Search

- **Vague:** Look for v in **sorted** sequence segment b[h..k].

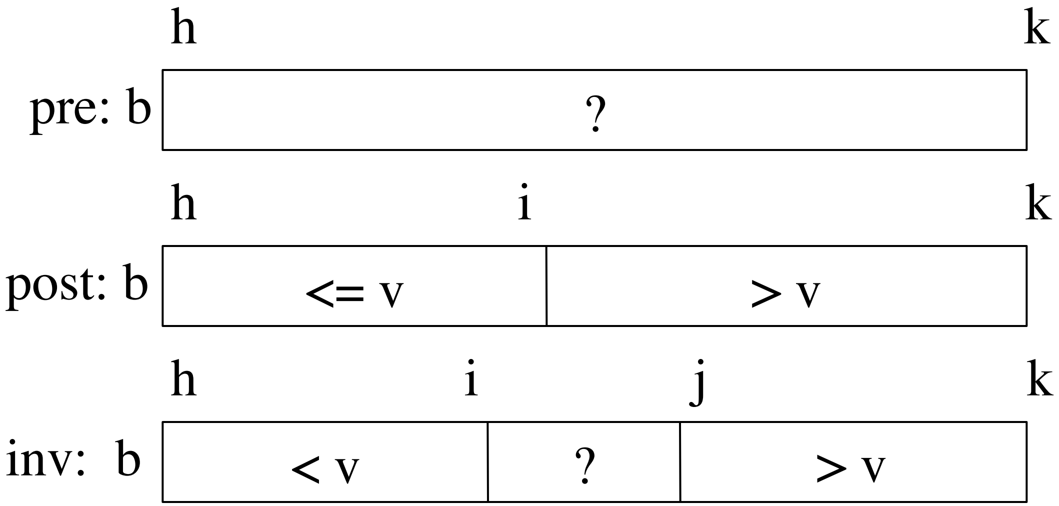
Binary Search

- **Vague:** Look for v in **sorted** sequence segment b[h..k].
- **Better:**
  - Precondition: b[h..k-1] is sorted (in ascending order).
  - Postcondition: b[h..i] <= v and v < b[i+1..k-1]
- Below, the array is in non-descending order:



Binary Search

- **Vague:** Look for v in **sorted** sequence segment b[h..k].
- **Better:**
  - Precondition: b[h..k-1] is sorted (in ascending order).
  - Postcondition: b[h..i] <= v and v < b[i+1..k-1]
- Below, the array is in non-descending order:



Called binary search because each iteration of the loop cuts the array segment still to be processed in half

Extras Not Covered in Class

Loaded Dice

- Sequence p of length n represents n-sided die
  - Contents of p sum to 1
  - p[k] is probability die rolls the number k

1	2	3	4	5	6
0.1	0.1	0.1	0.1	0.3	0.3

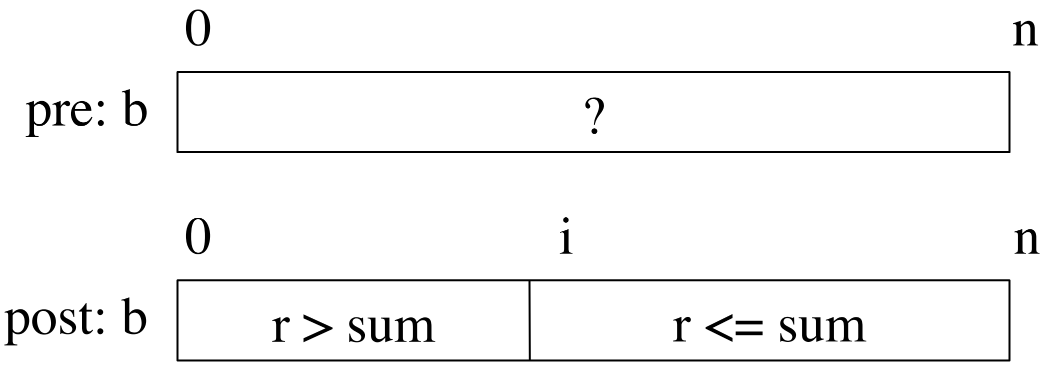
weighted d6, favoring 5, 6

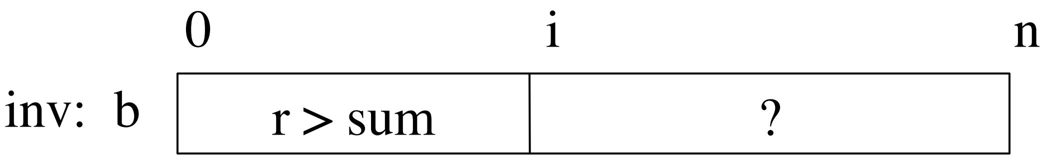
- Goal: Want to “roll the die”
  - Generate random number r between 0 and 1
  - Pick p[i] such that $p[i-1] < r \le p[i]$

0.1	0.1	0.1	0.1	0.3	0.3
0.1	0.2	0.3	0.4	0.7	1.0

Loaded Dice

- **Want**: Value i such that p[i-1] < r <= p[i]





- Same as precondition if i = 0
- Postcondition is invariant + false loop condition

Loaded Dice

```
def roll(p):
    """Returns: randint in 0..len(p)-1; i returned with prob. p[i]
    Precondition: p list of positive floats that sum to 1."""
    r = random.random()     # r in [0,1)
    # Think of interval [0,1] divided into segments of size p[i]
    # Store into i the segment number in which r falls.
    i = 0;    sum_of = p[0]
    # inv: r >= sum of p[0] .. p[i–1]; pEnd = sum of p[0] .. p[i]
    while r >= sum_of:
        sum_of = sum_of + p[i+1]
        i = i + 1

    # post: sum of p[0] .. p[i–1] <= r < sum of p[0] .. p[i]
    return i
```

Analyzing the Loop

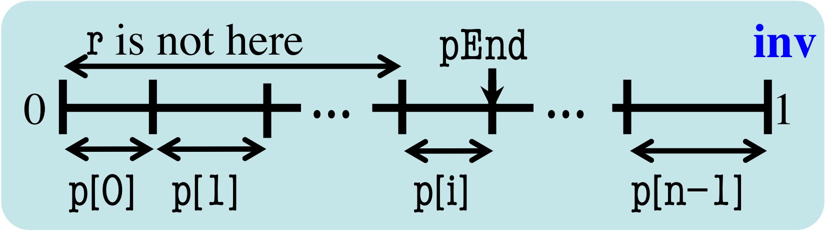
1. Does the initialization make **inv** true?

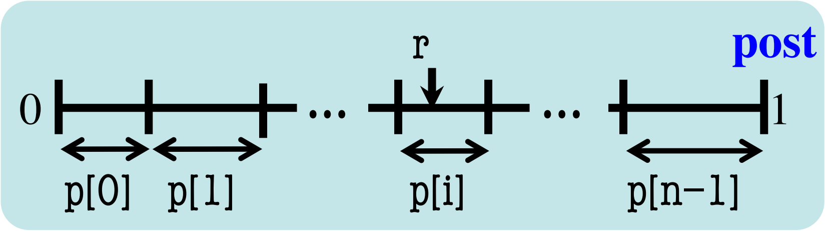
2. Is **post** true when **inv** is true and **condition** is false?

3. Does the repetend make progress?

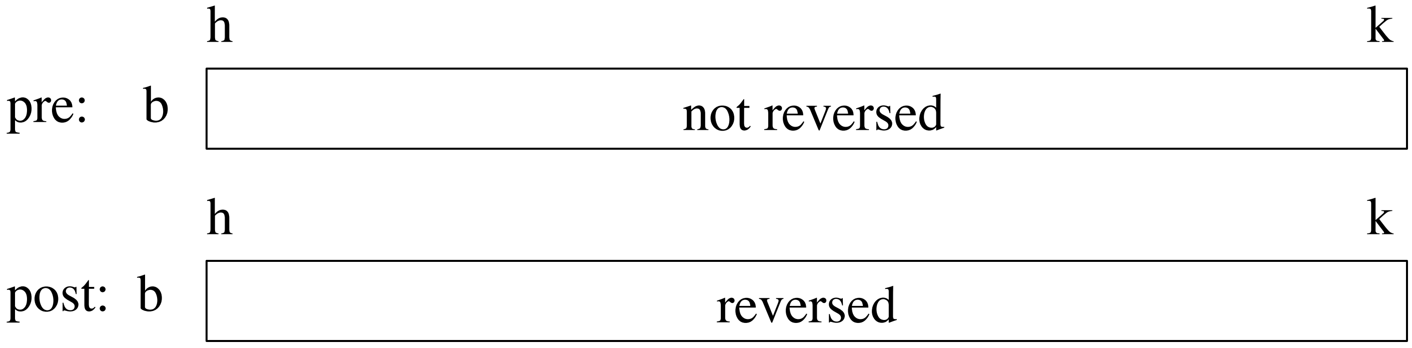
4. Does the repetend keep **inv** true?

r < sum





Reversing a Sequence

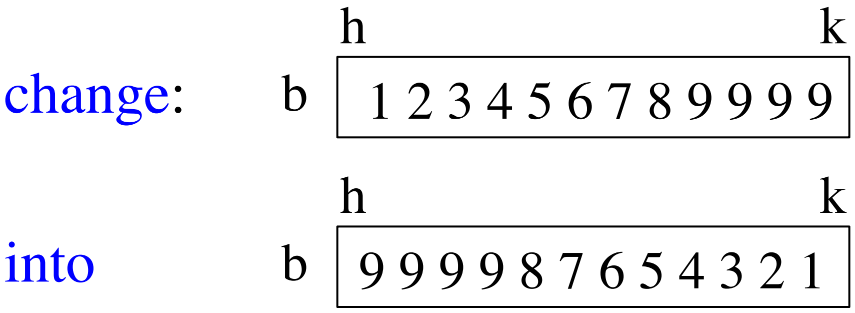


h k

pre: b not reversed

h k

post: b reversed

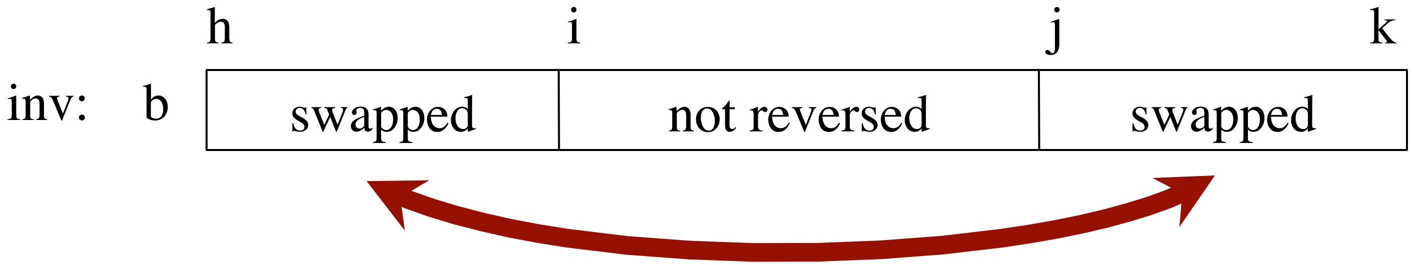


h k

change: b 1 2 3 4 5 6 7 8 9 9 9 9

h k

into b 9 9 9 9 8 7 6 5 4 3 2 1



h i j k

inv: b swapped not reversed swapped