Lecture 22

Loop Invariants

Announcements for This Lecture

Assignment Survey

- How much **completed**?
  - 82% finished B
  - 68% finished C
  - 50% finished D
  - 36% finished E
  - 4% completed it all
- How much **time**?
  - 52% spent ≤ 8 hours
  - 78% spent ≤ 12 hours

The Verdict

- This looks **okay, actually**
  - Part D is half-way(ish)
  - Wanted time to be ~15 hrs
- Not making big changes
  - But Part II is less points
  - **Can get B+/A-** w/o Part II
  - But finish all Part I!
- A6 will go out on time
  - Also about ~15 hours

Exam Info

- **Today**, 7:30–9:00PM
  - Last name **A – G** in Olin 155
  - Last name **H – K** in Olin 165
  - Last name **L – R** in Olin 255
  - Last name **S – Z** in Upson B17
  - **Extra-time Students**: Upson 5130 at 6:30 pm
- **Makeup**: Friday, 6:30-8pm in Upson 5130
  - Only if you have contacted me for permission
- Exams will be graded over the weekend

Some Important Terminology

- **assertion**: true-false statement placed in a program to *assert* that it is true at that point
  - Can either be a **comment**, or an **assert** command
- **precondition**: assertion placed before a statement
  - Same idea as **function precondition**, but more general
- **postcondition**: assertion placed after a statement
- **loop invariant**: assertion supposed to be true before and after each iteration of the loop
  - Distinct from **attribute invariant**
- **iteration of a loop**: one execution of its body

Some Important Terminology

- **assertion**: true-false statement placed in a program to *assert* that it is true at that point
  - Can either be a **comment**, or an **assert** comma
- **precondition**: assertion placed
  - Same idea as **fun** general
- **postc** d after a statement
- **l** assertion supposed to be true before an er each iteration of the loop
  - Distinct from **attribute invariant**
- **iteration of a loop**: one execution of its body

Gives methodology for designing loops

Assertions versus Asserts

- Assertions **prevent bugs**
  - Help you keep track of what you are doing
- Also **track down bugs**
  - Make it easier to check belief/code mismatches
- The assert statement is a (type of) assertion
  - One you are **enforcing**
  - Cannot always convert a comment to an assert

# x is the sum of 1..n

Comment form of the assertion.

The root of all bugs!

x	?	n	1
x	?	n	3
x	?	n	0

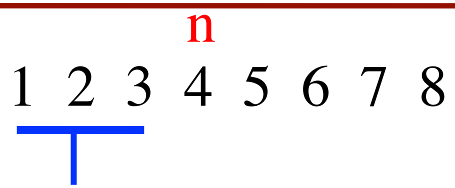
Preconditions & Postconditions

precondition

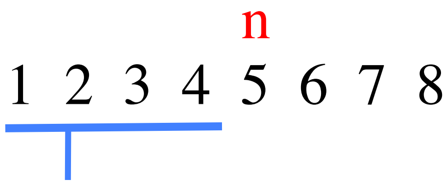
```
# x  = sum of 1..n-1
x = x + n
n = n + 1
# x =  sum of 1..n-1
```

postcondition

- **Precondition:** assertion placed before a segment
- **Postcondition:** assertion placed after a segment



x contains the sum of these (6)

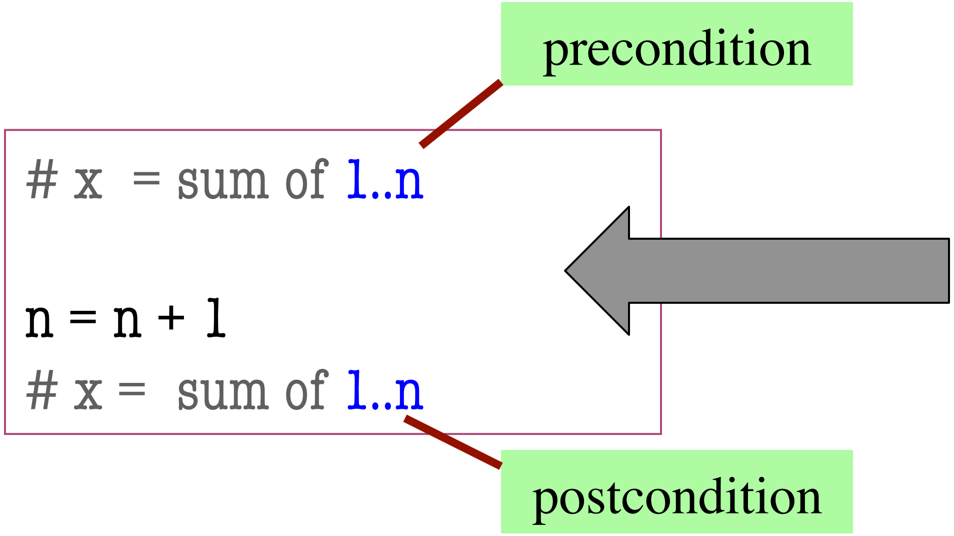


x contains the sum of these (10)

Relationship Between Two

If precondition is true, then postcondition will be true

Solving a Problem



precondition

```
# x  = sum of 1..n

n = n + 1
# x =  sum of 1..n
```

postcondition

What statement do you put here to make the postcondition true?

A: x = x + 1
B: x = x + n
C: x = x + n+1
D: None of the above
E: I don’t know

Solving a Problem

precondition

```
# x  = sum of 1..n

n = n + 1
# x =  sum of 1..n
```

postcondition

What statement do you put here to make the postcondition true?

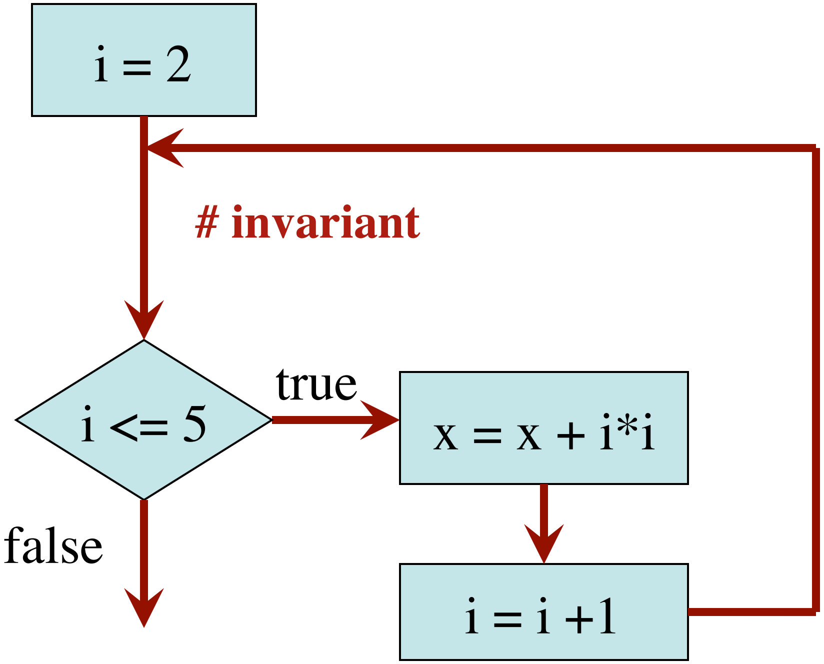
A: x = x + 1
B: x = x + n
C: x = x + n+1
D: None of the above
E: I don’t know

Remember the new value of n

Invariants: Assertions That Do Not Change

- **Loop Invariant:** an assertion that is true before and after each iteration (execution of repetend)

```
x = 0; i = 2
while i <= 5:
    x = x + i*i
    i = i +1
# x = sum of squares of 2..5
```



The loop processes the range 2..5

Invariant:

x = sum of squares of 2..i-1

in terms of the range of integers that have been processed so far

Invariants: Assertions That Do Not Change

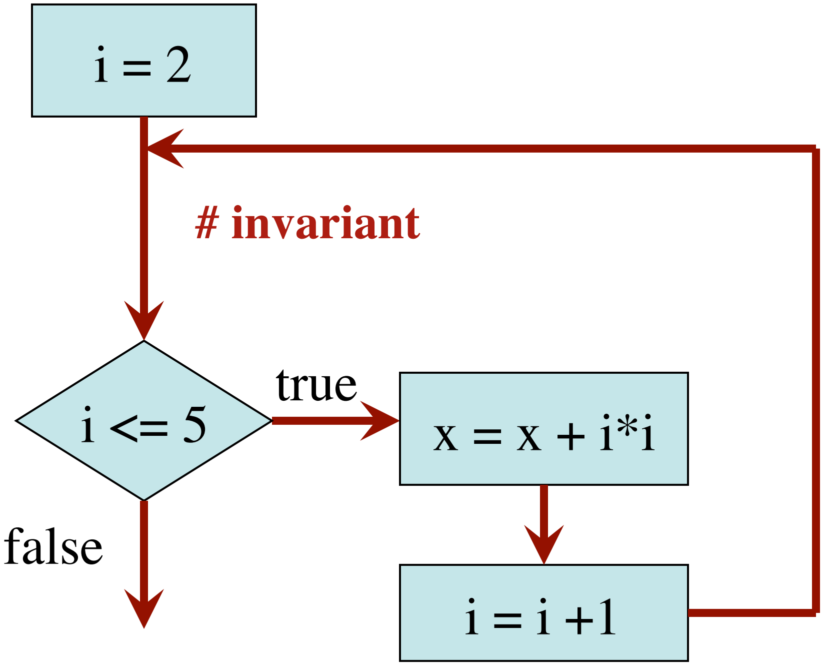
```
x = 0; i = 2
# Inv: x = sum of squares of 2..i-1
while i <= 5:
    x = x + i*i
    i = i +1
# Post: x = sum of squares of 2..5
```

Integers that have been processed:

Range 2..i-1:

x 0

i ?



The loop processes the range 2..5

Invariants: Assertions That Do Not Change

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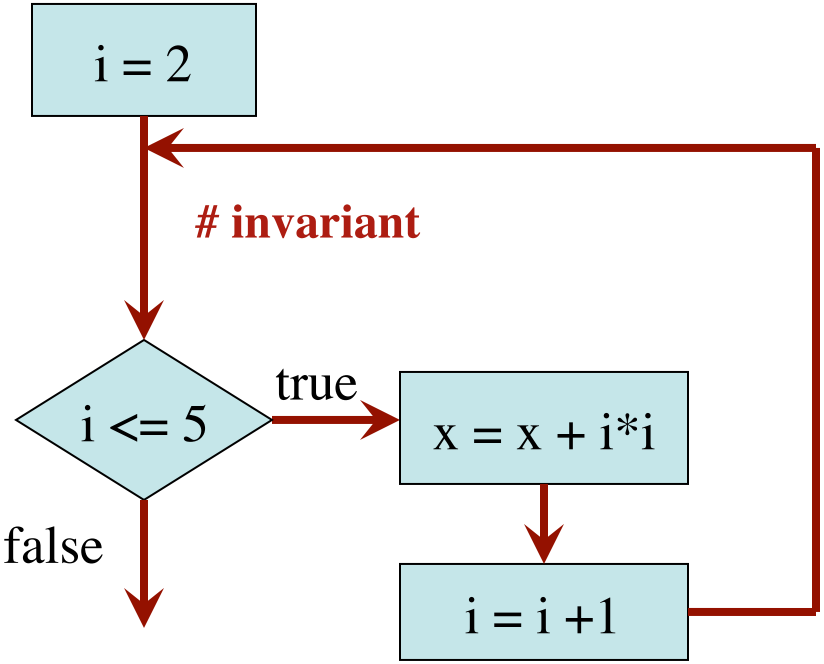
x = 0; i = 2
# Inv: x = sum of squares of 2..i-1
while i <= 5:
    x = x + i*i
    i = i +1
# Post: x = sum of squares of 2..5
```

Integers that have been processed:

Range 2..i-1: 2..1 (empty)

x 0

i ~~?~~ 2



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Invariants: Assertions That Do Not Change

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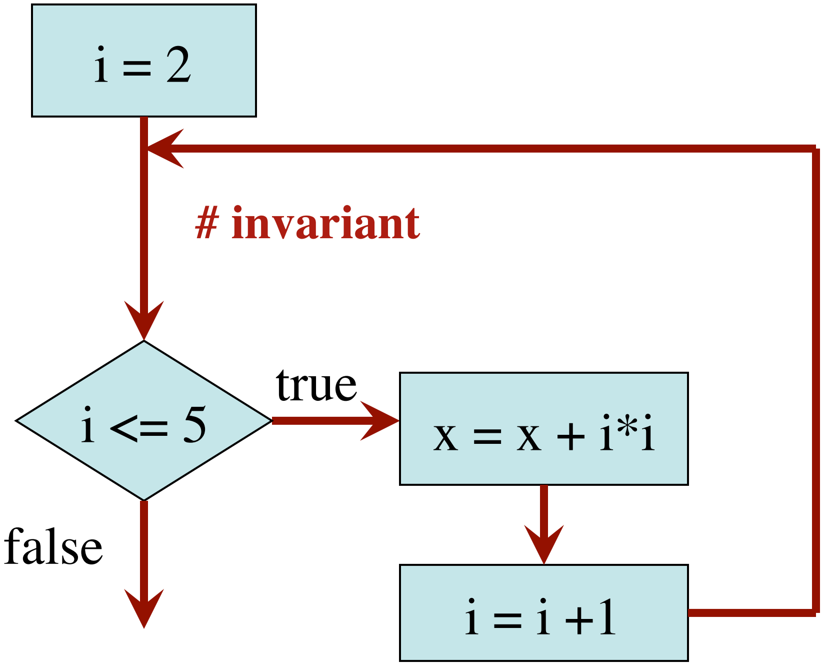
x = 0; i = 2
# Inv: x = sum of squares of 2..i-1
while i <= 5:
    x = x + i*i
    i = i +1
# Post: x = sum of squares of 2..5
```

x ~~0~~ 4

i ~~?~~ ~~2~~ 3

Integers that have been processed: 2

Range 2..i-1: 2..2



The loop processes the range 2..5

Invariants: Assertions That Do Not Change

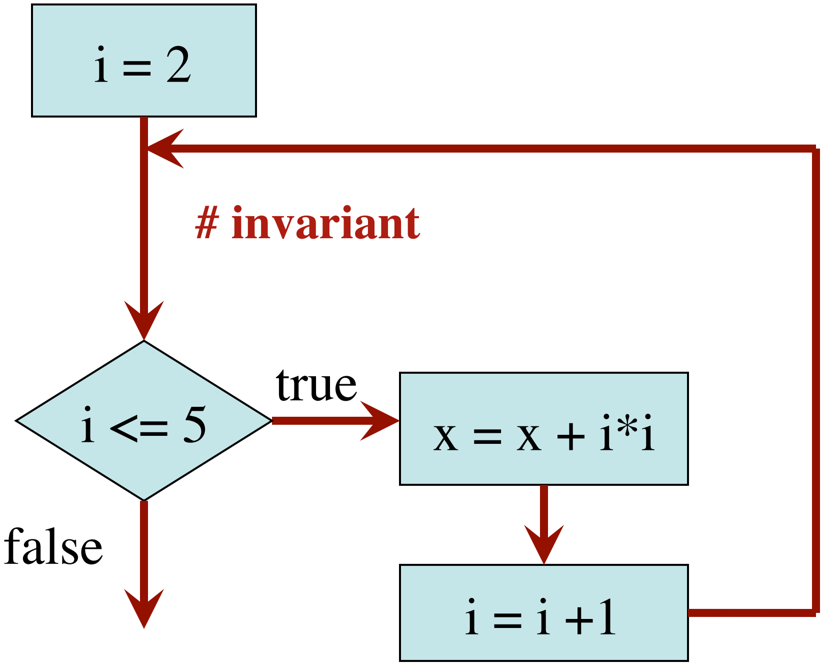
```
x = 0; i = 2
# Inv: x = sum of squares of 2..i-1
while i <= 5:
    x = x + i*i
    i = i +1
# Post: x = sum of squares of 2..5
```

x ~~0~~ ~~4~~ 13

i ~~?~~ ~~2~~ ~~3~~ 4

Integers that have
been processed: 2 , 3

Range 2..i-1: 2..3



The loop processes the range 2..5

Invariants: Assertions That Do Not Change

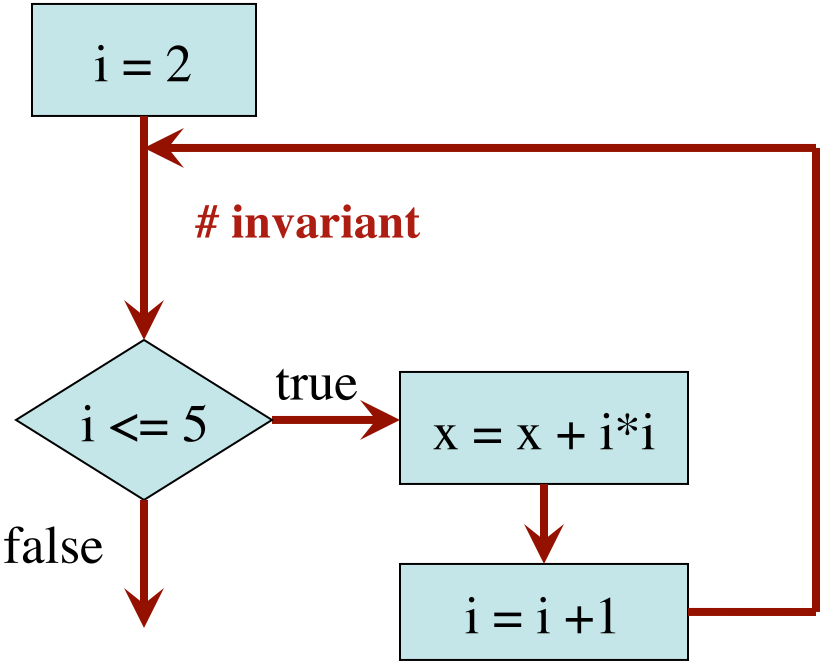
```
x = 0; i = 2
# Inv: x = sum of squares of 2..i-1
while i <= 5:
    x = x + i*i
    i = i +1
# Post: x = sum of squares of 2..5
```

x ~~0~~ ~~4~~ ~~13~~ 29

i ~~?~~ ~~2~~ ~~3~~ ~~4~~ 5

Integers that have
been processed: 2, 3, 4

Range 2..i-1: 2..4



The loop processes the range 2..5

Invariants: Assertions That Do Not Change

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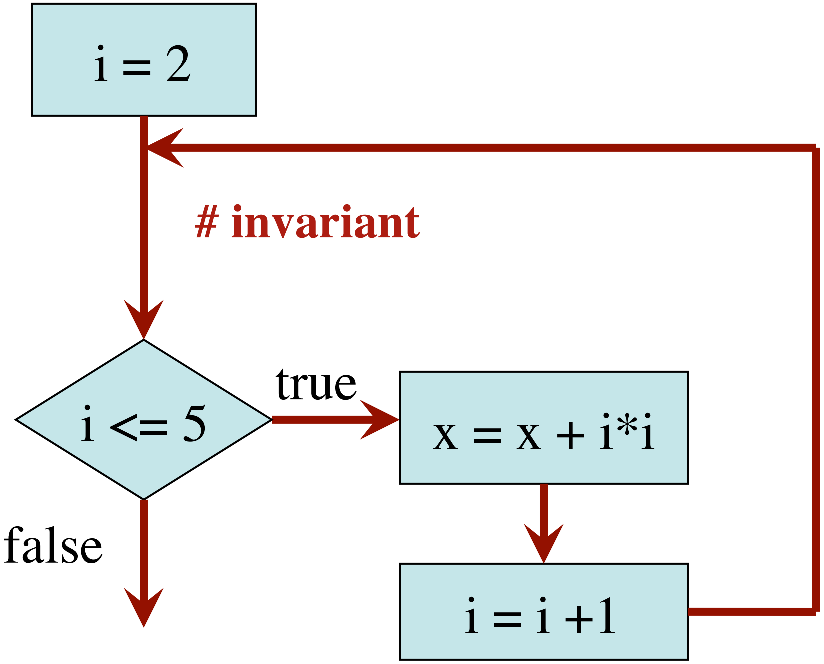
x = 0; i = 2
# Inv: x = sum of squares of 2..i-1
while i <= 5:
    x = x + i*i
    i = i +1
# Post: x = sum of squares of 2..5
```

Integers that have
been processed: 2 , 3, 4, 5

Range 2..i-1: 2..5

x ~~0~~ ~~4~~ ~~13~~ ~~29~~ 54

i ~~?~~ ~~2~~ ~~3~~ ~~4~~ ~~5~~ 6



The loop processes the range 2..5

Invariants: Assertions That Do Not Change

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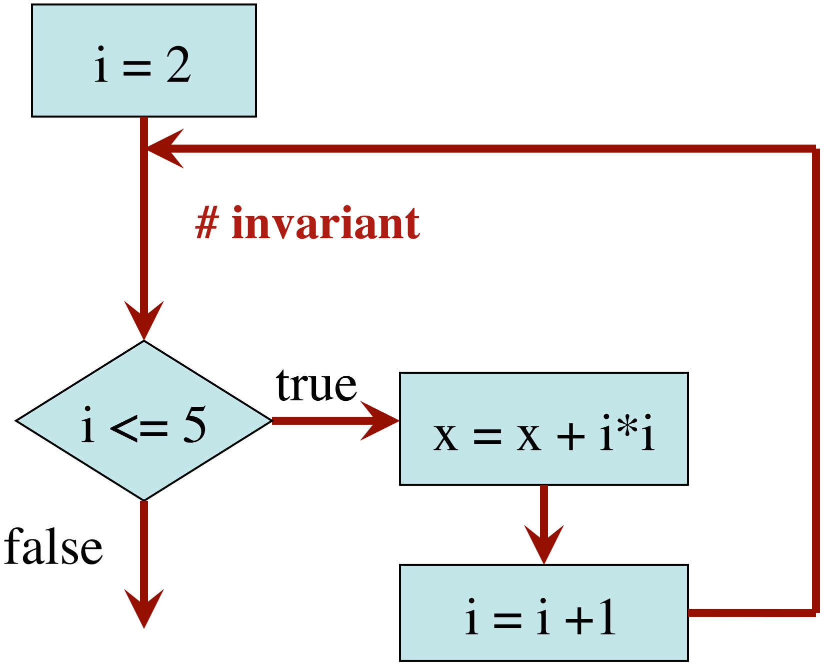
x = 0; i = 2
# Inv: x = sum of squares of 2..i-1
while i <= 5:
    x = x + i*i
    i = i +1
# Post: x = sum of squares of 2..5
```

x	~~0~~	~~4~~	~~13~~	~~29~~	54	
i	~~?~~	~~2~~	~~3~~	~~4~~	~~5~~	6

Integers that have been processed:	2 , 3, 4, 5
Range 2..i-1:	2..5

Invariant was always true just before test of loop condition. So it's true when loop terminates



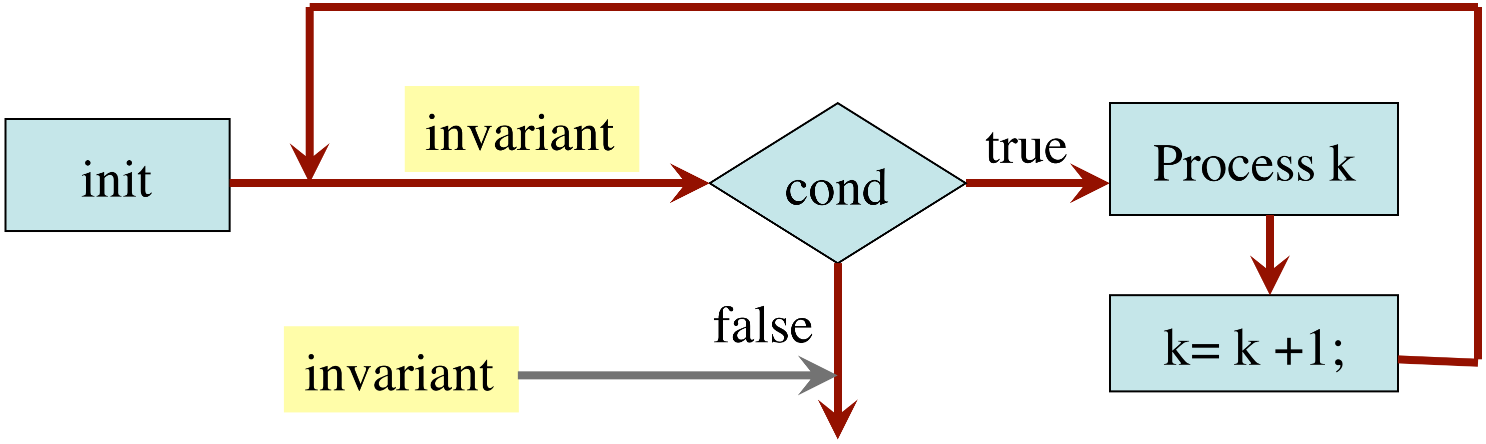
The loop processes the range 2..5

Designing Integer while-loops

```
# Process integers in a..b
# inv: integers in a..k-1 have been processed
k = a
while  k <= b:
    process integer k
    k = k + 1
# post: integers in a..b have been processed
```

Command to do something

Equivalent postcondition



Designing Integer while-loops

1. Recognize that a range of integers b..c has to be processed
2. Write the command and equivalent postcondition
3. Write the basic part of the while-loop
4. Write loop invariant
5. Figure out any initialization
6. Implement the repetend (process k)

Designing Integer while-loops

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```
# Process b..c
```

```
# Postcondition: range b..c has been processed
```

Designing Integer while-loops

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```
# Process b..c



while k <= c:

    k = k + 1
# Postcondition: range b..c has been processed
```

Designing Integer while-loops

1. Recognize that a range of integers b..c has to be processed
2. Write the command and equivalent postcondition
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6. Implement the repetend (process k)

```
# Process b..c


# Invariant: range b..k-1 has been processed
while k <= c:


    k = k + 1
# Postcondition: range b..c has been processed
```

Designing Integer **while**-loops

1. Recognize that a range of integers b..c has to be processed
2. Write the command and equivalent postcondition
3. Write the basic part of the while-loop
4. Write loop invariant
5. Figure out any initialization
6. Implement the repetend (process k)

```
# Process b..c
Initialize variables (if necessary) to make invariant true

# Invariant: range b..k-1 has been processed
while  k <= c:
    # Process k
    k = k + 1

# Postcondition: range b..c has been processed
```

Finding an Invariant

Command to do something

```
# Make b True if no int in 2..n-1 divides n, False otherwise
```

```
# b is True if no int in 2..n-1 divides n, False otherwise
```

Equivalent postcondition

What is the invariant?

Finding an Invariant

Command to do something

```
# Make b True if no int in 2..n-1 divides n, False otherwise



while k < n:
    # Process k;


    k = k +1
# b is True if no int in 2..n-1 divides n, False otherwise
```

Equivalent postcondition

What is the invariant?

Finding an Invariant

Command to do something

```
# Make b True if no int in 2..n-1 divides n, False otherwise



# invariant: b is True if no int in 2..k-1 divides n, False otherwise
while k < n:
    # Process k;



    k = k +1
# b is True if no int in 2..n-1 divides n, False otherwise
```

Equivalent postcondition

What is the invariant?

$1 \;\; \underline{2 \;\; 3 \;\; \ldots \;\; k\text{-}1} \;\; k \;\; k\text{+}1 \;\ldots\; n$

Finding an Invariant

Command to do something

```
# Make b True if no int in 2..n-1 divides n, False otherwise
b = True
k = 2
# invariant: b is True if no int in 2..k-1 divides n, False otherwise
while k < n:
    # Process k;



    k = k +1
# b is True if no int in 2..n-1 divides n, False otherwise
```

Equivalent postcondition

What is the invariant?

1 $\underline{2\ \ 3\ \ \ldots\ \ k\text{-}1}$ k k+1 … n

Finding an Invariant

Command to do something

```
# Make b True if no int in 2..n-1 divides n, False otherwise
b = True
k = 2
# invariant: b is True if no int in 2..k-1 divides n, False otherwise
while k < n:
    # Process k;
    if n % k == 0:
        b = False
    k = k +1
# b is True if no int in 2..n-1 divides n, False otherwise
```

Equivalent postcondition

What is the invariant?

$1\ \underline{2\ \ 3\ \ \ldots\ \ k\text{-}1}\ \ k\ \ k\text{+}1\ \ldots\ n$

Finding an Invariant

```
# set x to # adjacent equal pairs in s[0..len(s)-1]



while k < len(s):
    # Process k

    k = k + 1
# x = # adjacent equal pairs in s[0..len(s)-1]
```

Command to do something

for s = 'ebeee', x = 2

Equivalent postcondition

k: next integer to process.
Which have been processed?

A: 0..k
B: 1..k
C: 0..k–1
D: 1..k–1
E: I don't know

Finding an Invariant

```
# set x to # adjacent equal pairs in s[0..len(s)-1]



while k < len(s):
    # Process k

    k = k + 1
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```

Command to do something

for s = 'ebeee', x = 2

Equivalent postcondition

k: next integer to process.
Which have been processed?

A: 0..k

B: 1..k

C: [✓ 0..k–1]

D: 1..k–1

E: I don't know

What is the invariant?

A: x = no. adj. equal pairs in s[1..k]

B: x = no. adj. equal pairs in s[0..k]

C: x = no. adj. equal pairs in s[1..k–1]

D: x = no. adj. equal pairs in s[0..k–1]

E: I don't know

Finding an Invariant

```
# set x to # adjacent equal pairs in s[0..len(s)-1]


# inv: x = # adjacent equal pairs in s[0..k-1]
while k < len(s):
    # Process k

    k = k + 1
# x = # adjacent equal pairs in s[0..len(s)-1]
```

Command to do something

for s = 'ebeee', x = 2

Equivalent postcondition

k: next integer to process.
Which have been processed?

A: 0..k

B: 1..k

C: [✓ 0..k–1]

D: 1..k–1

E: I don't know

What is the invariant?

A: x = no. adj. equal pairs in s[1..k]

B: x = no. adj. equal pairs in s[0..k]

C: x = no. adj. equal pairs in s[1..k–1]

D: [✓ x = no. adj. equal pairs in s[0..k–1]]

E: I don't know

Finding an Invariant

```
# set x to # adjacent equal pairs in s[0..len(s)-1]
x = 0

# inv: x = # adjacent equal pairs in s[0..k-1]
while k < len(s):
    # Process k

    k = k + 1
# x = # adjacent equal pairs in s[0..len(s)-1]
```

Command to do something

for s = 'ebeee', x = 2

Equivalent postcondition

k: next integer to process.
What is initialization for k?

A: k = 0
B: k = 1
C: k = –1
D: I don’t know

Finding an Invariant

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# set x to # adjacent equal pairs in s[0..len(s)-1]
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k = 1
# inv: x = # adjacent equal pairs in s[0..k-1]
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Command to do something

for s = 'ebeee', x = 2

Equivalent postcondition

k: next integer to process.
What is initialization for k?

A: k = 0
B: [✓ k = 1]
C: k = –1
D: I don't know

Which do we compare to "process" k?

A: s[k] and s[k+1]
B: s[k-1] and s[k]
C: s[k-1] and s[k+1]
D: s[k] and s[n]
E: I don't know

Finding an Invariant

```
# set x to # adjacent equal pairs in s[0..len(s)-1]
x = 0
k = 1
# inv: x = # adjacent equal pairs in s[0..k-1]
while k < len(s):
    # Process k
    x = x + 1 if (s[k-1] == s[k]) else 0
    k = k + 1
# x = # adjacent equal pairs in s[0..len(s)-1]
```

Command to do something

for s = 'ebeee', x = 2

Equivalent postcondition

k: next integer to process.
What is initialization for k?

A: k = 0
B: [✓ k = 1]
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D: I don’t know

Which do we compare to “process” k?

A: s[k] and s[k+1]
B: [✓ s[k-1] and s[k]]
C: s[k-1] and s[k+1]
D: s[k] and s[n]
E: I don’t know

Reason carefully about initialization

```
# s is a string; len(s) >= 1
# Set c to largest element in s
c = ??
k = ??
# inv:
while k < len(s):
    # Process k
    k = k+1
#  c = largest char in s[0..len(s)–1]
```

Command to do something

Equivalent postcondition

1. What is the invariant?

Reason carefully about initialization

```
# s is a string; len(s) >= 1
# Set c to largest element in s
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Command to do something

Equivalent postcondition

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```

Command to do something

Equivalent postcondition

1. What is the invariant?
2. How do we initialize c and k?

A: k = 0; c = s[0]

B: k = 1; c = s[0]

C: k = 1; c = s[1]

D: k = 0; c = s[1]

E: None of the above

Reason carefully about initialization

```
# s is a string; len(s) >= 1
# Set c to largest element in s
c = ??
k = ??
# inv: c is largest element in s[0..k–1]
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Command to do something

Equivalent postcondition

1. What is the invariant?
2. How do we initialize c and k?

A: k = 0; c = s[0]

[✓ B: k = 1; c = s[0]]

C: k = 1; c = s[1]

D: k = 0; c = s[1]

E: None of the above

An empty set of characters or integers has no maximum. Therefore, be sure that 0..k–1 is not empty. You must start with k = 1.